1-бөлім Математика

Раздел 1 Математика

Section 1 Matematics

UDC 517. 968. 2

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Solvability and construction of solutions of integral equations

A class of integral equations with respect to one variable function as well as to multivariable function that are solvable for any right hand side of an equation has been singled out. A necessary and sufficient condition for existence of a solution has been obtained for the class of integral equations and the general form of their exact solutions has been found. Necessary and sufficient conditions for existence of solutions to the mentioned equations with a given right hand side are obtained by reducing them to solving an extremal problem. An algorithm for solving the extremal problem by constructing a minimizing sequence has been developed and a convergence rate estimation has been obtained. A solvability criterion as a requirement on infimum of functional has been formulated. A necessary and sufficient condition for solvability of an integral equation with parameter has been obtained and its general solution has been found.
Key words: integral equation, general solution, existence of a solution, necessary and sufficient condition, solvability criterion, extremal problem, minimizing sequence.

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Разрешимость и построение решений интегральных уравнений

Определен класс интегральных уравнений от искомой функции одной переменной, а также от нескольких переменных, разрешимых для любой правой части уравнения. Для данного класса интегральных уравнений получены необходимые и достаточные условия существования решения, найдены их общие решения в виде суммы частного решения и решения однородного уравнения. Показаны ортогональность частного решения и решения однородного уравнения, а также что частное решение является решением рассматриваемого уравнения с минимальной нормой. Получены необходимые и достаточные условия существования решений указанных уравнений при заданной правой части, путем сведения их к решению экстремальной задачи специального вида. Разработан алгоритм построения решения экстремальной задачи путем построения минимизирующей последовательности, получена оценка скорости сходимости ее к решению интегрального уравнения. Сформулирован критерий разрешимости интегрального уравнения в виде требования на значение нижней грани целевого функционала. Исследовано интегральное уравнение с ограничением на искомую функцию, подробно описаны способ проверки его разрешимости и метод построения его решения, а также доказаны их корректность. Для интегрального уравнения с параметром получены необходимые и достаточные условия разрешимости и найдено общее его решение.
Ключевые слова: интегральное уравнение, общее решение, существование решения, необходимое и достаточное условие, критерий разрешимости, экстремальная задача, минимизирующая последовательность.

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Интегралдық теңдеулердің шешімдерінің бар болуы және оларды құру

Оң жағында кез-келген функция жағдайы үшін шешілетін ізделінді бір айнымалының және көп айнымалының функцияларына қатысты интегралдық теңдеулер класы бөлiнiп алынды. Осы класс үшін шешімнің бар болуының қажетті және жеткілікті шарттары табылды, олардың жалпы шешімдері дербес шешімі мен біртекті бөлігінің жалпы шешімінің косындысы ретінде құрылды. Дербес шешім мен біртекті бөлігінің жалпы шешімінің ортогональдiгi және дербес шешім сол теңдеудің минималды нормалы шешімі екені көрсетілді. Аталған теңдеулердің оң жақтары берілген функция жағдайларында шешімдерінің бар болуының қажетті және жеткілікті шарттары теңдеулерді экстремалды есепке келтіру арқылы алынды. Экстремалды есепті минимумдаушы тізбекті құру арқылы шешу алгоритмі құрастырылды және тізбектің интегралдық теңдеудің шешіміне жинақталу жылдамдығы бағаланды. Интегралдық теңдеудің шешілетіндігінің критерийі функционалдың төменгі қырына қойылатын талап түрінде алынды. Ізделінді функцияға шектеуі бар интегралдық теңдеу зерттелген, оның шешілетіндігін тексеру жолы және шешімін құру әдісі сипатталған, сол әдістердің дұрыстығы дәлелденген. Параметрлі интегралдық теңдеу үшін шешімнің бар болуының қажетті және жеткілікті шарттары анықталды және жалпы шешімі табылды.
Түйін сөздер: интегралдық теңеу, жалпы шешім, шешімнің бар болуы, қажетті және жеткілікті шарт, шешілетіндігінің критерийі, экстремалды есеп, минимумдаушы тізбек.

