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On constructive nilpotent groups

In connection with the development of the theory of algorithms a study of the problems of computability of important classes of algebraic systems are currently relevant. Groups of unitriangular matrices over the ring are a classic representative of the class of nilpotent groups and have numerous applications both in group theory and in its applications. In this paper we investigate the questions of computability of nilpotent groups. The connection between the bases of the subgroup of the nilpotent torsion-free group and the quotient of this subgroup are found here. Sufficient condition of computability of nilpotent groups in terms of their subgroups and quotient by this subgroup are given. On the basis of these conditions, we exhibit find a large class of computable subgroups of the group of unitriangular matrices of degree 3 over the ring of polynomials in one variable with integer coefficients. In particular, it is proved that every Abelian subgroup of this group is computable. It has been established that any computable numbering of a non-Abelian subgroup of the group of all unitriangular matrices of degree 3 over the polynomial ring in one variable with integer coefficients induces in any computable numbering its maximal subgroups and quotient of it. We obtain a sufficient condition for computability of the quotient of a computable nilpotent group by its periodic part. We find a sufficient condition for computability of a nilpotent group, enriched by an additional predicate root extraction.

Key words: nilpotent group, unitriangular group of matrices over the ring of polynomials in one variable with integer coefficients, the center of the group.

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Есептелінетін нильпотент топтар туралы

Алгоритмдер теориясының дамуына байланысты алгебралық жүйелердің маңызды кластарының есептелімділік мәселелерін шешуді зерттеу өзекті мәселелердің біріне айналды. Сақинадағы униүшбұрышты матрицалар тобы нильпотентті топтар класстарының классикалық өкілі болып табылады және көптеген қолданылымдары тек қана топтар теориясында ғана емес, оның қосымшалары үшін де маңызды орын алған. Бұл жұмыста нильпотентті топтардың есептелімді болуы туралы сұрақ зерттелген. Айналымсыз нильпотентті топтардың ішкі топтарының базистері мен осы ішкі топ бойынша факторттоптардың арасында байланыс табылған. Нильпотентті топтардың есептелімділігінің жеткілікті шарты оның ішкі топтары мен осы ішкі топ бойынша факторттоптар тілінде берілген. Осы алынған нәтижелер негізінде бүтін коэффицентті бір айнымалы көпмүшеліктер сақинасында үшінші ретті барлық униүшбұрышты матрицалар топтарының есептелімді ішкі топтарының кең класы табылған. Дербес жағдайда, осы топтың кез келген абельдік ішкі тобы есептелімді екендігі дәлелденген. Бүтін коэффицентті бір айнымалы көпмүшеліктер сақинасында үшінші ретті барлық униүшбұрышты матрицалар топтарының абельдік емес ішкі топтарының кез келген есептелімді нөмірлеуі оның кез келген максималды ішкі топтары мен ол бойынша факторттоптарының есептелімді нөмірлеуін айқындайды. Есептелімді нильпотентті топтардың оның периодты бөлігі бойынша факторттоптарының есептелімділігінің жеткілікті шарты алынған. Нильпотентті топтардың түбір табу қосымша предикатымен толықтырылған есептелімділігінің жеткілікті шарты табылған.

Түйін сөздер: нильпотент топ, есептелінетін топ, бүтін коэффициентті бір айнымалы көпмүшеліктер сақинасы үстінде анықталған барлық үш дәрежелі униүшбұрышты матрицалар тобы, топтың центрі.

