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Applications of the Cayley-Hamilton Theorem in Linear Systems

The classical Cayley-Hamilton theorem says that every square matrix satisfies its own characteristic equation. Cayley Hamilton theorem is widely applicable in many fields not only related to mathematics, but in other scientific fields too. This theorem is used all over in linear algebra. It also is quite useful in modern control theory, especially in the linear systems. This paper introduced the applications of the Cayley-Hamilton theorem in linear systems, mainly from seven aspects: 1.The transfer function matrix is derived from the state-space description. 2.Equivalent representation of the uncontrollable subspace of the continuous system is presented. 3. The controllability canonical form and observability canonical form of the single input – single output system is obtained. 4. Controllable subspace of the discrete system is obtained. 5. The controllability of the linear time-invariant continuous systems after time discretization is presented. 6. The equivalent representation of the unobservable subspaces of a continuous system is obtained. 7. The observability of the linear time-invariant discrete system is derived.
Key words: Application, Cayley-Hamilton Theorem, Linear system.

Жұмақан К.
Сызықтық жүйелердегі Гамильтон-Кэли теоремасының қолданылымдары

Классикалық Гамильтон-Кэли теоремасы бойынша, әрқандай квадрат матрица өзінің характеристикалық теңдеуін қанағаттандырады. Гамильтон-Кэли теоремасы математикаға қатысты саладан сырт, басқа ғылым салаларында да кең көлемді қолданылымға ие. Бұл теорема сызықтық алгебраның барлық жерінде қолданылады. Ол сонымен қатар, заманауи басқару теоремаларында, әсіресе, сызықтық жүйелерінде ерекше қолданысқа ие. Бұл мақалада жеті негізгі аспектіден тұратын Гамильтон-Кэли теоремасының қолданылымы көрсетілген: 1. Табыстама матрица функциясы күйлер ортасынан шығады. 2. Үздіксіз эквивалент жүйесін сипаттайтын бақылауға алынбайтын орта көрсетілген. 3. Канондық форманың басқарылуы және канондық форманың бірінғай кіріс бақылауы - жалғыз кіріс жүйесі алынды 4. Дисркетті жүйенің бақылауға болатын ортасы алынды. 5. Уақыт дискретизациясынан кейінгі уақыттағы сызықтық басқармалы инвариант көрсетілды. 6. Үздіксіз жүйенің айқындалмаған орта эквиваленті алынды. 7. Уақыт инвариантының сызықтық дискретті жүйе бақылауы алынды.
Түйін сөздер: қолданылым, Гамильтон-Кэли теоремасы, сызықтық жүйе.

Жумакан К.
Применение теоремы Гамильтона-Кэли в линейных системах

Классическая теорема Гамильтона-Кэли утверждает, что каждый квадратная матрица удовлетворяет свой собственный характерный уравнение. Теорема Кэли Гамильтон широко применяется во многих областях не только связанных с математикой, но и в других научных областях тоже. Эта теорема используется в линейной алгебре. Он также является весьма полезным в современной теории управления, особенно в линейных системах. В этой статье представлена применение теоремы Гамильтона-Кэли в линейных системах,

в основном, из семи аспектах: 1. Передаточная функция матрицы происходит из описания пространства состояний. 2. Представлено неконтролируемой подпространство представление системы непрерывного эквивалента. 3. Получено система с одним выходом - управляемость канонических форм и наблюдаемость канонических форм единого входа. 4. Получено управляемая подпространство дискретной системы. 5. Представлена управляемость линейных инвариантов во времени непрерывных систем после дискретизации времени. 6. Получено эквивалент неочевидных подпространств непрерывной системы. 7. Получена наблюдаемость линейных дискретных систем инварианта времени.
Ключевые слова: приложение, теорема Гамильтона-Кэли, линейная система.