1 Problem statement

Controllability problems solving for dynamical systems [1-3], solving problems of mathematical theory of optimal processes [4-6], boundary value problems for differential equations with phase and integral constraints [7-9] are reduced to solvability and construction of a general solution to the integral equation

$$K_1 w = \int_a^b K(t_*, \tau) w(\tau) d\tau = \beta, \quad t_* \in [t_0, t_1] \tag{1}$$

here $K(t_*, \tau) = K(\tau) = \|K_{ij}(\tau)\|$, $i = \overline{1,n}$, $j = \overline{1,m}$ is a given matrix with elements from the space L_2, $t_* \in [t_0, t_1]$ is fixed, $K_{ij}(\tau) \in L_2(I_1, R^1)$, $w(\tau) \in L_2(I_1, R^m)$ is unknown function, $\beta \in R^n$, $I_1 = [a, b]$.

Note that (1) is a special case of the Fredholm integral equation of the first kind

$$Ku = \int_a^b K(t, \tau) u(\tau) d\tau = f(t), \quad t \in [t_0, t_1],$$

where $K(t, \tau) = \|K_{ij}(t, \tau)\|$, $i = \overline{1,n}$, $j = \overline{1,m}$ is a given $n \times m$ matrix, elements of the matrix $K(t, \tau)$ the functions $K_{ij}(t, \tau)$ are measurable and belong to the class L_2 on the set $S_1 = \{(t, \tau) \in R^2 \; / \; t_0 \le t \le t_1, \; a \le \tau \le b\}$,

$$\int_a^b \int_{t_0}^{t_1} |K_{ij}(t, \tau)|^2 dt d\tau < \infty,$$

the function $f(t) \in L_2(I, R^n)$ is given, $u(\tau) \in L_2(I_1, R^m) is an unknown function$, $I_1 = [a, b]$, t_0, t_1, a, b are fixed, $t_1 > t_0$, $b > a$, $K : L_2(I_1, R^m) \to L_2(I, R^n)$.

Problem 1. *Provide a necessary and sufficient condition for existence of a solution to integral equation* (1) *for any* $\beta \in R^n$.

Problem 2. *Find a general solution to integral equation* (1) *for any* $\beta \in R^n$.

Problem 3. *Provide a necessary and sufficient condition for existence of a solution to integral equation* (1) *with a given* $\beta \in R^n$.

Problem 4. *Find a solution to integral equation* (1) *with a given* $\beta \in R^n$.

Problem 5. *Provide a necessary and sufficient condition for existence of a solution to integral equation* (1) *with a given* $\beta \in R^n$, *and the unknown function* $w(\tau) \in W(\tau) \subset L_2(I_1, R^m)$;

Problem 6. *Find a solution to integral equation* (1) *with a given* $\beta \in R^n$, *and* $w(\tau) \in W(\tau) \subset L_2(I_1, R^m)$, *where* $W(\tau)$ *is a given set.*

Consider an integral equation with parameter of the following form

$$K_2(v) = \int_a^b K(t,\tau)v(t,\tau)d\tau = \mu(t), \quad t \in I = [t_0, t_1], \tag{2}$$

where $K(t,\tau) = \|K_{ij}(t,\tau)\|$, $i = \overline{1,n}$, $j = \overline{1,m}$ is a given matrix with elements from L_2, $v(t,\tau) \in L_2(S_1, R^m)$ is an unknown function, t is a parameter, $\mu(t) \in L_2(I, R^n)$.

Problem 7. *Provide a necessary and sufficient condition for existence of a solution to integral equation* (2) *for any* $\mu(t) \in L_2(I, R^n)$;

Problem 8. *Find a general solution to integral equation* (2) *for any* $\mu(t) \in L_2(I, R^n)$;

Consider an integral equation with respect to multivariable function

$$K_3 w = \int_a^b \int_c^d K(t,\tau)w(t,\tau)d\tau dt = \beta, \quad \beta \in R^n, \tag{3}$$

где $K(t,\tau) = \|K_{ij}(t,\tau)\|$, $i = \overline{1,n}$, $j = \overline{1,m}$ is a known $n \times m$ matrix, $K_{ij}(t,\tau) \in L_2(G, R^1)$, $w(t,\tau) \in L_2(G, R^m)$ is unknown function, $G = \{(t,\tau)/a \le t \le b, \quad c \le \tau \le d\}$, $\int_a^b \int_c^d |K_{ij}(t,\tau)|^2 d\tau dt < \infty$, $K_3 : L_2(G, R^m) \to R^n$.

Problem 9. *Provide a necessary and sufficient condition for existence of a solution to integral equation* (3) *for any* $\beta \in R^n$;

Problem 10. *Find a general solution to integral equation* (3) *for any* $\beta \in R^n$;

As it is obvious from the foregoing investigation of solvability and solving integral equations (1) – (3) are topical for solving boundary value problems for differential equations.