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О вычислимых нильпотентных группах

В связи с развитием теории алгоритмов актуальным является исследование проблем вычислимости важных классов алгебраических систем. Группы унитреугольных матриц над кольцом составляют важный класс нильпотентных групп, имеющий многочисленные применения как в самой теории групп, так и в её приложениях. В данной работе исследуются вопросы вычислимости нильпотентных групп. Найдена связь между базисами подгруппы нильпотентной группы без кручения и факторгруппы по этой подгруппе. Дано достаточное условие вычислимости нильпотентной группы на языке её подгруппы и факторгруппы по этой подгруппе. На основе этих результатов найден широкий класс вычислимых подгрупп группы всех унитреугольных матриц степени три над кольцом многочленов от одной переменной с целыми коэффициентами. В частности доказано, что любая абелева подгруппа этой группы вычислима. Установлено, что любая вычислимая нумерация неабелевой подгруппы группы всех унитреугольных матриц степени три над кольцом многочленов от одной переменной с целыми коэффициентами индуцирует вычислимые нумерации любой ее максимальной подгруппы и факторгруппы по ней. Получено достаточное условие вычислимости факторгруппы вычислимой нильпотентной группы по её периодической части. Найдено достаточное условие вычислимости нильпотентной группы, обогащенной дополнительным предикатом извлечения корней.
Ключевые слова: нильпотентная группа, вычислимая группа, унитреугольная группа матриц размерности 3 над кольцом многочленов от одной переменной с целыми коэффициентами, центр группы.

1. Introduction

The study of computable groups was begun in [1], where A.I. Mal'cev posed the general problem: Determine the constructive numberings that given abstractly defined groups admit. Also in the same work some description was given of computable torsion-free Abelian groups. This problem was studied by Yu.L. Ershov, S.S. Goncharov, and other mathematicians (см. [2]). For the results on constructivizations of nilpotent groups see, for example, [3], where it was proved that each constructivization of a torsion-free locally nilpotent group naturally extends to a constructivization of its completion; [4], [5], where, for each $n > 0$, a nilpotent group was constructed whose algorithmic dimension is equal to n and sufficient conditions were given for the autostability of these groups; [6], where an example of a computable nilpotent group was constructed whose quotient group is noncomputable in the periodic part. Connections between the constructivization of a commutative associative ring K with unity and matrix groups over K were studied in [7], [8]. It was proved in [7] that the matrix groups $GL_n(K)$, $SL_n(K)$ and $UT_n(K)$ for $n \geq 3$ are constructive if and only if the ring K is constructive. In [8] an example was constructed of a nonconstructive ring K for which the group $GL_2(K)$ is constructive. In [9]–[13], some criteria were obtained for the existence of a positive (constructive) numbering of a torsion-free nilpotent group. In this paper, we obtain conditions of the computability of the group and on its basis is described a broad class of computable subgroups of the group $UT_3(Z[x])$ of all unitriangular matrices of degree 3 over the ring of polynomials in one variable with integer coefficients. It is proved that any Abelian subgroup of the group $UT_3(Z[x])$ and the quotient of this group by any maximal Abelian subgroup is computable. It is obtained the sufficient condition of not computability of periodic nilpotent group of class 2. It is proved that there is an algorithm of root extraction in a finitely generated torsion-free nilpotent group. Sufficient conditions for the existence of this algorithm for nilpotent groups without torsion stage 2 and computability of the quotient of computable nilpotent group on its periodic part. All notions of the theory of constructive groups, can be found in [2] and [15] ; and those of the theory of abstract groups, can be found in – [16], [17]. We only recall some of them. Suppose that ω is the set of all naturals, G is a group, and $\nu : \omega \to G$ is a mapping from ω onto G. The pair (G, ν) is called an *numbered group*. A numbered group is called *constructive* if there is an algorithm that, from all naturals n, m and s, verifies the validity of the equalities $\nu n = \nu m$ and $\nu n \cdot \nu m = \nu s$. A group G is called *computable* if there exists an numbering $\nu : \omega \to G$ of G such that (G, ν) is a constructive group. A subgroup H in an numbered group (G, ν) is called *computable* (*computably enumerated*) in (G, ν) if the set $\nu^{-1}H$ is computable (computably enumerated). If (G, ν) is a constructive group then ν is called a *computable numbering* of G. By $UT_3(Z[x])$ we denote the group of all unitriangular matrices of degree 3 over the ring of polynomials in one variable with integer coefficients, $\mathbb{Z}(G)$ is the center of the group G, $C(g)$ is centralizer of element g.