1. Introduction

Modern control theory describes system with the state space. Its theory and the linear algebra theory have the close relation. In the linear system, many concepts, conclusion statements have the very big similarity with the linear algebra, and many conclusions are the direct applications of the linear algebra theory. The theories and methods of linear algebra is the important mathematical tool for the study of the linear systems theory. The classical Cayley-Hamilton theorem says that every square matrix satisfies its own characteristic equation [1]. The Cayley-Hamilton theorem and its generalizations have been used in control systems[3], electrical circuits[2], systems with delays[5], singular systems[6], 2-D linear systems[4], etc. This paper introduced the applications of the Cayley-Hamilton theorem in linear systems, mainly from seven aspects: 1.The transfer function matrix is derived from the state-space description. 2.Equivalent representation of the uncontrollable subspace of the continuous system is presented. 3. The controllability canonical form and observability canonical form of the single input – single output system is obtained. 4. Controllable subspace of the discrete system is obtained. 5. The controllability of the linear time-invariant continuous systems after time discretization is presented. 6. The equivalent representation of the unobservable subspaces of a continuous system is obtained. 7. The observability of the linear time-invariant discrete system is derived.

2. Preliminaries

First,we introduce some basic concepts. Let A be an $n \times n$ square matrix, if there exists nonzero vector X,such that $AX = \lambda X$, then λ is called the eigenvalue of A, X is called the eigenvector of A corresponding to eigenvalue λ, $f(\lambda) = |\lambda I - A|$ is called the characteristic polynomial of A. The eigenvalue of A is the root of $f(\lambda)$, the eigenvector of A corresponding to λ is the all nonzero solutions of the equation system.

Theorem 1 *(Cayley-Hamilton theorem) Let $f(\lambda)$ be the characteristic polynomial of matrix A. Then $f(A) = 0$.*

From Cayley-Hamilton theorem,we can readily obtain the following results:

Lemma 1 *$A^k (k \geq n)$ can be written as a linear combination of $I, A, A^2, \ldots, A^{n-1}$.*

Theorem 2 *X is the eigenvector of A corresponding to λ ,and $|A| \neq 0$, then X is the eigenvector of A^{-1} corresponding to $\frac{1}{\lambda}$.*

Lemma 2 *. If A is nonsingular,then A does not have zero eigenvalue.*

Theorem 3 *Let A be an $s \times n$ matrix. If P is an $s \times s$ invertible matrix,Q is an $n \times n$ invertible matrix,then $r(A) = r(PA) = r(AQ)$*

3. Main Results

We introduce seven important applications of the Cayley-Hamilton theorem in linear systems.

3.1 Deriving the transfer function matrix from the state-space description

Theorem 4 *Given coefficient matrix (A,B,C) of the state-space description, it is obtained that*

$$a(s) = det(sI - A) = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 \tag{1}$$

$$\begin{cases} E_{n-1} = CB, \\ E_{n-2} = CAB + a_{n-1}CB, \\ \cdots\cdots\cdots \\ E_1 = CA^{n-2}B + a_{n-1}CA^{n-3}B + a_2 CB, \\ E_0 = CA^{n-1}B + a_{n-1}CA^{n-2}B + \ldots + a_1 CB, \end{cases} \tag{2}$$

Then the corresponding transfer function matrix can be written as

$$G(s) = \frac{1}{a(s)}(E_{n-1}s^{n-1} + E_{n-2}s^{n-2} + \ldots + E_1 s + E_0) \tag{3}$$

Proof. First we derive a relation of $(sI - A)^{-1}$. Let $P = (sI - A)^{-1}$, noticing that $(sI - A)P = I$, we obtain

$$\begin{array}{l} sP = AP + I, \\ s^2P = sAP + sI = A^2P + A + sI, \\ \cdots\cdots \\ s^lP = A^lP + A^{l-1} + A^{l-2}s + \ldots + As^{l-2} + s^{l-1}I \\ \cdots\cdots \\ s^nP = A^nP + A^{n-1} + A^{n-2}s + \ldots As^{n-2} + s^{n-1}I. \end{array} \tag{4}$$