The aim of this paper is to provide new methods for investigation of solvability and construction general solutions to integral equations (1) – (3).

This paper is an extension of scientific research presented in [10-12].

2 Integral equation solvable for any right hand side

Consider problems 1, 2. The following theorem provides a necessary and sufficient condition for existence of a solution to integral equation (1).

Theorem 1. *A necessary and sufficient condition for existence a solution to integral equation* (1) *for any* $\beta \in R^n$ *is that the* $n \times n$ *matrix*

$$C = \int_a^b K(\tau)K^*(\tau)d\tau \tag{4}$$

be positive definite for all a, b, $b > a$, *where the superscript* $(*)$ *means transposed.*

Proof. *Sufficiency.* Let the matrix C be positive definite. Show that integral equation (1) has a solution for any $\beta \in R^n$. Choose $w(\tau) = K^*(\tau)C^{-1}\beta$, $\tau \in I_1 = [a, b]$. Then

$$K_2w = \int_a^b K(\tau)K^*(\tau)d\tau C^{-1}\beta = \beta.$$

Consequently in the case $C > 0$, integral equation (1) has at least one solution $w(\tau)=K^*(\tau)C^{-1}\beta$, $\tau \in I_1$, here $\beta \in R^n$ is an arbitrary vector. The sufficiency is proved.

Necessity. Let us assume that integral equation (1) has a solution for any fixed $\beta \in R^n$. Show that the matrix $C > 0$. Since $C \geq 0$, it is sufficient to show that the matrix C is nonsingular.

Assume the converse. Then the matrix C is singular. Therefore there exists a vector $c \in R^n$, $c \neq 0$ such that $c^*Cc = 0$. Define the function $v(\tau) = K^*(\tau)c$, $\tau \in I_1$, $v(\cdot) \in L_2(I_1, R^m)$. Note that

$$\int_a^b v^*(\tau)v(\tau)d\tau = c^* \int_a^b K(\tau)K^*(\tau)d\tau c = c^*Cc = 0.$$

This means that the function $v(t) = 0$, $\forall\tau$, $\tau \in I_1$. Since integral equation (1) has a solution for any $\beta \in R^n$, in particular, there exists a function $\overline{w}(\cdot) \in L_2(I_1, R^m)$ such that $(\beta = c)$

$$\int_a^b K(\tau)\overline{w}(\tau)d\tau = c.$$

Then we have

$$0 = \int_a^b v^*(\tau)\overline{w}(\tau)d\tau = c^* \int_a^b K(\tau)\overline{w}(\tau)d\tau = c^*c.$$

This contradicts the fact that $c \neq 0$. The necessity is proved. The theorem if proved.

The following theorem provides a general solution to the integral equation(3).

Theorem 2. *Let the matrix* C *defined by* (4) *be positive definite. Then for any* $\beta \in R^n$

$$w(\tau) = K^*(\tau)C^{-1}\beta + p(t) - K^*(\tau)C^{-1} \int_a^b K(\eta)p(\eta)d\eta, \quad \tau \in I_1 = [a, b], \tag{5}$$

is a general solution to integral equation (2), *where* $p(\cdot) \in L_2(I_1, R^m)$ *is an arbitrary function,* $\beta \in R^n$ *is an arbitrary vector.*

Proof. Let us introduce the sets

$$W = \{w(\cdot) \in L_2(I_1, R^m) \ / \ \int_a^b K(\tau)w(\tau)d\tau = \beta\}, \tag{6}$$

$$\begin{aligned} Q = \{w(\cdot) \in L_2(I_1, R^m) \ / \ w(\tau) = K^*(\tau)C^{-1}\beta + p(t) - \\ -K^*(\tau)C^{-1} \int_a^b K(\eta)p(\eta)d\eta, \ \ \forall p(\cdot), \ p(\cdot) \in L_2(I_1, R^m)\}. \end{aligned} \tag{7}$$

The set W contains all solutions of the integral equation (1) under the condition $C > 0$. The theorem states that the function $w(\cdot) \in L_2(I_1, R^m)$ belongs to the set W if and only if it is contained in Q, i.e. $W = Q$. Show that $W = Q$. In order to prove this it is sufficient to show that $Q \subset W$ and $W \subset Q$.