2. On computable subgroups of nilpotent groups

Here, it is obtained a sufficient condition for the computability of a certain class of nilpotent groups. Based on this it is proven computability of subgroup G of group of all unitriangular matrices of dimension 3 over the ring of polynomials in one variable with integer coefficients such that for some integers m and any matrix $g \in G$ of degree $g_{11}(x)(g_{23}(x))$ is not more than m.

Theorem 1 *Let G be nilpotent torsion-free group of stage 2, $g_0 \in G\backslash\mathbb{Z}(G)$. Then for the sequence of elements*

$$g_1, g_2, g_3, \ldots \tag{1}$$

from G and subgroup

$$G_0 = gr(\{g_1, g_2, g_3, \ldots\} \cup C(g_0))$$

is valid equivalence:

sequence of commutators

$$[g_0, g_1], [g_0, g_2], [g_0, g_3], \ldots \tag{2}$$

is basis of subgroup $[g_0, G]$ if and only, if

$$\bar{g}_1, \bar{g}_2, \bar{g}_3, \ldots \tag{3}$$

is basis of quotient $\overline{G}_0 = G_0/C(g_0)$

Proof. $\Rightarrow$. Let the sequence (2) be a basis of subgroup $[g_0, G]$. Suppose that

$$g_s^k = g_{i_1}^{k_1} \cdot \ldots \cdot g_{i_n}^{k_n} c,$$

where $k, k_1, \ldots, k_n$ are integers, nonzero and $c \in C(g_0)$, $0 < s$, $0 < i_1 < \ldots < i_n$, $s \notin \{i_1, \ldots, i_n\}$. Then

$$[g_0, g_s]^k = [g_0, g_{i_1}]^{k_1} \cdot \ldots \cdot [g_0, g_{i_n}]^{k_n},$$

i.e. the sequence (2) is linearly dependent, which is contrary to the assumption. Hence the sequence (3) is linearly dependent modulo $C(g_0)$.

Let a element $g \in G_0 \backslash C(g_0)$ is given. Since (2) is basis of subgroup $[g_0, G]$, then

$$[g_0, g]^k = [g_0, g_1]^{k_1} \cdot \ldots \cdot [g_0, g_n]^{k_n},$$

for some numbers $k, k_1, \ldots, k_n$, where $k \neq 0$. Then

$$[g_0, g^k g_1^{-k_1} \cdot \ldots \cdot g_n^{-k_n}] = 1,$$

i.e.

$$g^k = g_1^{k_1} \cdot \ldots \cdot g_n^{k_n} (mod\ C(g_0)).$$

Hence the sequence (3) is basis of a group $\overline{G}_0$.

$\Leftarrow$. Let the sequence (3) is basis of the group $\overline{G}_0$. Let's prove, that the sequence (2) is linearly independent. Assume the contrary, i.e.

$$[g_0, g_{i_1}]^{k_1} \cdot \ldots \cdot [g_0, g_{i_n}]^{k_n} = 1.$$

Then

$$[g_0, g_{i_1}^{k_1} \cdot \ldots \cdot g_{i_n}^{k_n}] = 1,$$

i.e. $g_{i_1}^{k_1} \cdot \ldots \cdot g_{i_n}^{k_n} \in C(g_0)$, hence

$$\overline{g}_{i_1}^{k_1} \cdot \ldots \cdot \overline{g}_{i_n}^{k_n} = 1,$$

that it is impossible. Hence the sequence (2) is linearly independent.

