UDC 517.938

Mirzakulova A. E., Aldazharova M.M., Moldabek Zh.T., Aldibekov T.M*.

Kazakh National University after al-Farabi, Almaty, Kazakhstan
* E-mail: tamash59@mail.ru

Generalized singular exponents linear system of differential equations *

Consider a finite-dimensional linear homogeneous system of differential equations with continuous bounded coefficients in an infinite interval in critical cases of singular exponents. We introduce generalized singular upper and generalized singular lower exponents of the finite-dimensional linear homogeneous system of differential equations with continuous and tending to zero coefficients in an infinite interval. Formulas for calculating the generalized upper and generalized lower singular exponents of the linear homogeneous system of differential equations with continuous and tending to zero coefficients in an infinite interval were found. Introduced the asymptotic characteristics of linear homogeneous systems of differential equations are used for researches of nonlinear systems of differential equations. With the first approximation method investigated non-linear system of differential equations and uniform upper bounds of solutions of nonlinear differential equations in a defined class of nonlinear differential systems were established. We found sufficient conditions for asymptotic stability of the zero solution of the nonlinear system of differential equations. The generalized exponential stability of the zero solution of the nonlinear system of differential equations was established.
Key words: linear differential systems, singular exponents, nonlinear differential systems, stability, asymptotic stability

Мирзакулова А.Е., Алдажарова М.М., Молдабек Ж.Т., Алдибеков Т.М.
Обобщенные особые показатели линейной системы дифференциальных уравнений

Рассматривается конечномерная линейная однородная система дифференциальных уравнений с непрерывными ограниченными коэффициентами на бесконечном промежутке в критических случаях особых показателей. Вводятся обобщенное особое верхнее и обобщенное особое нижнее показатели конечномерной линейной однородной системы дифференциальных уравнений с непрерывными, со стремящейся к нулю коэффициентами на бесконечном промежутке. Найдены формулы для вычисления обобщенной верхней и обобщенной нижней особых показателей линейной однородной системы дифференциальных уравнений с непрерывными и со стремящимися к нулю коэффициентами на бесконечном промежутке. Введенные асимптотические характеристики линейной однородной системы дифференциальных уравнений применяются для исследования нелинейной системы дифференциальных уравнений. Методом первого приближения исследована нелинейная система дифференциальных уравнений и установлена равномерная оценка сверху решений нелинейной системы дифференциальных уравнений в определенном классе нелинейных дифференциальных систем. Найдено достаточное условие асимптотической устойчивости нулевого решения нелинейной системы дифференциальных уравнений. Приведен обобщенная экспоненциальная устойчивость нулевого решения нелинейной системы дифференциальных уравнений.
Ключевые слова: линейные дифференциальные системы, особые показатели, нелинейные дифференциальные системы, устойчивость, асимптотическая устойчивость

*This work was supported by funding of the program in G2015.

Мирзакулова А.Е., Алдажарова М.М., Молдабек Ж.Т., Әлдибеков Т.М.

Сызықты дифференциалдық теңдеулер жүйесінің жалпылама ерекше көрсеткіштері

Коэффициенттері үзіліссіз шенелген ақырлы өлшемді сызықты біртекті дифференциалдық теңдеулер жүйесінің ақырсыз аралықта ерекше көрсеткіштері сыни жағдайларда қарастырылады. Коэффициенттері үзіліссіз, нөлге ұмтылатын ақырлы өлшемді сызықты біртекті дифференциалдық теңдеулер жүйесінің ақырсыз аралықта жалпылама ерекше жоғарғы және жалпылама ерекше төменгі ерекше көрсеткіштері ендіріледі. Коэффициенттері үзіліссіз және нөлге ұмтылатын сызықты біртекті дифференциалдық теңдеулер жүйесінің ақырсыз аралықта жалпылама ерекше жоғарғы және жалпылама ерекше төменгі ерекше көрсеткіштерін есептеудің формулалары табылған. Келтірілген сызықты біртекті дифференциалдық теңдеулер жүйелерінің асимптотикалық сипаттауыштары сызықты емес дифференциалдық теңдеулер жүйелерін зерттегенде қолданылады. Сызықты емес дифференциалдық жүйелердің анықталған класында бірінші жуықтау әдісімен сызықты емес дифференциалдық теңдеулер жүйесінің шешімдерінің жоғарыдан бірқалыпты бағалауы орнатылған және сызықты емес дифференциалдық теңдеулер жүйелері зерттелген. Сызықты емес дифференциалдық теңдеулер жүйесінің нөлдік шешімінің асимптотикалық орнықтылығының жеткілікті шарты табылған. Сызықты емес дифференциалдық теңдеулер жүйесінің нөлдік шешімінің жалпылама экспоненциалдық орнықтылығының белгісі келтірілген.

