UDK 519.683.5

Usenov I.A.

Kyrgyz National University named after Zhusup Balasagyn, Kyrgyz Republic, Bishek
E-mail: iausen@mail.ru

About some methods of solution regularization of the first kind nonliner operator equation in hilbert space

Many applied problems of physics and geophysics are reduced to operator equations of the first kind. In cases when the expression for the Green's function is unknown, inverse problems of math physics are also reduced to these equations. The inverse problem of wells' electric logging which defines fields and calculates the reserves of mineral resources is the exact example of such problems. The applied problems written above are urgent problems of modern science, the solution of these problems can discover new verges of current state of mankind's development. In this regard, it stresses the importance of research of ill-posed problems. In the article [1], M.M. Lavrent'ev offered the regularization method for the solution of first kind linear operator equation in Hilbert space with the replacement of the original equation that is close to it, for which the problem of finding a solution is resistant to small changes in the right part and solvable for any right part, i.e. the equation replaces by another equation

$$\alpha z + Az = u,$$

where A- linear operator $\alpha > 0$ - regularization parameter.In the article [2] Newton's method of approximate solution of equations was distributed by L.V.Kantorovich for functional equations

$$K(z) = 0,$$

where $K(z)$- non-linear, twice differentiable operator by Frechet, acting from the one Banach space into another. In this article the combined method of the new type regularization is offered, combining the ideas of M. M. Lavrent'ev's method [1], Newton-Kantorovich method [2] for the solution regularization of first kind nonlinear operator equation in Hilbert space.
Key words: nonlinear operator, regularization, Hilbert space, the Frechet differential, linear operator, bounded operator, Lipschitz conditio.

Усенов И.А.

Гильберт кеңістігіндегі бірінші текті сызықты емес операторлық теңдеудің шешімін регуляризациялау әдісі

Физика мен геофизиканың көптеген есептері бірінші текті операторлық теңдеулермне келтірілгеді. Мұндай теңдеулерге Грин функциясының мәні белгісіз болған жағдайындағы математикалық физикалық кері есептері де келтмріледі. Мұндай есептердің мысалы ретінде пайдалы қазба кеніштері мен қорларының санын аныктайтын бұрғылау скважиналарының геофизикалық зерттеуінің кері есептерін қарастыру керек. Жоғарыда аталған қолданбалы есептер заманауи ғылымның өзекті мәселесі болып табылады. Мұндай есептерді шешу адамзат дамуының заманауи күйлерінің жаңа қырларын ашуға мүмкіндік жасайды. Осыған орай, қисынсыз қойылған есептерді зерттеудің маңыздылығы айқындалады. Лавреньтев 1-текті сызықты операторлық теңдеудің гильберттік кеңстігіндегі шешімін регурялизациялап, бастапқы теңдеуге жақын, шешімді табу мәселесі оң жағының кішігірім өзгерістеріне орнықты және кез келген оң жағы үшін шешімді болатын теңдеумен ауыстырылатын, ягни теңдеу [1]

$$\alpha z + Az = u,$$

теңдеуімен ауыстырылатын әдісті ұсынды, мұндағы A- сызықты оператор,$\alpha > 0$- регуряризациялау параметрі. Жұмыста теңдеуді жуықтап шешудің Ньютон әдісін Л.В. Канторович келесі функционалдық теңдеуінде келтірген [2]

$$K(z) = 0,$$

мұндағы $K(z)$- банах кеңістігіндегі сызықты емес, Фреше бойынша екі рет дифференциалданатын оператор. Бұл жұмыста М.М. Лаврентьев және Ньютон-Канторович әдістерінің идеясын біріктіретін гильберт кеңістігіндегі бірінші текті сызықты емес операторлық теңдеуінің шешімін регуряризациялайтын жаңа түрдегі регуляризациялаудың аралас әдісі ұсынылады. осы жұмыста Ньютон-Канторовичтің 1-текті сызықты емес лператорлық теңдеудің гильберттік кеңістігіндегі шешімін регурялизациялау әдісі мен Лаврентьев әдісінің идеяларын біріктіретін регурялизациялаудың жаңа түрдегі құрамдастырылған әдісі ұсынылады.
Түйін сөздер: сызықты емес операторлар, регуляризация, гильберт кеңістігі, Фреше дифференциалы, сызықты оператор, оператордың шектеулігі, Липшица шарты.