Applying Cayley-Hamilton Theorem, we have

$$A^n + a_{n-1}A^{n-1} + \ldots + a_1 A + a_0 I = 0 \tag{5}$$

Thus, from (1), (4) and (5), it can be obtained that

$$\begin{array}{l} a(s)P = A^nP + A^{n-1} + A^{n-2}s + \ldots + As^{n-2} + s^{n-1}I \\ +a_{n-1}(A^{n-1}P + A^{n-2} + A^{n-3}s + \ldots + As^{n-3} + s^{n-2}I) \\ + \ldots + a_2(A^2P + A + sI) + a_1(AP + I) + a_0 P \\ = (A^n + a_{n-1}A^{n-1} + \ldots + a_1 A + a_0 I)P, \\ +(A^{n-1} + a_{n-1}A^{n-2} + \ldots + a_2 A + a_1 I) \\ +(A^{n-2} + a_{n-1}A^{n-3} + \ldots + a_2 I)s \\ + \ldots + (A + a_{n-1}I)s^{n-2} + Is^{n-1} \end{array} \tag{6}$$

Further from (5) we obtain that

$$\begin{array}{l} P=(sI-A)^{-1}=\frac{1}{a(s)}[(A^{n-1}+a_{n-1}A^{n-2}+\ldots+a_2A+a_1I) \\ +(A^{n-2}+a_{n-1}A^{n-3}+\ldots+a_2I)s+\ldots, \\ +(A+a_{n-1}I)s^{n-2}+Is^{n-1}] \end{array} \quad (7)$$

Then by substituting (7) into

$G(s)=C(sI-A)^{-1}B$

We can obtain (3). The proof is completed.

Lemma 3 *$(sI-A)^{-1}$ can be abbreviated as*

$$(sI-A)^{-1}=\frac{1}{\alpha(s)}\sum_{j=0}^{n-1}s^j\sum_{i=j+1}^{n}a_iA^{i-j-1}=\frac{1}{\alpha(s)}\sum_{j=0}^{n-1}A^j\sum_{i=j+1}^{n}a_is^{i-j-1}$$

where $a_n=1$. Therefore the transfer function matrix $G(s)$ can also be written as

$$G(s)=C\cdot\frac{\sum_{j=0}^{n-1}S^j\sum_{i=j+1}^{n}a_iA^{i-j-1}}{a(S)}\cdot B=C\cdot\frac{\sum_{j=0}^{n-1}A^j\sum_{i=j+1}^{n}a_iS^{i-j-1}}{a(S)}\cdot B.$$

3.2 The Equivalent Representation of the Uncontrollable Subspaces of a Continuous System

Consider the known system

$$\dot{x}=Ax+Bu, x(0)=x_0, \quad (8)$$

Its uncontrollable subspace X_{NC} is the constant solution space of

$$a^Te^{-At}B=0, t\in[0,T] \quad (9)$$

Theorem 5 *The uncontrollable subspace X_{NC} of the system is the solution space of*

$$\alpha^T[B \quad AB \quad \ldots \quad A^{n-1}B]=0, \quad (10)$$

Proof. The uncontrollable subspace X_{NC} is contained in the solution space of (10). Conversely, if a is the solution of (10), then we have

$$a^TA^jB=0, j=0,1,\ldots,n-1, \quad (11)$$

From Cayley-Hamilton Theorem, we know that all $A^k(k\geq n)$ can be written as a linear combination of $I,A,A^2,\ldots,A^{n-1}$.Therefore we have

$$e^{At}=\sum_{k=0}^{\infty}\frac{(At)^k}{k!}=\sum_{j=0}^{n-1}A^ja_j(t), \quad (12)$$

So that

$$e^{-At} = e^{A(-t)} = \sum_{j=0}^{n-1} A^j a_j(-t) = \sum_{j=0}^{n-1} A^j \beta_j(t), \tag{13}$$

where $\alpha_j(t), \beta_j(t)$ are the polynomials of t, then from (13), we get

$$a^T e^{-At} B = a^T \sum_{j=0}^{n-1} A^j \beta_j(t) B = \sum_{j=0}^{n-1} a^T A^j B \beta_j(t) = 0$$

That is, a is the constant solution of (9). Therefore $a \in X_{NC}$. The proof is completed.