Түйін сөздер: сызықты дифференциалдық жүйелер, ерекше көрсеткіштер, сызықты емес дифференциалдық жүйелер, орнықтылық, асимптотикалық орнықтылық

1 Introduction

The upper and lower singular exponents of the differential system introduced in the works [1, 2]. History of the discovery of these important asymptotic characteristics is contained in [3]. Detailed information about singular exponents of a homogeneous system of linear differential equations with bounded continuous coefficients is contained in the book [4] and review in [5]. In this paper, singular exponents of the differential system is investigated in critical cases, i.e., for zero values of these characteristics. Definition of generalized exponential stability of the zero solution of the nonlinear system of differential equations is given in [6].The work is closely related with the work [7].

2 Linear system of differential equations

Consider linear homogeneous system of differential equations

$$\dot{x} = A(t)x, x \in R^n, t \geqslant t_0 \tag{1}$$

where $A(t)$ is a continuous matrix and satisfies the condition

$$\|A(t)\| \leq C_A\varphi(t), t \geqslant t_0, \tag{2}$$

where C_A is a constant depending on the choice of matrix A, $\varphi(t)$ is a positive continuous function on the interval $[t_0, +\infty)$ and these function such, that,

$\lim\limits_{t\to\infty} \varphi(t) = 0$ and integral $I(\varphi) = \int\limits_{t_0}^{+\infty} \varphi(s)ds$ are diverges.

Denote by

$$q(t) = \int_{t_0}^{t} \varphi(s)ds. \tag{3}$$

Definition 1. Constants $n(q)$ and $N(q)$ are called respectively generalized lower and generalized upper exponents with respect to q for system (1) with condition (2), if for any $\varepsilon > 0$, for all nonzero solutions $x(t)$ of the system (1) is performed the estimation

$$d_{n,\varepsilon} exp\{(n(q) - \varepsilon)q(t)\} \leq \frac{\|x(t)\|}{\|x(s)\|} \leq D_{N,\varepsilon} exp\{(N(q) + \varepsilon)q(t)\} \tag{4}$$

for all $t \geq s \geq t_0$, where $D_{N,\varepsilon}$ $d_{n,\varepsilon}$ are constants, depending on the choice of $N(q)$, $n(q)$ $\varepsilon > 0$ and function $q(t)$ defined by the formula (3).

The set $\{N(q)\}$ generalized upper constants of the system (1) is called upper class of the system with respect to q and denoted by the symbol $B_0(A, q)$.

The set $\{n(q)\}$ generalized lower constants of the system (1) is called lower class of the system (1) with respect to q and denoted by the symbol $H_0(A, q)$.

Definition 2. Number

$$\Omega_0(A, q) = \inf_{N(q) \in B_0(A,q)} N(q) \tag{5}$$

is called a generalized upper singular exponent of the system (1) with respect to q.

Number

$$\omega_0(A, q) = \sup_{n(q) \in H_0(A,q)} n(q) \tag{6}$$

is called a generalized lower singular exponent of the system (1) with respect to q.

From definition 1 it follows, that is valid the inequality

$$\omega_0(A, q) \leq \Omega_0(A, q) \tag{7}$$

Remark 1. If we consider linear system (1) with continuous bounded coefficients without the condition (2) and $q(t) = t$, then the generalized singular exponents will convert to numbers, entered by Bohl-Persidskii.

Note, that for any $\varepsilon > 0$ exist $D_\varepsilon > 0$ and $d_\varepsilon > 0$ and for Cauchy matrix of the linear system (1) with the condition (2) are valid the inequalities

$$\|X(t, t_0)\| \leqslant D_\varepsilon e^{(\Omega_0(A,q)+\varepsilon)(q(t)-q(t_0))} \tag{8}$$

and

$$d_\varepsilon e^{(\omega_0(A,q)-\varepsilon)(q(t)-q(t_0))} \leqslant \|X(t, t_0)\| \tag{9}$$

for all $t \geqslant t_0$.

Remark 2. Singular exponents of Bohl and Persidskii of the system (1) satisfying the condition (2) are equal to zero, i.e., occurs critical cases.