Усенов И.А.

О некотором методе регуляризации решения нелинейного операторного уравнения первого рода в гильбертовом пространстве

Многие прикладные задачи физики и геофизики сводятся к операторным уравнениям первого рода. К таким уравнениям сводятся также обратные задачи математической физики в тех случаях, когда выражение для функции Грина неизвестно. Обратная задача электрокаротажа скважин, определяющая месторождения и подсчет запасов полезных ископаемых является примером таких задач. Вышеперечисленные прикладные задачи являются актуальными задачами современной науки, решение этих задач может открыть новые грани современного состояния развития человечества. В связи с этим подчеркивается важность исследования некорректно поставленных задач. В работе [1], М.М. Лаврентьевым предложен метод регуляризации решения линейного операторного уравнения первого рода в гильбертовом пространстве с заменой исходного уравнения близким ему, в некотором смысле, для которого задача нахождения решения устойчива к малым изменениям правой части и разрешима для любой правой части, т.е. уравнение заменяется уравнением

$$\alpha z + Az = u,$$

where A- линейный оператор $\alpha > 0$ - параметр регуляризации. В работе [2] метод Ньютона приближенного решения уравнений был распространен Л.В. Канторовичем на функциональные уравнения

$$K(z) = 0,$$

где $K(z)$- лнелинейный, дважды дифференцируемый по Фреше оператор, действующий из одного пространства Банаха в другой. В данной работе предлагается комбинированный метод регуляризации нового типа, объединяющий идеи метода М.М. Лаврентьева [1], метода Ньютона-Канторовича [2] для регуляризации решения нелинейного операторного уравнения первого рода в гильбертовом пространстве.
Ключевые слова: нелинейный оператор, регуляризация, пространство гильберта, дифференциал Фреше, линейный оператор, ограниченность оператора, условия Липшица.

1. Statement of objectives

Lets consider the nonlinear operator equation of the first kind type

$$K(z) = u, \tag{1}$$

where $K : H \to H$ - non-linear operator that defined in the set $Q \subset H$ and Frechet differentiable. In the space we use the notation of the ball $S(z_0, r) = z : \|z - z_0\| \leq r_z$.

We assume that 1) when $u = u^*$ a unique solution z^* of equation exists (1), i.e. the identity occurs

$$K(z^*) \equiv u^*; \tag{2}$$

2) the operator K is continuous in the ball $S(z_0, r_z)$, and has a Frechet K' derivative continuous in point z_0.
3) linear operator $K'(z_0)$ is reversible, but unlimited, i.e. $\|[K'(z_0)]^{-1}\| = \infty$;
4) element u^* is unknown to us, but instead the element u_δ is known such that

$$\|u_\delta - u^*\| \leq \delta, \tag{3}$$

where $\delta > 0$ - numeric parameter. Thus, equation (1) in the ball $S(z_0, r_z)$ is an ill-posed problem. In the article [4] of A. Saadabaev this method studied when the derivative operator satisfies the Lipschitz condition. In this article the combined method of regularization is being studied for a wider class of equations, when he derivative operator satisfies the Holder condition

$$\|K'(z_1) - K'(z_2)\| \leq N\|z_1 - z_2\|^\beta, \tag{4}$$

where N and β - constants, $0 < \beta \leq 1$.

2. The Regularization of a problem's solution Along with the equation (1) let's consider the equation

$$\alpha z^\alpha + K(z^\alpha) = u, \tag{5}$$

where $\alpha > 0$ - small regularization parameter. We output one estimate for the remainder of Taylor's formula for functions of a real variable function f(t). Let the derivative $f^{(n)}(t)$ satisfy the Holder condition in the range of (a,b) in a fixed-point a:

$$\|f^{(n)}(t) - f^{(n)}(a)\| \leq N\|t - a\|^\beta, 0 < \beta \leq 1. \tag{6}$$

Then

$$\|f(t) - \sum_{m=0}^{n} \frac{f^{(m)}(a)}{m!}(t-a)^m\| \leq \frac{N}{(1+\beta)(2+\beta)...(n+\beta)}\|t - a\|^{n+\beta}. \tag{7}$$