Let element $g \notin C(g_0)$ is given. Since the sequence (3) is a basis of the group $\overline{G}$, then

$$\overline{g}^k = \overline{g}_1^{k_1} \ldots \overline{g}_n^{k_n},$$

for some integers $k, k_1, \ldots, k_n$, where $k \neq 0$. Thence $g^k = g_1^{k_1} \ldots g_n^{k_n} c$ for some element $c \in C(g_0)$. Then

$$[g_0, g]^k = [g_0, g_1]^{k_1} \cdot \ldots \cdot [g_0, g_n]^{k_n},$$

i.e. (2) is a basis of a subgroup $[g_0, G]$ □

Theorem 2 *Let G be a group and H be its normal subgroup such, that the followings are valid:*

1^0. *There exist computable numberings α and β of groups H and $\overline{G} = G/H$ respectively;*

2^0. *There exists computably enumerable sequence of elements in $(\overline{G}, \beta)$*

$$\overline{g_0}, \overline{g_1}, \overline{g_2}, \ldots \tag{4}$$

such that

$$\overline{G} = (\overline{g_0}) \oplus (\overline{g_1}) \oplus \ldots \tag{5}$$

3^0. $[g_i, g_j] = \alpha f(i,j)$ *for some computable function $f(i,j)$.*

Then a group G is computable.

Proof. For any element $g \in G\backslash H$ there exists a sequence of integers

$$\pi_g = \langle i_0, k_0, \dots, i_{s-1}, k_{s-1}, m\rangle$$

such that

$$g = g_{i_0}^{k_0} \cdot \dots \cdot g_{i_{s-1}}^{k_{s-1}} \alpha m$$

and an element g is uniquely determined by the sequence π_g. From 1^0, 2^0 follows that a set $\pi = \{\pi_g | g \in G\}$ is computable. Let element

$$h = g_{j_0}^{l_0} \cdot \dots \cdot g_{j_{t-1}}^{l_{t-1}} \alpha n$$

is given.

Then from the (5) and 3^0 follow that by sequences π_g and π_h ones can effectively determines the sequence π_{gh}, and also an equality of elements g and h. □

Theorem 3 *Let G be a subgroup of a group of all unitriangular matrices of degree 3 over the ring of polynomials in one variable with integer coefficients such that the following conditions are true:*

1. There exists a natural number m such that for any matrix $g \in G$ a degree $deg\ g_{12} \leq m$;

2. There exists matrix $g \in G$ such that $deg\ g_{23} \geq n$ for any number $n \in \omega$.

Then the group G is computable.

We preface the proof the following statement.

Lemma 1 *There exists a subgroup G_0 in G such that for it the condition 2 of the theorem is true and for any matrix $g \in G_0$ the $g_{12}(x) = 0$ is true.*

Proof. From conditions 1, 2 of the theorem follow that there exists the smallest number m_0 such that there are infinitely many matrices

$$g^{(0)}, g^{(1)}, g^{(2)}, \dots$$

from G, for which the following is true

$$deg\ g_{12}^{(i)}(x) = m_0$$

$$deg\ g_{23}^{(0)}(x) < deg\ g_{23}^{(1)}(x) < \dots$$

$i = 0, 1, 2, \dots$

Let's prove that $m_0 = 0$. Suppose that $m_0 \neq 0$. Then for any $i > 0$ there exist such numbers k_i and l_i, that

$$deg(g^{(0)^{k_i}} \cdot g^{(i)^{l_i}})_{12}(x) < m_0,$$
$$deg(g^{(0)^{k_i}} \cdot g^{(i)^{l_i}})_{23}(x) = deg\ g_{23}^{(i)}(x),$$

which contradicts of the choice of number m_0.Hence $m_0 = 0$ and subgroup $G_0 = \{g_{12}(x) = 0 \mid g \in G\}$ will be required. □

Lemma 2 *Let G_0 be a subgroup of the group G, which is defined in Lemma 1 and the element $g^{(0)} \in G_0\backslash\{1\}$. Then a dimension of subgroup $[g^{(0)}, G]$ is finite.*