Show that $Q \subset W$. Indeed, if $w(\tau) \in Q$, then as it follows from (7), the following equality holds

$$\int_a^b K(\tau)w(\tau)d\tau = \int_a^b K(\tau)K^*(\tau)d\tau C^{-1}\beta + \int_a^b K(\tau)p(\tau)d\tau - \int_a^b K(\tau)K^*(\tau)d\tau C^{-1} \times$$

$$\times \int_a^b K(\eta)p(\eta)d\eta = \beta + \int_a^b K(\tau)p(\tau)d\tau - \int_a^b K(\eta)p(\eta)d\eta = \beta.$$

This implies that $w(\tau) \in W$.

Show that $W \subset Q$. Let $w_*(\tau) \in W$, i.e. the equality(6) holds for the function $w_*(t) \in W$:

$$\int_a^b K(\tau)w_*(\tau)d\tau = \beta.$$

Note that the function $p(t) \in L_2(I_1, R^m)$ is an arbitrary in the relation (5). In particular, we can choose $p(t) = w_*(\tau)$, $\tau \in I_1$. Now the function $w(\tau) \in Q$ can be rewritten in the form

$$w(\tau) = K^*(\tau)C^{-1}\beta + w_*(\tau) - K^*(\tau)C^{-1} \int_a^b K(\tau)w_*(\tau)d\tau = K^*(\tau)C^{-1}[\int_a^b K(\tau)w_*(\tau)d\tau] +$$

$$+ w_*(\tau) - K^*(\tau)C^{-1} \int_a^b K(\tau)w_*(\tau)d\tau = w_*(\tau), \ \ \tau \in I_1.$$

Consequently $w_*(t) = w(t) \in Q$. This yields that $W \subset Q$. It follows from the inclusions $Q \subset W$, $W \subset Q$ that $W = Q$. The theorem is proved.

The main properties of solutions of the integral equation (1):

1. The function $w(\tau)$, $\tau \in I_1$ can be represented in the form $w(\tau) = w_1(\tau) + w_2(\tau)$, where $w_1(\tau) = K^*(\tau)C^{-1}\beta$ is a particular solution of the integral equation (2), $w_2(\tau) =$

$= p(t) - K^*(\tau)C^{-1}\int\limits_a^b K(\eta)p(\eta)d\eta$, $\tau \in I_1$, is a solution of the homogeneous integral equation $\int\limits_a^b K(\tau)w_2(\tau)d\tau = 0$, where $p(t) \in L_2(I_1, R^m)$ is an arbitrary function.

Indeed,

$$\int\limits_a^b K(\tau)w_1(\tau)d\tau = \int\limits_a^b K(\tau)K^*(\tau)C^{-1}\beta d\tau = \beta, \ \ \forall\beta, \ \ \beta \in R^n,$$

$$\int\limits_a^b K(\tau)w_2(\tau)d\tau = \int\limits_a^b K(\tau)p(\tau)d\tau - \int\limits_a^b K(\tau)K^*(\tau)C^{-1}d\tau \int\limits_a^b K(\eta)p(\eta)d\eta = 0.$$

2. The functions $w_1(\tau) \in L_2(I_1, R^m)$, $w_2(\tau) \in L_2(I_1, R^m)$ are orthogonal in L_2, i.e. $w_1 \perp w_2$. Indeed,

$$< w_1, w_2 >_{L_2} = \int\limits_a^b w_1^*(\tau)w_2(\tau)d\tau = \int\limits_a^b \beta^*C^{-1}K(\tau)p(\tau)d\tau - \int\limits_a^b \beta^*C^{-1}K(\tau)K^*(\tau)d\tau \times$$

$$\times C^{-1}\int\limits_a^b K(\eta)p(\eta)d\eta = \beta^*C^{-1}\int\limits_a^b K(\tau)p(\tau)d\tau - \beta^*C^{-1}\int\limits_a^b K(\eta)p(\eta)d\eta = 0.$$

3. The function $w_1(\tau) = K^*(\tau)C^{-1}\beta$, $\tau \in I_1$ is a solution of the integral equation (1) with minimal norm in $L_2(I_1, R^m)$. Indeed, $\|w(\tau)\|^2 = \|w_1(\tau)\|^2 + \|w_2(\tau)\|^2$. Hence $\|w(\tau)\|^2 \geq \|w_1(\tau)\|^2$. If the function $p(\tau) = 0$, $\tau \in I_1$, then the function $w_2(\tau) = 0$, $\tau \in I_1$. Hence $w(\tau) = w_1(\tau)$, $\|w\| = \|w_1\|$;