In further studies we will use an analogue of the rating for functional operators.
Lemma. If K(z)- Frechet differentiable operator from H in H and for $K'(z)$ the weakened condition of Holder is offered with a fixed element $z_2 \equiv z_\alpha$, then

$$\|K(z) - K(z_\alpha) - K'(z_\alpha)(z - z_\alpha)\| \leq \frac{N}{1+\beta}\|z - z_\alpha\|^{1+\beta}. \tag{8}$$

It is sufficient to satisfy the Holder condition only (4) on the segment $z_\tau = z_\alpha + \tau(z - z_\alpha), 0 \leq \tau \leq 1$.
Proof. The lemma follows from the identity

$$K(z) - K(z_\alpha) - K'(z_\alpha)(z - z_\alpha) = \int_0^1 \left\{K'(z_\alpha + \tau(z - z_\alpha)) - K'(z_\alpha)\right\}(z - z_\alpha)d\tau. \tag{9}$$

To construct a regularizing operator inside the ball $S(z_0, r_z)$ picking the number z_α we create the ball $S(z_\alpha, r_z(\alpha))$, where the center of the ball at $\alpha \to 0$ seeks to the center of the ball $S(z_\alpha, r_z)$ approaches, ie $z_\alpha(u_0) \to z_0$, consequently $r_z(\alpha) \to 0$. The ball elements $S(z_\alpha, r_z(\alpha))$ satisfy the inequality $\|z - z_\alpha\| \le r_z(\alpha)$, then between the elements z_α and z^* occurs $\|z^* - z_\alpha\| \le \gamma r_z(\alpha)$ where $0 < \gamma < 1/2$.

Theorem 1. Let 1) the condition occur 2), and 2) the linear operator $K'(z_\alpha)$ is continuous positive self-adjoint operator; 3) the operator $K'(z)$ holds the weakened of Holder condition r with a fixed element $z_2 \equiv z_\alpha$

$$\|K'(z) - K'(z_\alpha)\| \le N\|z - z_\alpha\|^\beta \; at \; S(z_\alpha, r_z(\alpha)); \tag{10}$$

4) hold a limit relation

$$\lim_{r_z(\alpha)\to 0} \frac{r_z(\alpha)}{\alpha(r_z(\alpha))} = 0, \quad \alpha(r_z(\alpha)) = \sigma r_z^{\sigma\beta}(\alpha), \quad 0 < \sigma < \left(\frac{1}{2}\right)^{\frac{\beta(1-\sigma)}{2-\beta-\beta\alpha}} \left(\frac{N}{1+\beta}\right)^{\frac{1-\beta}{2-\beta-\beta\alpha}}, \tag{11}$$

then there is a number $\tilde{r} > 0$, that at $r_z(\alpha) < \tilde{r}$ operator
$I[\tau] = (\alpha E + K'(z_\alpha))\left(E + (\alpha E + K'(z_\alpha))^{-1}\left(\int_0^1 \{K'(z_\alpha + \tau(z - z_\alpha)) - K'(z_\alpha)\} d\tau\right)\right)$
(where $0 \le \tau \le 1$, z - fixed element) in a ball $S(z_\alpha, r_z(\alpha))$ has an inverse bounded operator.
Proof. Assess

$$\|(\alpha E + K'(z_\alpha))^{-1}(\int_0^1 \{K'(z_\alpha + \tau(z - z_\alpha)) - K'(z_\alpha)\} d\tau)\| \le \frac{1}{\alpha}\frac{N}{1+\beta}\|z - z_\alpha\|^\beta \le \frac{N r_z^\beta(\alpha)}{(1+\beta)\alpha}, \tag{12}$$

where $\|(\alpha E + K'(z_\alpha))^{-1}\| \le \frac{1}{\alpha}$, [1]. From condition (4) it follows that there exists $\tilde{r} > 0$ such that