Proof. Let $g \in G\backslash G_0$ be any element . Then, by condition 1, we have $deg\ g_{12}(x) \leq m$. Thence

$$[g^{(0)}, g] = \begin{pmatrix} 1 & 0 & g_{23}^{(0)}(x)g_{12}(x) \\ & 1 & 0 \\ & & 1 \end{pmatrix}.$$

Then, by condition 1, we have

$$deg[g^{(0)}, g]_{13} \leq g_{23}^{(0)}(x) + m$$

Hence the dimension of the Abelian group $[g^{(0)}, G]$ is finite. □

Now we prove that for the group G all the conditions of Theorem 2 are valid. By the lemma 2 the dimension of a subgroup $[g_0, G]$ is finite. Let

$$[g_0, g_1], \ldots, [g_0, g_n] \tag{6}$$

be a basis of the subgroup $[g_0, G]$, and $G_0 \leq C(g_0)$, where G_0 be subgroup, which is defined in lemma 1 and $\overline{G} = G/G_0$. From (6) and the theorem 1 follow that

$$\overline{g}_1, \ldots, \overline{g}_{n-1}$$

is a basis of subgroup $\overline{G}$. If a matrix $g \notin C(g_0)$, then $g_{12}(x) \neq 0$. From this and condition 1 follow that $deg\ g_{12}(x) \leq m$. Then it is easy to check that $\overline{G}$ is a subgroup of direct sum of m-copies of the infinitive cycle group. Therefore, there are matrices $h_0, \ldots, h_{s-1}, s \leq m$ such that

$$\overline{G} = (\overline{h}_0) \oplus \ldots \oplus (\overline{h}_{s-1}) \tag{7}$$

The centralizer $C(g_0)$ consists from all matrices $g \in G$, where $g_{12}(x) = 0$. Therefore $C(g_0) \leq Z[x] \oplus Z[x]$, i.e. $C(g_0) \simeq Z^w$. Hence the subgroup $C(g_0)$ is computable. Since $\overline{G}$ has a finite basis, then $[h_i, h_j]$ is effectively found. Therefore all conditions of the theorem 2 are valid. Hence the group G is computable. □

Remark 1 *The theorem 3 is true and for the case when condition 2 is replaced by "there are such number n, that for any matrix g the degree $deg\ g_{23}(x) \leq n$".*

So the following is true.

Theorem 4 *Let $G \leq UT_3(Z[x])$ and there exists such number m, that for any matrix g is true $deg\ g_{12}(x) \leq m(deg\ g_{23}(x) \leq m)$. Then the group G is computable.*

3. On computability of Abelian subgroups of a group $UT_3(Z[x])$

Theorem 5 *Any Abelian subgroup of the group $UT_3(Z[x])$ is isomorphic to the direct sum of infinite cyclic groups.*

Proof. Let G be Abelian subgroup of the group $UT_3(Z[x])$. Let's consider the possible cases:

1^0. There exists matrix $g \in G$ such that $g_{12}(x) = 0$, $g_{23}(x) \neq 0$. We prove that for any matrix $h \in G$ is true

$$h_{12}(x) = 0 \tag{8}$$

Let's assume the contrary, i.e. $h_{12}(x) \neq 0$. Then

$$[g,h] = \begin{pmatrix} 1 & 0 & -g_{23}(x)h_{12}(x) \\ & 1 & 0 \\ & & 1 \end{pmatrix} \neq 1,$$

i.e. the group G is not Abelian. We have get a contradiction. Hence (8) is valid.

Let a subgroup $G_0 \leq G$ consists from matrices $g \in G$ such that

$$g_{12}(x) = g_{23}(x) = 0.$$

Then G_0 is isomorphic to the some subgroup $Z[x]$. Then G_0 is isomorphic to the direct sum of cyclic groups and it is pure in G.

Quotient $\overline{G} = G/G_0$ is also isomorphic to the direct sum of cyclic groups. From this and the L.Ya. Kulikov's theorem [17] follow that $G \simeq G_0 \oplus \overline{G}$, i.e. G is isomorphic to the direct sum of cyclic groups.