Lemma 1. Generalized upper singular exponent of the system (1) with respect to q, satisfying the condition (2) defined by the formula

$$\Omega_0(A,q) = \overline{\lim_{t-s\to+\infty}} \frac{\ln\|X(t,s)\|}{q(t)-q(s)}.$$

Proof. From the inequality (8) follows, that for any $\varepsilon > 0$ for all $t \geqslant s \geqslant t_0$ is valid the inequality

$$\frac{\ln\|X(t,s)\|}{q(t)-q(s)} \leq \frac{\ln D_\varepsilon}{q(t)-q(s)} + \Omega_0(A,q) + \varepsilon \tag{10}$$

Consequently, take place the inequality

$$\exists\mu \equiv \overline{\lim_{t-s\to+\infty}} \frac{\ln\|X(t,s)\|}{q(t)-q(s)} \leq \Omega_0(A,q) + \varepsilon$$

Hence, turn to $\varepsilon \to 0$ we obtain the inequality

$$\mu \leq \Omega_0(A,q) \tag{11}$$

We will prove that take place and the converse inequality. Such that

$$\mu = \overline{\lim_{t-s\to+\infty}} \frac{\ln\|X(t,s)\|}{q(t)-q(s)}$$

then, for any $\varepsilon > 0$ exists $\overline{t} - \overline{s} \geq 0$ and for all $t - s > \overline{t} - \overline{s} \geq 0$ is valid the inequality

$$\frac{\ln\|X(t,s)\|}{q(t)-q(s)} \leq \mu + \varepsilon.$$

It follows that,

$$\|X(t,s)\| \leq e^{(\mu+\varepsilon)(q(t)-q(s))}$$

In respect that the segment $[t_0, \overline{t}]$, we obtain, that for $\varepsilon > 0$ exist $\overline{D}_\varepsilon > 0$ and for Cauchy matrix of the linear system (1) with the condition (2) is valid the inequality

$$\|X(t,s)\| \leq \overline{D}_\varepsilon e^{(\mu+\varepsilon)(q(t)-q(s))}$$

for all $t \geq s \geq t_0$ Such number $\Omega_0(A,q)$ is a infimum of numbers performing such estimation, then occurs the inequality

$$\Omega_0(A,q) \leq \mu \tag{12}$$

Combining the inequalities (11), (12) we obtain the required assertion. Lemma 1 is proved.

Lemma 2. Generalized lower singular exponent of the system (1) with respect to q , satisfying the condition (2) is defined by the formula

$$\omega_0(A,q) = \underline{\lim_{t-s\to+\infty}} \frac{\ln\|X(t,s)\|}{q(t)-q(s)}.$$

Proof. From the inequality (9) follows, that for any $\varepsilon > 0$ for all $t \geqslant s \geqslant t_0$ is valid the inequality

$$\frac{\ln d_\varepsilon}{q(t)-q(s)} + \omega_0(A,q) - \varepsilon \leq \frac{\ln \|X(t,s)\|}{q(t)-q(s)}$$

Consequently, take place the inequality

$$\exists \gamma \equiv \varliminf_{t-s\to+\infty} \frac{\ln \|X(t,s)\|}{q(t)-q(s)} \geq \omega_0(A,q) - \varepsilon$$

Hence, turn to $\varepsilon \to 0$ we obtain the inequality

$$\omega_0(A,q) \leq \gamma \tag{13}$$

We will prove that take place and the converse inequality. Such that

$$\gamma = \varliminf_{t-s\to+\infty} \frac{\ln \|X(t,s)\|}{q(t)-q(s)}$$

then, for any $\varepsilon > 0$ exists $\overline{t} - \overline{s} \geq 0$ and for all $t - s > \overline{t} - \overline{s} \geq 0$ is valid the inequality

$$\frac{\ln \|X(t,s)\|}{q(t)-q(s)} \geq \gamma - \varepsilon.$$

or

$$\|X(t,s)\| \geq e^{(\gamma-\varepsilon)(q(t)-q(s))},$$

for all $t - s > \overline{t} - \overline{s} \geq 0$ In respect that the segment $[t_0, \overline{t}]$, we obtain, that for $\varepsilon > 0$ exist $\overline{d}_\varepsilon > 0$ and for Cauchy matrix of the linear system (1) with the condition (2) is valid the inequality

$$\|X(t,s)\| \geq \overline{d}_\varepsilon e^{(\gamma-\varepsilon)(q(t)-q(s))}$$

for all $t \geq s \geq t_0$ Such number $\omega_0(A,q)$ is a supremum of numbers performing such estimation, then take place the inequality

$$\omega_0(A,q) \geq \gamma \tag{14}$$

Combining the inequalities (13), (14) we obtain the required assertion. Lemma 2 is proved.