$$\frac{N r_z^\beta(\alpha)}{(1+\beta)\alpha(r_z(\alpha))} < \frac{N\tilde{r}}{(1+\beta)\alpha(\tilde{r})} = q < \sigma \quad at \quad r_z(\alpha) < \tilde{r}, \tag{13}$$

where $0 < \sigma < \left(\frac{1}{2}\right)^{\frac{\beta(1-\sigma)}{2-\beta-\beta\sigma}} \left(\frac{N}{1+\beta}\right)^{\frac{1-\beta}{2-\beta-\beta\sigma}}$. When selecting $\alpha(r_z(\alpha)) = \sigma r_z^{\sigma\beta}(\alpha)$, the condition (4) satisfies, and choose a specific number $\tilde{r} > 0$, that

$$r_z(\alpha) < \left(\frac{(1+\beta)\sigma^2}{N}\right)^{\frac{1}{\beta(1-\sigma)}} = \tilde{r} \quad and \quad \alpha(\tilde{r}) = \sigma\left(\frac{(1+\beta)\sigma^2}{N}\right)^{\frac{\sigma}{1-\sigma}}. \tag{14}$$

Consequently,

$$q = \frac{N\tilde{r}}{(1+\beta)\alpha(\tilde{r})} = \frac{\left(\frac{(1+\beta)\sigma^2}{N}\right)^{\frac{1}{\beta(1-\sigma)}} N}{(1+\beta)\sigma\left(\frac{(1+\beta)\sigma^2}{N}\right)^{\frac{\sigma}{1-\sigma}}} = \left(\frac{(1+\beta)}{N}\right)^{\frac{1-\beta}{\beta(1-\sigma)}} \sigma^{\frac{2-\beta-\beta\sigma}{\beta(1-\sigma)}} < \frac{1}{2}. \tag{15}$$

By virtue of the Banach theorem [3] in the ball $S(z_\alpha, r_z(\alpha))$ of the operator $\Phi[\tau] = E+(\alpha E+ K'(z_\alpha))^{-1}\left(\int\limits_0^1 \{K'(z_\alpha+\tau(z-z_\alpha)) - K'(z_\alpha)\}d\tau\right)$ reversible and the estimate

$$\|\Phi[\tau]\| = \|(E+(\alpha E+K'(z_\alpha))^{-1}(\int\limits_0^1 \{K'(z_\alpha+\tau(z-z_\alpha)) - K'(z_\alpha)\}d\tau))^{-1}\| \leq \frac{1}{1-q}. \quad (16)$$

Thus, the operator is reversible $I[\tau]$, and the inverse operator has the form $I[\tau] = (\Phi[\tau])^{-1}(\alpha E+ K'(z_\alpha))^{-1}$ and the estimate

$$\|(I[\tau])^{-1}\| \leq \frac{1}{1-q}\frac{1}{\alpha}. \quad (17)$$

QED. Equation (4) is equivalent to write in the form

$$z^\alpha = z^\alpha - (I[\tau])^{-1}(\alpha z^\alpha + K(z^\alpha) - u). \quad (18)$$

Theorem 2. Let the equation (5) follow the conditions: 1) at the point z_α linear operator $K'(z_\alpha)$ is continuous positive self-adjoint; 2) the operator $(I[\tau])^{-1}(\alpha z_\alpha + K(z_\alpha) - u)$ satisfies the condition $\|(I[\tau])^{-1}(\alpha z_\alpha + K(z_\alpha) - u)\| \leq \frac{N}{(1-q)\sigma^{\frac{1}{\sigma}}}\alpha^{\frac{1-\sigma}{\sigma}} = \eta$; 3) for the operator $K'(z_\alpha)$ it is satisfied the Holder condition (4) with a fixed element $z_2 \equiv z_\alpha$ and for any z_1 from region

$$\|z - z_\alpha\| \leq t\eta, \quad where \;\; 1 < t \leq \frac{2+\beta}{\beta}; \quad (19)$$

4) permanent $\frac{1}{(1-q)\alpha}, \eta, N, \beta$ and t in conditions 1), 2) and 3) are such that

$$h = \frac{N^{1+\beta}}{(1-q)^{1+\beta}\sigma^{\frac{\beta}{\sigma}}}\alpha^{\frac{\beta-(1+\beta)\sigma}{\sigma}} \leq \frac{(1+\beta)(t-1)}{2t^{1+\beta}}, \quad (20)$$

then in a ball (19), equation (5) has a unique solution $z^{*,\alpha}$.