4. The solution set for the integral equation (1) is convex. As it follows from the proof of theorem 2 the set of all solutions to the equation (1) is Q. Show that Q is a convex set. Let

$$\overline{w}(\tau) = K^*(\tau)C^{-1}\beta + \overline{p}(\tau) - K^*(\tau)C^{-1}\int\limits_a^b K(\eta)\overline{p}(\eta)d\eta,$$

$$\overline{\overline{w}}(\tau) = K^*(\tau)C^{-1}\beta + \overline{\overline{p}}(\tau) - K^*(\tau)C^{-1}\int\limits_a^b K(\eta)\overline{\overline{p}}(\eta)d\eta$$

be arbitrary elements of the set Q. The function

$$w_\alpha(\tau) = \alpha\overline{w}(\tau) + (1-\alpha)\overline{\overline{w}}(\tau) = K^*(\tau)C^{-1}\beta + p_\alpha(\tau) -$$

$$-K^*(\tau)C^{-1}\int\limits_a^b K(\eta)p_\alpha(\eta)d\eta \in Q, \ \ \forall\alpha, \ \alpha \in [0,1],$$

where $p_\alpha(\tau) = \alpha\overline{p}(\tau) + (1-\alpha)\overline{\overline{p}}(\tau) \in L_2(I_1, R^m)$.

Example 1. Consider the integral equation $K_2 w = \int\limits_0^1 w(\tau)d\tau = \beta$, where $K(\tau) = 1$, $w(\cdot) \in L_2(I_1, R^1)$, $I_1 = [0,1]$. For this example $C = \int\limits_0^1 d\tau = 1 > 0$. Consequently this integral equation has a solution for any $\beta \in R^1$. By formula (5), the general solution is $w(\tau) = \beta + p(\tau) - \int\limits_0^1 p(\eta)d\eta$, $\tau \in I_1$, where $p(\tau) \in L_2(I_1, R^1)$ is an arbitrary function. The particular solution $w_1(\tau) = \beta$, the solution of the homogeneous integral equation $\int\limits_0^1 w_2(\tau)d\tau = 0$ is $w_2(\tau) = p(\tau) - \int\limits_0^1 p(\eta)d\eta$, $\tau \in I_1 = [0,1]$, $\forall p(\cdot)$, $p(\cdot) \in L_2(I_1, R^1)$.

Consider problems 9, 10. The results described above hold true for integral equations with respect to multivariable unknown function. In particular, for the integral equation (3) we have the following theorems.

Theorem 3. *A necessary and sufficient condition for existence a solution of the integral equation* (3) *for any $\beta \in R^n$ is that the $n \times n$ matrix*

$$T(a,b,c,d) = \int\limits_a^b \int\limits_c^d K(t,\tau)K^*(t,\tau)d\tau dt \tag{8}$$

be positive definite.

Theorem 4. *Let the matrix $T(a,b,c,d)$ defined by* (8) *be positive definite. Then for any $\beta \in R^n$*

$$\begin{gathered} w(t,\tau) = v(t,\tau) + K^*(t,\tau)T^{-1}(a,b,c,d)\overline{a} - K^*(t,\tau)T^{-1}(a,b,c,d)\times \\ \times \int\limits_a^b \int\limits_c^d K(\eta,\xi)v(\eta,\xi)d\xi d\eta, \end{gathered} \tag{9}$$

is a general solution of the integral equation (3)*, here $v(t,\tau) \in L_2(G, R^m)$ is an arbitrary function, $\beta \in R^n$ is an arbitrary vector.*

The main properties of the solution. The general solution of the integral equation (3) defined by (9) has the following properties:

1. The function $w(t,\tau) = w_1(t,\tau) + w_2(t,\tau)$, $(t,\tau) \in G$ where $w_1(t,\tau) = K^*(t,\tau)T^{-1}(a,b,c,d)\beta \in L_2(G, R^m)$ is a particular solution to the integral equation (3), and the function $w_2(t,\tau)$ is a solution of the homogeneous integral equation

$$\int\limits_a^b \int\limits_c^d K(t,\tau)w_2(t,\tau)d\tau dt = 0.$$

2. The functions $w_1(t,\tau) \in L_2(G, R^m)$ and $w_2(t,\tau) \in L_2(G, R^m)$ are orthogonal $w_1 \perp w_2$.
3. The function $w_1(t,\tau) \in L_2(G, R^m)$ is a solution with minimal norm for the integral equation (3).
4. The solution set for the integral equation (3) is convex.