2^0. There exists matrix $g \in G$ such that $g_{23}(x) = 0$, $g_{12}(x) \neq 0$. This case is similar.

3^0. For any matrix $g \in G$ is true $g_{12}(x) = g_{23}(x) = 0$. Then $G \leq < Z[x], + >$. Therefore G is a direct sum of cyclic groups.

4^0. Case 1^0 - 3^0 does not hold. Then any matrix G is hold:

$$g_{12}(x) = 0 \text{ if and only if } \quad g_{23}(x) = 0.$$

Let g and h are matrices from G such that

$$g_{12}(x) \neq 0, h_{12}(x) \neq 0.$$

Then $g_{23}(x) \neq 0$, $h_{23}(x) = 0$ are true. Let's prove that

$$\frac{g_{12}(x)}{g_{23}(x)} = \frac{h_{12}(x)}{h_{23}(x)} \tag{9}$$

Indeed, the commutator

$$[g,h] = \begin{pmatrix} 1 & 0 & g_{12}(x)h_{23}(x) - h_{12}(x)g_{23}(x) \\ & 1 & 0 \\ & & 1 \end{pmatrix} = 1,$$

From this we have the equality (9). If we fix some matrix $g \in G$ such that $g_{12}(x) \neq 0$ and let

$$\frac{g_{12}(x)}{g_{23}(x)} = \lambda(x)$$

then for any matrix $h \in G$, where $h_{23}(x) \neq 0$, we have

$$h_{12}(x) = h_{23}(x)\lambda(x),$$

From this follows, that quotient G/G_0 is isomorphic to the some subgroup of the group $< Z[x], + >$. Again applying the mentioned L.Ya.Kulikov's theorem we get that G is isomorphic to the direct sum of the infinite cyclic groups.

Corollary 1 *Any Abelian subgroup of the group $UT_3(Z[x])$ is computable.*

Indeed, let $G \leq UT_3(Z[x])$ and G Abelian groups. Then by the theorem 5 G is isomorphic to the direct sum of no more than a countable number of copies of the infinite cyclic group. Therefore the G is computable.

Corollary 2 *Any maximal Abelian subgroup $G \leq UT_3(Z[x])$ is the centralizer of any of its elements $g \in G \backslash \mathbb{Z}(G)$.*

Corollary 3 *Let G be not Abelian subgroup of the group $UT_3(Z[x])$. Then any maximal Abelian subgroup A of the group G is the the centralizer of any its elements $a \in A \backslash \mathbb{Z}(G)$.*

Proof. A subgroup $A \neq Z(G)$. Indeed, otherwise, a group would not be maximal. Therefore there exists an element $a \in A \backslash \mathbb{Z}(G)$. Suppose that $a_{12}(x) = 0$, $a_{23}(x) \neq 0$. Then for the element a there is a case 1^0 of the proof of the theorem. Therefore for any element $b \in G$ is true.

$$[b, a] = 1 \Leftrightarrow b_{12}(x) = 0.$$

From this $A = \{b \mid [b, a] = 1\}$, i.e. the group A consists from all centralizers of the element $a \in A$. Similarly, the remaining cases are considered proof. □

Similarly to the corollary 3 the followings are proved.

Corollary 4 *Let G be not Abelian subgroup of the group $UT_3(Z[x])$. Then any maximal Abelian subgroup of the group G is normal in G.*

Corollary 5 *Let G be not Abelian subgroup of the group $UT_3(Z[x])$ and $\nu :\to G$ be some its computable and numbering. Then any maximal Abelian subgroup is computable (recursive) in (G, ν), and therefore the subgroup A and quotient G/A have computable numberings ν_0 u ν_1, induced by the numbering ν.*

4. On a sufficient condition of the computability of the quotient of a nilpotent group of its periodic part

Theorem 6 *Let G be computable nilpotent group and T is its periodic part. If the dimension of the commutant of a quotient G/T is finite, then G/T is computable group.*