Proof. Based on the initial approximation z_α, we construct a sequence of elements $\{z_n\}_{n=0}^\infty$

$$z_{n+1} = z_n - (I[\tau])^{-1}(\alpha z_n + K(z_n) - u). \quad (21)$$

Firstly, by induction let's show that all z_n are in the ball (19). From the condition 2) we have $\|z_1 - z_\alpha\| = \|(I[\tau])^{-1}(\alpha z_\alpha + K(z_\alpha) - u)\| \leq \eta < t\eta$, and z_1 is in the ball (19). Let some natural n executed $\|z_n - z_\alpha\| \leq t\eta$. Using this statement let's show that z_{n+1} is a part of the ball (19). From (21) we have

$$z_{n+1}-z_\alpha = z_n - z_\alpha - (I[\tau])^{-1}(\alpha z_n + K(z_n) - u) == -I[\tau])^{-1}(K(z_n) - K(z_\alpha) - K'(z_\alpha)(z_n - z_\alpha))$$
$$-(I[\tau])^{-1}(\alpha z_\alpha + K(z_\alpha) - u) + (I[\tau])^{-1}(\int\limits_0^1 \{K'(z_\alpha+\tau(z-z_\alpha)) - K'(z_\alpha)\}d\tau|(z_n - z_\alpha)). \quad (22)$$

From (8), we consider the conditions of 2), (3) and (4) theorem 2:

$$\|z_{n+1} - z_\alpha\| \le t\eta, \tag{23}$$

and, therefore, z_{n+1} is in the ball (19). Let's estimate the difference between two successive iterations $\|z_{n+1} - z_n\|$. From (21) we have $z_{n+1} - z_n = z_n - z_{n-1} - (I[\tau])^{-1}(\alpha z_n + K(z_n) - u) +$
$+ (I[\tau])^{-1}(\alpha z_{n-1} + K(z_{n-1}) - u) = (I[\tau])^{-1}(\alpha(z_n - z_{n-1}) + K'(z_\alpha)(z_n - z_{n-1})) +$
$+ (I[\tau])^{-1}(\int\limits_0^1 \{K'(z_\alpha + \tau(z - z_\alpha)) - K'(z_\alpha)\} d\tau (z_n - z_{n-1})) -$
$- (I[\tau])^{-1}(\alpha z_n - K(z_n) - \alpha z_{n-1} - K(z_{n-1}))$.
Including the operator $B_\alpha(z)$

$$B_\alpha(z) = z - (I[\tau])^{-1}(\alpha z + K(z) - u). \tag{24}$$

Transforming the operator $B_\alpha(z)$ we have

$$B_\alpha(z) = (I[\tau])^{-1}((\alpha E + K'(z_\alpha))\Phi[\tau]z - \alpha z - K(z) + u). \tag{25}$$

The derivative of operator $B_\alpha(z)$ at the point $\forall z \in S(z_\alpha, r(\alpha))$ has the form

$$B'_\alpha(z) = (I[\tau])^{-1}(K'(z_\alpha) - K'(z) + \int\limits_0^1 \{K'(z_\alpha + \tau(z - z_\alpha)) - K'(z_\alpha)\} d\tau). \tag{26}$$

Let's estimate the norm of operator $B'_\alpha(z)$

$$\|B'_\alpha(z)\| \le \frac{N(2+\beta)}{\alpha(1+\beta)(1-q)} \|z - z_\alpha\|^\beta \le \frac{2+\beta}{1+\beta} h t^\beta. \tag{27}$$

From the estimate (27) it follows that $\forall z_1, z_2 \in S(z_\alpha, r(\alpha))$ the operator $B_\alpha(z)$ holds the Lipschitz condition

$$\|B_\alpha(z_1) - B_\alpha(z_2)\| \le \frac{2+\beta}{1+\beta} h t^\beta \|z_1 - z_2\|. \tag{28}$$