Lemma 3. Equality

$$\Omega_0(A,q) = -\omega_0(A,q)$$

holds if and only if

$$\{N(q)\} = -\{n(q)\}$$

Proof. From the inequality (4) follows, that for any $\varepsilon > 0$, for all nonzero solutions $x(t)$ of the system (1) occurs the inequality

$$d_{n,\varepsilon} exp\{(-N(q)-\varepsilon)q(t)\} \leq \frac{\|x(t)\|}{\|x(s)\|} exp\{(-1)(N(q)+n(q))q(t)\} \leq$$

$$\leq D_{N,\varepsilon} exp\{(-n(q)+\varepsilon)q(t)\}$$

Hence, using definition of the generalized singular exponents, we obtain the required assertion. Lemma 3 is proved.

3 Nonlinear system of differential equations

Consider nonlinear system of differential equations

$$\dot{x} = A(t)x + f(t,x), x \in R^n, t \in I \equiv [t_0, +\infty), \tag{15}$$

where $A(t)$ is a continuous matrix for $t \geqslant t_0$ and satisfy the condition (2), $f(t,x)$ is a continuous vector function in the domain $G = I \times R^n$ and $f(t,0) = 0$.

Denote by $L(\varphi(t))$ class of vector function $f(t,x)$ satisfying the inequality

$$\|f(t,x)\| \leqslant \delta(t)\|x\|, \tag{16}$$

where $\delta(t)$ is a continuous perturbation norm for $t \geqslant t_0$ and satisfy the condition

$$\lim_{t\to+\infty} \frac{\delta(t)}{\varphi(t)} = 0. \tag{17}$$

Theorem 1. If a nonlinear system (15), first approximation of the system (1) satisfy the condition (2) and perturbation $f(t,x) \in L(\varphi(t))$, then for any $\varepsilon > 0$ exist $D_\varepsilon > 0$ such, that uniformly for all nonzero solutions of the system (15) are valid the inequality

$$\|x(t)\| \leq D_\varepsilon \|x(t_0)\| e^{(\Omega_0(A,q)+\varepsilon)(q(t)-q(t_0))} \tag{18}$$

for all $t \geq t_0$.

Proof. As is well known solutions of the perturbed system (15) satisfy the equation

$$x(t) = X(t,t_0)x(t_0) + \int_{t_0}^{t} X(t,s)f(s,x(s))ds. \tag{19}$$

We fix $\varepsilon > 0$. From (8) follows the inequality

$$\|X(t,t_0)\| \leq D_{\varepsilon_1} e^{(\Omega_0(A,q)+\varepsilon_1)(q(t)-q(t_0))} \tag{20}$$

where

$$\varepsilon_1 = \frac{\varepsilon}{3}, \quad D_{\varepsilon_1} > 0.$$

Using (16) and (20) estimating the norm (19) we obtain the inequality

$$\|x(t)\| \leq D_{\varepsilon_1} e^{(\Omega_0(A,q)+\varepsilon_1)(q(t)-q(t_0))}\|x(t_0)\| + \int_{t_0}^{t} D_{\varepsilon_1} e^{(\Omega_0(A,q)+\varepsilon_1)(q(t)-q(s))}\delta(s)\|x(s)\|ds \tag{21}$$

for all $t \geqslant s \geqslant t_0$.

From (21) follows the following inequality

$$y(t) \leqslant D_{\varepsilon_1}\|x(t_0)\| + \int_{t_0}^{t} D_{\varepsilon_1}\delta(s)y(s)ds \tag{22}$$

where

$$y(t) = \|x(t)\| e^{-(\Omega_0(A,q)+\varepsilon_1)(q(t)-q(t_0))}.$$

From (22) we obtain, that

$$y(t) \leqslant D_{\varepsilon_1} \|x(t_0)\| e^{\int\limits_{t_0}^{t} D_{\varepsilon_1}\delta(\tau)d\tau}$$

or

$$\|x(t)\| \leqslant D_{\varepsilon_1} \|x(t_0)\| e^{(\Omega_0(A,q)+\varepsilon_1)(q(t)-q(t_0))+\int\limits_{t_0}^{t} D_{\varepsilon_1}\delta(\tau)d\tau}.$$

Consequently, by virtue of (3) follows, that

$$\|x(t)\| \leqslant D_{\varepsilon_1} \|x(t_0)\| e^{(\Omega_0(A,q)+\varepsilon_1)(q(t)-q(t_0))+\int\limits_{t_0}^{t} D_{\varepsilon_1}\frac{\delta(\tau)}{\varphi(\tau)}dq(\tau)}. \quad (23)$$