Proof. Let (G, ν) be constructive (i.e. computable numbered group). A subgroup T is computably enumerable in (G, ν), i.e. a set $\nu^{-1}T = \{n | \nu_n \in T\}$ is computably enumerable. Then quotient G/T is computably enumerable (i.e. recursively enumerable) defined nilpotent torsion-free group. By condition of the theorem the dimension of the commutant of this group is finite. In work [13] is proved the following

Theorem 7 *A recursively enumerable defined nilpotent torsion-free group, the dimension of its commutant is finite, constructive, i.e. computable.*

Thus for the group G the all conditions of the theorem are true, and therefore it is computable. □

5. The condition of not computability of periodic nilpotent group

Theorem 8 *If (G, ν) is computably numbered periodic nilpotent group of stage 2, then a set*

$$G_p = \{g \in G \mid \exists\ ng^{p^n} = 1\}$$

is computable subgroup in (G, ν) for any prime number p.

Proof. Firstly, let's establish that the G_p is a subgroup of the group G. Let elements $x, y \in G_p$ are given . Then

$$x^{p^m} = y^{p^n} = 1$$

for some numbers $0 < m \leq n$. Then

$$(xy)^{p^n} = x^{p^n} \cdot y^{p^n}[y^{p^n}, x] = (x^{p^n})^{p^{n-m}} \cdot 1[1, x] = 1,$$

$$(x^{-1})^{p^m} = (x^{p^m})^{-1} = 1.$$

From this follow, that the G_p is a subgroup. Similar it is proved that the G_p is a normal subgroup.

Let's prove that the subgroup G_p is computable in (G, ν). Let m be a natural number. We need to effectively determine that element νm is belongs to the subgroup G_p. It could be considered $\nu m \neq 1$. Since the group G is periodic, then we can effectively find such number $k > 0$, that $(\nu m)^k = 1$. Now let's factorize k into prime factors. If $k = p^s$ for some s, then $\nu m \in G_p$, and if not, then $\nu m \notin G_p$. □

Corollary 6 *If G is computable periodic nilpotent group of stage 2, then for any prime number p primary component G_p and quotient $\overline{G}_p = G/G_p$ are computable.*

Corollary 7 *If G is a periodic nilpotent group of stage 2 and ones can find a prime number p such that either G_p, or quotient $\overline{G}_p$ are not computable, then and the group G is not computable.*

Corollary 8 *If G is a periodic nilpotent group of stage 2 and for some its primary component the subgroup $G_p \bigcap G'$ is not computable, then the subgroup G is not computable, where G' is commutant of the group G.*

Indeed, suppose, that G is computable. Then ones can find a computable numbering ν of the group G. Then by the theorem 1 a primary component G_p of the group G is computable in (G, ν), and commutant G' is computably enumerable in (G, ν). From this $G_p \bigcap G'$ is computably enumerable subgroup in (G, ν), and therefore it is computable. Hence the group G is not computable.

6.On an algorithm of root extraction for a nilpotent torsion-free group

If (G, ν) is a numbered group and from any natural numbers k and n ones can effectively determine whether there are $\sqrt[k]{\nu n}$, then we can say that in (G, ν) there exists an algorithm of root extraction.

Theorem 9 *There exists an algorithm of root extraction in a finitely generated nilpotent group G.*

Proof. Let G be nilpotent group of stage s and are given the following equation

$$x^n = g \tag{10}$$

where $n \in \omega$. If $s = 1$, i.e. the group G is Abelian, then G is a direct sum of finite number of cyclic groups (a_i), $i \leq m$. From this

$$g = a_0^{k_0} a_1^{k_1} \cdot \ldots \cdot a_m^{k_m}$$

Then by the degrees k_i and the order of element a_i ones can effectively find out whether the equation (10) has solution, and if it has, then find out how many solutions.