Now let's estimate $\|z_{n+1} - z_n\|$. Using the estimate (28) we have

$$\|z_{n+1} - z_n\| = \|B_\alpha(z_n) - B_\alpha(z_n - 1)\| \le \frac{2+\beta}{1+\beta} h t^\beta \|z_n - z_n - 1\|, \tag{29}$$

where $q_1 = \frac{2+\beta}{1+\beta} h t^\beta < 1$, because at $t < \frac{2+\beta}{\beta}$ we have from (20)
$q_1 = \frac{2+\beta}{1+\beta} h t^\beta \le \frac{(2+\beta)(t-1)}{2t} < 1$
Continuing evaluation (29), we obtain

$$\|z_{n+1} - z_n\| \le q_1^n \|z_1 - z_\alpha\| \le q_1^n \eta. \tag{30}$$

Next let's show that the sequence $z_{nn} = 0^\infty$ is a fundamental sequence. Valid for all, n and p we have, using the triangle inequality and (30)

$$\begin{aligned} \|z_{n+p} - z_n\| &\le \|z_{n+p} - z_{n+p-1}\| + \|z_{n+p-1} - z_{n+p-2}\| + ... + \|z_{n+1} - z_n\| \le \\ &\le q_1^{n+p-1}\|z_1 - z_\alpha\| + q_1^{n+p-2}\|z_1 - z_\alpha\| + ... + q_1^n\|z_1 - z_\alpha\| = \\ &= (q_1^{n+p-1} + q_1^{n+p-2} + ... + q_1^n)\|z_1 - z_\alpha\|. \end{aligned} \tag{31}$$

From inequality (31) using the sum of a geometric progression,we have $q_1 < 1$

$$\|z_{n+p} - z_n\| \leq \|z_1 - z_\alpha\| q_1^n \sum_{i=1}^{p} q_1^{i-1} \leq q_1^n \frac{q_1^p}{1-q_1} \|z_1 - z_\alpha\|. \quad (32)$$

Because of the completeness of H there is

$$\lim_{n\to\infty} z_n = z^\alpha. \quad (33)$$

Let's fix in inequality (32) the index n and tend p to infinity. Then we obtain the estimate

$$\|z_n - z^*\| \leq \frac{q_1^n}{1-q_1}\eta, \quad q_1 = \frac{2+\beta}{1+\beta} h t^\beta < 1. \quad (34)$$

This estimate (34) is the speed of convergence of the method. This is because at $n \to \infty$ the error decreases as q^n.
Thus, for the solution of equation (5) in the ball (19) we have the estimate

$$\|z^\alpha - z_\alpha\| \leq \frac{(2+\beta)N}{\beta(1-q)\sigma^{\frac{1}{\sigma}}} \alpha^{\frac{1-\sigma}{\sigma}}. \quad (35)$$

Theorem 2 is proved.

Theorem 3. Let 1) all the conditions of Theorem 2 satisfy; 2) between the elements z_α and z^* we have the estimate $\|z^* - z_\alpha\| \leq \gamma r_2(\alpha)$, where $0 < \gamma < 1/2$.
Then a unique continuous solution $z^{\alpha,*}$ of equation (18) at $u = u^*$ converges in the norm of H the exact solution z^* of equation (1) at $\alpha \to 0$.
Proof. Using the triangle inequality, the inequality (35) for $u = u^*$ and the second condition of Theorem 3, estimating the norm $\|z^{\alpha,*} - z^*\|$, we have

$$\|z^{\alpha,*} - z^*\| \leq \|z^{\alpha,*} - z_\alpha\| + \|z^* - z_\alpha\| \leq (1+\gamma) \frac{(2+\beta)N}{\beta(1-q)\sigma^{\frac{1}{\sigma}}} \alpha^{\frac{1-\sigma}{\sigma}}. \quad (36)$$