3 Solvability of an integral equation with fixed right hand side

The question naturally arises: if the matrix C is not positive definite, has the integral equation (1) a solution? The answer is unambiguous, in this case the integral equation (1) can have solution, but not for any vector $\beta \in R^n$. The condition $C > 0$ is a rigid for the kernel of the integral equation. The analogue of this condition is an existence of the inverse matrix A^{-1} for the linear algebraic equation $Ax = b$, which provides an existence of a solution for any $b \in R^n$. The algebraic equation $Ax = b$ can have solution in the case of non-existence of the inverse matrix too, but not for any vector $b \in R^n$ ($rangA = rang(A, b)$, by the Kronecker-Capelli theorem).

Solutions of problems 3,4. An investigation of the extremal problem is needed in order to solve problems 3, 4:

$$J(w) = |\beta - \int_a^b K(\tau)w(\tau)d\tau|^2 dt \to \inf \tag{10}$$

under the condition

$$w(\cdot) \in L_2(I_1, R^m), \tag{11}$$

where $\beta \in R^n$ is a given vector.

Theorem 5. *Let a kernel of the operator $K(\tau)$ be measurable and belong to the class L_2. Then:*

1) the functional (10) *under the condition* (11) *is continuously Frechet differentiable, the gradient of the functional $J'(w) \in L_2(I_1, R^m)$ for any point $w(\cdot) \in L_2(I_1, R^m)$ is defined by*

$$J'(w) = -2K^*(\tau)\beta + 2\int_a^b K^*(\tau)K(\sigma)w(\sigma)d\sigma, \quad \tau \in I_1; \tag{12}$$

2) the gradient of the functional $J'(w) \in L_2(I_1, R^m)$ satisfies the Lipchitz condition

$$\|J'(w + h) - J'(w)\| \le l\|h\|, \quad \forall w, \ w + h \in L_2(I_1, R^m); \tag{13}$$

3) the functional (10) *under the condition* (11) *is convex, i.e.*

$$J(\alpha w + (1 - \alpha)u) \le \alpha J(w) + (1 - \alpha)J(u), \quad \forall w, u \in L_2(I_1, R^m), \quad \forall \alpha, \ \alpha \in [0, 1]; \tag{14}$$

4) the second Frechet derivative is defined by

$$J''(w) = 2K^*(\sigma)K(\tau), \quad \sigma, \ \tau \in I_1; \tag{15}$$

5) if the inequality

$$\int_a^b\int_a^b \xi^*(\sigma)K^*(\sigma)K(\tau)\xi(\tau)d\tau d\sigma = \left[\int_a^b K(\tau)\xi(\tau)d\tau\right]^2 \geq \geq \mu\int_a^b |\xi(\tau)|^2 d\tau,\ \ \mu>0,\ \ \forall\xi,\ \xi(\tau)\in L_2(I_1,R^m), \tag{16}$$

holds, then the functional (10) *under the condition* (11) *is strongly convex.*

Proof. As it follows from (10), the functional

$$J(w)=\beta^*\beta-2\beta^*\int_a^b K(\sigma)w(\sigma)d\sigma+\int_a^b\int_a^b w^*(\tau)K^*(\tau)K(\sigma)w(\sigma)d\sigma d\tau.$$

Then the increment of the functional ($w,\ w+h\in L_2(I_1,R^m)$)

$$\Delta J = J(w+h)-J(w)=\int_a^b <-2K(\sigma)\beta+2\int_a^b K^*(\sigma)K(\tau)w(\tau)d\tau, h(\sigma)>d\sigma+ +\int_a^b\int_a^b h^*(\tau)K^*(\tau)K(\sigma)h(\sigma)d\sigma d\tau =<J'(w),h>_{L_2}+o(h), \tag{17}$$

where

$$|o(h)|=|\int_a^b\int_a^b h^*(\tau)K^*(\tau)K(\sigma)h(\sigma)d\sigma d\tau \leq c_1\|h\|^2_{L_2}.$$

It follows from (17) that $J'(w)$ is defined by (12). As

$$J'(w+h)-J'(w)=2K^*(\tau)\int_a^b K(\sigma)h(\sigma)d\sigma,$$

we have

$$|J'(w+h)-J'(w)|\leq 2\|K^*(\tau)\|\int_a^b \|K(\sigma)\|\ |h(\sigma)|d\sigma \leq$$
$$\leq c_2(\tau)\|h\|_{L_2},\ \ \tau\in I_1.$$

Hence

$$\|J'(w+h)-J'(w)\|_{L_2}=\left(\int_a^b |J'(w+h)-J'(w)|^2d\tau\right)^{1/2}\leq l\|h\|,$$

for any $w,\ w+h\in L_2(I_1,R^m)$. This implies the inequality (13).