Let class of the nilpotent group G is equal $s + 1$ and $\overline{G} = G/C^n$, where C is the center of the group G. Let's prove that: if the equation (10) has solution in $\overline{G}$, then it has solution and in G.

Let $\overline{g} = \bar{g}_0^n$. Then $g = g_0^n c$, $c \in C^n$. From this $c = c_0^n$ for some element $c_0 \in C$, and therefore $g = (g_0 c_0)^n$, i.e. the equation (10) has solution and in G. Obliviously, that if (10) does not have solution in $\overline{G}$, then it does not have solution and in G. Therefore it is s sufficient to prove that there exists required algorithm in the group $\overline{G}$. For this let's consider quotient.

$$\widetilde{G} = \overline{G}/\overline{C}$$

where $\overline{C} = C/C^n$. Since a class of nilpotent group $\widetilde{G}$ is less than s, then by induction hypothesis there exists the required algorithm in $\widetilde{G}$. If the equation (10) does not have solution in $\widetilde{G}$, then it does not have solution and in $\overline{G}$. Let the equation (10) has solution and $\tilde{g} = \tilde{g}_0^n$.

Since quotient $\overline{C}$ is finite, then ones can effectively check is there any element $\bar{c}_0 \in \overline{C}$, that $(\bar{g}_0 \bar{c}_0)^n = \bar{g}$.

From this in $\overline{G}$, and therefore in the G, has the required algorithm. □

Theorem 10 *Let (G, ν) be computable torsion-free nilpotent group and its central series is given.*

$$1 = G_0 < G_1 < G_2 < G \tag{11}$$

where G_1 is the center of the group G u $\nu^{-1} G_1$ is computable set. Then the following is true: if in the factors $\overline{G}_i = G_i/G_{i-1}$, $i = 1, 2$ there exists an algorithm of root extraction, then there exists such algorithm also in (G, ν).

Proof. Let $\nu n = g$ are some element. If $g \in G_1$, then by the condition of the theorem ones can effectively determine the existence of element $\sqrt[k]{\nu n}$ for any $k \in \omega$.

Let $g \notin G_1$. Since in $\overline{G}_2$ there exists an algorithm of root extraction,then ones can effectively determine is there $\sqrt[k]{\bar{g}}$.

Let's consider the possible cases:

1. Let for any element $\bar{g}_1 \in \overline{G}_2$ is true following

$$\bar{g}_1^k = \bar{g} \tag{12}$$

Then the element $c \in G_1$ can be found such that

$$g_1^k c = g \tag{13}$$

Since in G_1 there exists an algorithm of root extractions, then ones can effectively define is there $\sqrt[k]{c}$. Let's see possible cases:

a) there exists, i.e. $c = c_0^k$, where $c_0 \in G_1$. From this and (13) we have $g_1^k c_0^k = (g_1 c_0)^k = g$, i.e. there exists $\sqrt[k]{g}$ in G.

b) there does not exist $\sqrt[k]{c}$ in G_1.

Let's prove,that then there does not exist $\sqrt[k]{g}$. Let's assume the contrary, i.e for some element $g_0 \in G$ is true $g = g_2^k$

From this and (13) we have $g_1^k c = g_2^k$, i.e.

$$g_1^k g_2^{-k} = c^{-1} \tag{14}$$

Since G is torsion-free nilpotent group of class 2, then

$$g_1^k g_2^{-k} = (g_1 g_2^{-1})^k [g_1 g_2^{-1}]^{k^2}.$$

From this and (14) we get

$$c^{-1} = (g_1 g_2^{-1})^k [g_1 g_2^{-1}]^{k^2},$$

i.e. there exists $\sqrt[k]{c^{-1}}$. Then there exists $\sqrt[k]{c}$, which contradicts the condition b) Thus $\sqrt[k]{g}$ does not exists. Hence for any element $g \in G$ ones can effectively define is there $\sqrt[k]{g}$ in G, where $k \in \omega \backslash \{0, 1\}$. □

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