3.3 The controllability canonical form and observability canonical form of the single input –single output system

Consider the completely controllable single input-single output linear time-invariant system

$$x\dot{x} = Ax + bu \tag{14}$$

$$y = cx$$

where A is an $n \times n$ constant matrix, b and c are $n \times 1$ and $1 \times n$ constant matrices respectively. As the system is completely controllable, according to the Controllability Matrix Test [7], we have

$$rank[b \quad Ab \quad \dots \quad A^{n-1}b] = n.$$

Let the characteristic polynomial of the system be

$$det(sI - A) = a(s) = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0.$$

Construct matrix

$$P = [e_1 \quad e_2 \quad \dots \quad e_n] = [b \quad Ab \quad \dots \quad A^{n-1}b] \begin{bmatrix} a_1 & \dots & a_{n-1} & 1 \\ \vdots & \ddots & \ddots & \\ a_{n-1} & \ddots & & \\ 1 & & & \end{bmatrix},$$

As the system is completely controllable, we know that P is nonsingular. Let us define $\beta_{n-1} = ce_n, \beta_{n-2} = ce_{n-1}, \ldots, \beta_0 = ce_1$.

Theorem 6 *By applying a nonsingular transformation $x = P\bar{x}$ to system(14), we can get its controllability canonical form as*

$$\dot{\bar{x}} = A_c\bar{x} + b_c u$$
$$y = c_c\bar{x}$$

where

$$A_c = P^{-1}AP = \begin{bmatrix} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix}, bc = P^{-1}b = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

$c_c = cP = [\beta_0 \quad \beta_1 \quad \dots \quad \beta_{n-1}]$

Proof. Expand $e_1, e_2, \dots, e_n$ as

$$\begin{aligned} &e_1 = a_1 b + a_2 Ab + \dots + a_{n-1}A^{n-2}b + A^{n-1}b \\ &e_2 = a_2 b + a_3 Ab + \dots + a_{n-1}A^{n-3}b + A^{n-2}b \\ &\vdots, \\ &e_{n-1} = a_{n-1}b + Ab \\ &e_n = b \end{aligned} \tag{15}$$

(1) First, we derive A_c. Using $A_c = P^{-1}AP$, it can be derived that

$$PA_c = AP = A[e_1 \quad e_2 \quad \dots e_n] = [Ae_1 \quad Ae_2 \quad \dots Ae_n)] \tag{16}$$

Applying Cayley-Hamilton Theorem, we have

$$a(A) = A^n + a_{n-1}A^{n-1} + \dots + a_1 A + a_0 I = 0$$

Further from (15), it can be obtained that

$$\begin{aligned} &Ae_1 = (a_0 b + a_1 Ab + a_2 A^2 b + \dots + a_{n-1}A^{n-1}b + A^n b) - a_0 b = -a_0 e_n \\ &Ae_2 = (a_1 b + a_2 Ab + a_3 A^2 b + \dots + a_{n-1}A^{n-2}b + A^{n-1}b) - a_1 b = e_1 - a_1 e_n \\ &\vdots \\ &Ae_{n-1} = (a_{n-2}b + a_{n-1}Ab + A^2 b) - a_{n-2}b = e_{n-2} - a_{n-2}e_n \\ &Ae_n = Ab = e_{n-1} - a_{n-1}e_n \end{aligned} \tag{17}$$

Substituting (17) into (16) yields

$$PA_c = [-a_0 e_n \quad e_1 - a_1 e_n \quad e_{n-2} - a_{n-2}e_n \quad e_{n-1} - a_{n-1}e_n)] =$$

$$[e_1 \quad e_2 \quad \dots \quad e_n] \begin{bmatrix} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix}$$

Because $[e_1 \quad e_2 \quad e_n] = P$, through left multiplying the both sides of above equation by P^{-1}, we can get the expression of A_c.
(2) Now, we derive b_c. Using $b_c = P^{-1}b$ and $e_n = b$, we can derive

$$Pb_c = b = e_n = [e_1 \;\; e_2 \;\; \dots \;\; e_n] \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = P \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

By left multiplying above equation by P^1, we can get the expression of b_c.
(3) We derive c_c. From $c_c = cP$, we can obtain

$$c_c = cP = c[e_1 \;\; e_2 \;\; \dots \;\; e_n][ce_1 \;\; ce_2 \;\; \dots \;\; ce_n] = [\beta_0 \;\; \beta_1 \;\; \dots \;\; \beta_{n-1}].$$

The proof is completed.