From inequality (36) it follows that $z^{\alpha,*} \to z^*$ where $\alpha \to 0$ by the norm of H. Thus, the solution $z^{\alpha,*}$ of equation (18) at $u = u^*$ is an approximate solution of equation (1).
QED. The main problem that is subject to study is to construct such a sequence of solutions $z^{\alpha,\delta}$ by the approximate solutions u_δ, which converges in the space H to the exact solution z^* of equation (1), provided that the initial convergence $u_\delta \to u^*$ at $\delta \to 0$. To do this we estimate the difference $z^{\alpha,\delta} - z^*$, where $z^{\alpha,\delta}$ - the solution of equation (18) at $u = u_\delta$. It is represented in the form

$$z^{\alpha,\delta} = z^{\alpha,\delta} - (I[\tau])^{-1}(\alpha z^{\alpha,\delta} + K(z^{\alpha,\delta}) - u_\delta). \quad (37)$$

Using the triangle inequality, we have

$$\|z^{\alpha,\delta} - z^*\| \leq \|z^{\alpha,\delta} - z^{\alpha,*}\| + \|z^{\alpha,*} - z^*\|, \quad (38)$$

where $z^{\alpha,*} = z^{\alpha,*} - (I[\tau])^{-1}(\alpha z^{\alpha,*} + K(z^{\alpha,*}) - u^*)$ the solution of equation (18) at $u = u^*$. Let's consider the difference $z^{\alpha,\delta} - z^{\alpha,*}$

$$z^{\alpha,\delta} - z^{\alpha,*} = -(I[\tau])^{-1}(K'(z^{\alpha,*} - K'(z_\alpha))(z^{\alpha,\delta} - z^{\alpha,*}) -$$
$$- (I[\tau])^{-1} \int_0^1 \{(K'(z^{\alpha,*} + \tau(z^{\alpha,\delta} - z^{\alpha,*})) - K'(z^{\alpha,*}))\}(z^{\alpha,\delta} - z^{\alpha,*})d\tau +$$
$$+ (I[\tau])^{-1} \int_0^1 \{(K'(z_\alpha + \tau(z - z_\alpha)) - K'(z_\alpha))\}(z^{\alpha,\delta} - z^{\alpha,*})d\tau +$$
$$+ (I[\tau])^{-1}(u_\delta - u^*). \quad (39)$$

Assessing the converted difference, we have

$$\|z^{\alpha,\delta} - z^{\alpha,*}\| \leq \frac{(2 + 2^\beta + \beta)N}{(1-q)(1+\beta)} \cdot \frac{r_z^\beta(\alpha)}{\alpha}\|z^{\alpha,\delta} - z^{\alpha,*}\| + \frac{1}{1-q} \cdot \frac{\delta}{\alpha}. \quad (40)$$

Let's show that when choosing $\alpha(r_z(\alpha)) = \sigma_1 r_z^{\beta\sigma_1}(\alpha)$ where $\sigma < \sigma_1$ there exists $\widehat{r} > 0$ such as

$$\frac{(2 + 2^\beta + \beta)N}{(1-q)(1+\beta)} \cdot \frac{r_z^\beta(\alpha)}{\alpha} < \frac{(2 + 2^\beta + \beta)N}{(1-q)(1+\beta)} \cdot \frac{\widehat{r}}{\alpha(\widehat{r})} = q_2 \leq \sigma_1. \quad (41)$$

at $r_z < \widehat{r}$, where $0 < \sigma_1 < \left(\frac{1}{2}\right)^{\frac{\beta(1-\sigma_1)}{2-\beta-\beta\sigma_1}} \left(\frac{(2+2^\beta+\beta)N}{(1-q)(1+\beta)}\right)^{\frac{1-\beta}{\beta(1-\sigma_1)}}$.
Further, from (41) we have

$$r_z < \left(\frac{\sigma_1^2(1-q)(1+\beta)}{(2+2^\beta+\beta)N}\right)^{\frac{1}{\beta(1-\sigma_1)}} \equiv \widehat{r} \;\; and \;\; \alpha(\widehat{r}) = \sigma_1 \left(\frac{\sigma_1^2(1-q)(1+\beta)}{(2+2^\beta+\beta)N}\right)^{\frac{\sigma_1}{1-\sigma_1}}. \quad (42)$$