From the condition (17) follows, that exist such $T \geq t_0$, that for all $t \geq T \geq t_0$ have place the inequality

$$D_{\varepsilon_1}\frac{\delta(t)}{\varphi(t)} < \varepsilon_1.$$

Then from (23) we obtain the following inequality

$$\|x(t)\| \leqslant D_{\varepsilon_1}\overline{D_\varepsilon} \|x(t_0)\| e^{(\Omega_0(A,q)+2\varepsilon_1)(q(t)-q(t_0))},$$

where

$$\overline{D_\varepsilon} = \frac{exp(\int\limits_{t_0}^{T} D_{\varepsilon_1}\delta(\tau)d\tau)}{exp(\varepsilon_1(q(T)-q(t_0)))}.$$

Now supposing $D_\varepsilon \equiv D_{\varepsilon_1}\overline{D_\varepsilon}$ and considering, that $2\varepsilon_1 < \varepsilon$ we obtain, that for all nonzero solutions of the system (15) uniformly the inequality (18) for all $t \geqslant t_0$. Theorem 1 is proved.

Note, that generalized upper singular exponent with respect to q, is stable up of a asymptotic characteristic linear system (1) with the condition (2) in the class $L(\varphi(t))$. The following assertion holds.

Theorem 2. If a linear system (1) with the condition (2) have negative generalized upper singular exponent with respect to q, then zero solution of the nonlinear system (15) asymptotically Lyapunov stable as $t \to +\infty$, where perturbation $f(t,x) \in L(\varphi(t))$.

Proof. In fact, for all nonzero solutions of the system (15) is valid uniformly the inequality (18) for all $t \geq t_0$ and linear system (1) with the condition (2) have negative generalized upper singular exponent with respect to q, hence follows assertion. Theorem 2 is proved.

Such (18) means by the definition exponential stability with respect to q, as $t \to +\infty$, zero solution of nonlinear system (15) then we have the following statement to clarify Theorem 2.

Theorem 3. If linear system (1) with the condition (2) have negative generalized upper singular exponent with respect to q, then zero solution of nonlinear system (15) exponential stable with respect to q, as $t \to +\infty$.

Example. Consider the differential equation

$$\dot{x} = \frac{sint - 2}{t} x + f(t,x), \quad x \in R$$

where $f(t,x)$ is a continuous function in $t \geq t_0 \geq 1$ and continuously differentiable with x function such, that $f(t,0) = 0$.

Equation of the first approximation

$$\dot{x} = \frac{sint - 2}{t} x + f(t,x)$$

has generalized upper singular exponent

$$\Omega_0(q) = -1, \ q(t) = lnt, \ t \geq t_0 \geq 1.$$

Therefore, when the following condition is satisfied

$$|f(t,x)| \leq \delta(t)|x|, \ t\delta(t) \to 0, \ t \to +\infty$$

zero solution of the nonlinear differential equation is asymptotically Lyapunov stable as $t \to +\infty$.

4 Conclusion

Defined the upper and lower generalized singular exponents of linear systems of differential equations in critical cases of singular exponents. Established the uniform upper bound for solutions of nonlinear system of differential equations in a defined class of nonlinear differential systems. We found a sufficient condition for the asymptotic stability of the zero solution of the nonlinear system of differential equations. The generalized exponential stability of the zero solution of the nonlinear system of differential equations was established.

References

[1] *Bohl P.* Uber Differentialgleichungen // J. Reine und angew. Math. – 1913. –B. 144. – p. 284-318.

[2] *Persidskii K.P.* On the stability of motion in the first approximation // Matematical collection. - 1933. - V. 40, №3. – p. 284-292.(in Russian)

[3] *Daletskii Y.L., Krein M.G.* Stablitity of solutions of differential equations in a Banach space. - M., Nauka, 1970. – 536 p. (in Russian)

[4] *Bylov B.F., Vinograd R.E., Grobman D.M., Nemyskii V.V.* The theory of Lyapunov exponents and its application to stability issues. - M., 1966.- 576 p. (in Russian)

[5] *Izobov N.A.* Linear systems of ordinary differential equations // Matematical analysis. The results of science and technology. - 1974. – Vol. 12. – p. 71-146. (in Russian)

[6] *T.M. Aldibekov, M.M. Aldazharova.* On the Stability by the First Approximation of Lyapunov Characteristic Exponents in Critical Cases // Differential Equations, 2014.– Vol. 50, No. 10.– p. 1-5

[7] *T.M. Aldibekov, A.E.Mirzakulova, M.M. Aldazharova* About stable asymptotic characteristics of differential systems // KazNU Bulletin, ser. math., mech., inf. - 2015. - № 2(85). - S.33-41. (in Russian)