Similarly, the completely observable single input-single output linear time-invariant system

$$\dot{x} = Ax + bu \qquad\qquad y = cx \tag{18}$$

has its observability canonical form, where A is an $n \times n$ constant matrix, b and c are $n \times 1$ and $1 \times n$ constant matrices respectively:

Theorem 7 *By applying a nonsingular transformation $x = Q^{-1}\bar{x}$ to system(18), we can get its observability canonical form as*

$$\begin{aligned} \dot{x} &= A_c\bar{x} + b_c u \\ y &= c_c\bar{x} \end{aligned}$$

where

$$A_c = QAQ^{-1} = \begin{bmatrix} 0 & \dots & 0 & -a_0 \\ 1 & \cdots & & -a_1 \\ & \ddots & & \vdots \\ & & 1 & -a_{n-1} \end{bmatrix}, b_c = Qb = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{n-1} \end{bmatrix}, c_c = cQ^{-1} = [0 \dots 0 \; 1]$$

3.4 Controllable subspace of the discrete system

It is known that, when G is nonsingular, the controllable subspace X_c of the discrete system (G, H) is

$$X_c = span[G^{-n}H \;\; G^{-(n-1)}H \;\; \dots \;\; G^{-1}H],$$

Using the Cayley-Hamilton Theorem, we have

Theorem 8 *If G is nonsingular, then the controllable subspace X_c of the discrete system (G, H) is*

$$X_c = span[H \;\; GH \;\; \dots \;\; G^{n-1}H]$$

Proof. Because of the known conditions above, we only need to proof that

$span[G^{-n}H \;\; G^{-(n-1)}H \;\; \ldots \;\; G^{-1}H] = span[H \;\; GH \;\; \ldots \;\; G^{n-1}H)]$

Suppose that the characteristic polynomial of G is

$det(sI - G) = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0,$
Applying the Cayley-Hamilton Theorem, we have

$G^n + a_{n-1}G^{n-1} + \ldots + a_1 G + a_0 I = 0$

As G is nonsingular, then according to Lemma3, G does not have a zero eigenvalue, therefore we have $a_0 \neq 0$,thus we obtain

$I = \frac{-1}{a_0}(G^n + a_{n-1}G^{n-1} + a_1 G)$
Furthermore,we have

$G^{-1} = \frac{-1}{a_0}(G^{n-1} + a_{n-1}G^{n-2} + \ldots + a_1 I)$

That is, G^{-1} can be written as a linear combination of $I, G, \ldots, G^{n-1}$. Furthermore,we have

$G^{-2} = \frac{-1}{a_0}(G^{n-2} + a_{n-1}G^{n-2} + \ldots + a_2 I + a_1 G^{-1})$

Because G^{-1} can be written as a linear combination of $I, G, \ldots, G^{n-1}$,then G^{-2} can be written as a linear combination of $I, G, \ldots, G^{n-1}$. Proceeding forward like this, we can conclude that G^{-n} can be written as a linear combination of $I, G, \ldots, G^{n-1}$. Therefore we have

$span[G^{-n}H \;\; G^{-(n-1)}H \;\; \ldots \;\; G^{-1}H] \subseteq span[H \;\; GH \;\; \ldots \;\; G^{n-1}H]$

Also from

$[H \;\; GH \;\; \ldots \;\; G^{n-1}H] = G^n[G^{-n}H \;\; G^{-(n-1)}H \;\; \ldots \;\; G^{-1}H]$

and according to Theorem3,we can get

$rank[H \;\; GH \;\; \ldots \;\; G^{n-1}H] = rank[G^{-n}H \;\; G^{-(n-1)}H \;\; \ldots \;\; G^{-1}H]$

Therefore

$span[G^{-n}H \;\; G^{-(n-1)}H \;\; \ldots \;\; G^{-1}H] = span[H \;\; GH \;\; \ldots \;\; G^{n-1}H]$

The proof for the theorem is completed.