Under these conditions (42), $q_2 < \frac{1}{2}$.
Then from (40) we have

$$\|z^{\alpha,\delta} - z^{\alpha,*}\| \leq C_1 \frac{\delta}{\alpha}, \quad (43)$$

where $C_1 = \frac{1}{(1-q_2)(1-q)}$.
In view of (36) and (43) from (38) we have

$$\|z^{\alpha,\delta} - z^*\| \leq \|z^{\alpha,\delta} - z^{\alpha,*}\| + \|z^{\alpha,*} - z^*\| \leq C_2 \alpha^{\frac{1-\sigma}{\sigma}} + C_1 \frac{\delta}{\alpha}, \quad (44)$$

where $C_2 = (1+\gamma)\frac{(2+\beta)N}{\beta(1-q)\sigma^{\frac{1}{\sigma}}}$.
Minimizing the right part of inequality (44) we obtain the dependence of the regularization α parameter on the right part of the parameter error δ

$$\alpha(\delta) = \left(\frac{C_1}{C_2} \cdot \frac{\sigma}{1-\sigma}\delta\right)^\sigma. \quad (45)$$

Then the estimate (44) has the form

$$\|z^{\alpha,\delta} - z^*\| \leq C_1^{1-\sigma} C_2^{\sigma} (1-\sigma)^{1-\sigma} \left(\frac{1}{\sigma}\right) \delta^{1-\sigma}. \tag{46}$$

From (46) it follows that $z^{\alpha,\delta} \to z^* \quad \delta \to 0$ in the norm of space H.
Thus, it is proved

Theorem 4. Let 1) all the conditions of Theorem 3 satisfy; 2) the element satisfies condition (3); 3) at $\alpha(r_z(\alpha)) = \sigma_1 r_z^{\beta\sigma_1}(\alpha)$ where $\sigma < \sigma_1$ the condition occurs $\frac{(2+2^\beta+\beta)N}{(1-q)(1+\beta)} \cdot \frac{r_z^\beta(\alpha)}{\alpha} < \frac{(2+2^\beta+\beta)N}{(1-q)(1+\beta)} \cdot \frac{\widehat{r}}{\alpha(\widehat{r})} = q_2 \leq \sigma_1$ at $r_z < \widehat{r}$, where $0 < \sigma_1 < \left(\frac{1}{2}\right)^{\frac{\beta(1-\sigma_1)}{2-\beta-\beta\sigma_1}} \left(\frac{(2+2^\beta+\beta)N}{(1-q)(1+\beta)}\right)^{\frac{1-\beta}{\beta(1-\sigma_1)}}$; 4) the regularization parameter α satisfies the condition (45). Then the solution of equation (18) $u = u_\delta$, converges at $\delta \to 0$ to the exact solution z^* of equation (1). The speed of convergence satisfies (46).

3. Conclusion

It is proved that based on the combination of the regularization method for solution the operator equation in Hilbert space it is regularizing. Justification of the new combined regularization method proposed in this work consists in the following study results:

1. Regularizing operator is constructed for the solution of the first kind operator equations in Hilbert space;
2. The convergence of regularized solutions is proved to the exact solution of the original equation;
3. The parametr regularization now recieves the choice depending on the error of the right part;
4. The rate of convergence of the regularized solutions to the exact solution is recieved.

For the first time, based on the combined method of regularization regularizing operator is constructed for the solution of operator equation of the first kind in Hilbert space. Convergence and convergence rate of regularized solution to the exact solution are established. This work has a theoretical orientation. The result can be used in cases when solving the applied problems of geophysics, thermal physics, filtering, optics, medical imaging and so on.

References

[1] *Lavrentiev M.M.* About some ill-posed problems of mathematical physics. - Novosibirsk: USSR, 1962.

[2] *Krasnosel'skii. M.A. and others.* The approximate solution of operator equations. M-69.

[3] *Trenogin V.A.* Functional analysis. - M.: FIZMATLIT, 3rd ed. - 2002.

[4] *Saadabaev A.* Convergence of Newton's method in nonlinear ill-posed problems // News of high schools № 1-2, Bishkek-2003.

[5] *Usenov I.A.* Regularization of the solution of implicit operator equations of the first kind // "Functional analysis and its applications,"Astana-2012.