3.5 The controllability of the linear time-invariant continuous systems after time discretization

Consider the linear time-invariant continuous system
$\dot{x} = Ax + Bu$,
The discrete-time system with a sampling period T is
$x(k+1) = Gx(k) + Hu(k)$,

where

$G = e^{AT}$ and $H = \int_0^T e^{At}dtB$

Theorem 9 *If the discrete-time system (G, H) is controllable,then the continuous system (A, B) is controllable.*

Proof. If (G, H) is controllable,then according to the Controllability Matrix Test[7], we have
$rank[H \;\; GH \;\; \ldots \;\; G^{n-1}H] = n$,

Therefore, as long as

$$span[H \;\; GH \;\; \ldots \;\; G^{n-1}H] \subseteq span[B \; AB \ldots A^{n-1}B)] \tag{19}$$

we can readily get
$rank[A \;\; AB \;\; \ldots \;\; A^{n-1}B)] = n$,

thus (A, B) is controllable.

Next we prove that (19) holds. From the Cayley-Hamilton Theorem, we have

$$G = e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!} = \sum_{j=0}^{n-1} a_j(T)A^j \tag{20}$$

$$H = \int_0^T e^{AT}dt \cdot B = \int_0^T \sum_{j=0}^{n-1} a_j(t)A^j dt \cdot B = \sum_{j=0}^{n-1} A^j B \int_0^T a_j(t)dt = \sum_{j=0}^{n-1} A^j B r_j(T) \tag{21}$$

where $a_j(T)$, $r_j(T)$ are the scalars. From (21) we get

$$spanH \subseteq span[B \;\; AB \;\; \ldots \;\; A^{n-1}B)] \tag{22}$$

From (20), (21), we can derive

$$GH = \sum_{j=0}^{n-1} a_j(T)A^j H, A^k H = \sum_{j=0}^{n-1} A^{j+k} B r_j(T) \tag{23}$$

Therefore
$spanA^k H \subseteq span[B \;\; AB \;\; \ldots \;\; A^{n-1}B)]$,

Thus

$$spanGH \subseteq span[B \;\; AB \;\; \dots \;\; A^{n-1}B] \tag{24}$$

Proceeding forward like this,we can obtain

$$spanG^{n-1}H \subseteq span[B \;\; AB \;\; \dots \;\; A^{n-1}B] \tag{25}$$

Integrating (22),(24),(25),we can readily obtain that (19) holds.The proof is completed.

3.6 The Equivalent Representation of the Unobservable Subspaces of a Continuous System

Consider the known system

$$\begin{cases} \dot{x} = Ax, x(t_0) = x_0 \\ y = Cx \end{cases}$$

Its unobservable subspace X_{NO} is the constant solution space of $Ce^{A(t-t_0)}a = 0, t \in [0, T]$

Theorem 10 *The unobservable subspace X_{NO} of the system is the solution space of*

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} a = 0,$$

The proof of this theorem is similar to that of Theorem 5.

3.7 The Observability of the Linear Time-Invariant Discrete System

Consider the linear time-invariant discrete system

$$\begin{array}{l} x(k+1) = Gx(k), k = 0, 1, 2, \dots \\ y(k) = Cx(k) \end{array} \tag{26}$$

where $x(k)$ is the n-dimensional state variables, $y(k)$ is the q-dimensional output variables, G and C are the $n \times n$ and $q \times n$ constant matrices respectively.

According to the definition, for the system (26), there is a given non-zero initial state $x(0) = x_0$, if the output $y(k)$,$k = 0, 1, 2, \dots, n-1, \dots$ of the system trajectory $x(k)$ starting from x_0 is constantly zero, then x_0 is called the unobservable state. If x_0 is the unobservable state of the system (26), then

$$\begin{array}{r} y(0) = Cx(0) = Cx_0 = 0, \\ y(1) = CGx(0) = CGx_0 = 0, \\ \dots \\ y(n-1) = CG^{n-1}x(0) = CG^{n-1}x_0 = 0, \end{array}$$

From the Cayley-Hamilton Theorem,we know that, when $k \geq n$, then G^k can be written as a linear combination of $I, G, \ldots, G^{n-1}$. Therefore, x_0 can be the unobservable state, as long as $y(0) = y(1) = \ldots = y(n-1) = 0$ holds, that is

$$\begin{bmatrix} C \\ CG \\ \vdots \\ CG^{n-1} \end{bmatrix} x_0 = 0, \tag{27}$$

Thus, the unobservable subspace X_{NO} of the system is the solution space of equation system(27).

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