Show that the functional (10) is convex. Since the functional $J(w){\in}C^{1,1}(L_2(I_1,R^m))$, for the functional (10) to be convex it is necessary and sufficient to have

$$< J'(w_1)-J'(w_2), w_1-w_2 >_{L_2}=< 2\int\limits_a^b K^*(\tau)K(\sigma)[w_1(\sigma)-w_2(\sigma)]d\sigma,$$

$$w_1(\tau)-w_2(\tau) >_{L_2}= 2\int\limits_a^b\int\limits_a^b [w_1(\tau)-w_2(\tau)]^* K^*(\tau)K(\sigma)[w_1(\sigma)-w_2(\sigma)]d\sigma d\tau \geq 0.$$

This means that the functional (10) is convex, i.e. the inequality (14) holds. As it follows from (12), the increment

$$J'(w+h)-J'(w) =< J''(w), h >=< 2K^*(\sigma)K(\tau), h(\sigma) >_{L_2}=$$

$$= 2\int\limits_a^b K^*(\tau)K(\sigma)h(\sigma)d\sigma.$$

Consequently $J''(w)$ is defined by (15). It follows from (15), (16) that

$$< J''(w)\xi, \xi >_{L_2} \geq \mu\|\xi\|^2, \;\; \forall w, \; w \in L_2(I_1,R^m), \;\; \forall \xi, \; \xi \in L_2(I_1,R^m).$$

This means that the functional $J(w)$ is strongly convex in $L_2(I_1,R^m)$. The theorem is proved.

Theorem 6. *Let the sequence* $\{w_n(\tau)\} \in L_2(I_1,R^m)$ *be constructed for extremal problem* (10), (11) *by the rule*

$$\begin{aligned} &w_{n+1}(\tau) = w_n(\tau) - \alpha_n J'(w_n), \;\; g_n(\alpha_n) = \min_{\alpha>0} g_n(\alpha), \\ &\quad g_n(\alpha) = J(w_n - \alpha J'(w_n)), \;\; n = 0,1,2,\ldots. \end{aligned} \tag{18}$$

Then the numerical sequence $\{J(w_n)\}$ *decreases monotonically, the limit* $\lim\limits_{n\to\infty} J'(w_n) = 0$.

If besides the set $M(w_0) = \{w(\tau) \in L_2(I_1,R^m)/J(w) \leq J(w_0)\}$ *is bounded, then:*

1) the sequence $\{w_n(\tau)\}$ *is minimizing, i.e.*

$$\lim_{n\to\infty} J(w_n) = J_* = \inf J(w), \;\; w(\cdot) \in L_2(I,R^m), \;\; w_n \overset{weakly}{\to} w_* \;\; as \;\; n \to \infty,$$

where $w_* = w_*(\tau) \in W_*$,

$$W_* = \{w_*(\tau) \in L_2(I_1,R^m)/J(w_*) = \min_{w\in M(w_0)} J(w) = J_* = \inf_{w\in L_2(I_1,R^m)} J(w)\};$$

2) the following convergence rate estimation holds

$$0 \leq J(w_n) - J(w_*) \leq \frac{m_0}{n}, \;\; m_0 = const > 0, \;\; n = 1,2,\ldots \tag{19}$$

3) there exists a solution to the integral equation (1) *iff* $J(w_*) = 0$, $w_* \in W_*$. *In this case* $w_* \in W_*$ *is a solution of the integral equation* (1).

4) if $J(w_*) > 0$, *then the integral equation* (1) *hasn't a solution.*

5) if the inequality (16) *holds, then* $\|w_n - w_*\| \to 0$ *as* $n \to \infty$.

Proof. Minimization methods in Hilbert space [13] can be applied to a proof of the theorem. The conditions $g_n(\alpha_n) \le g_n(\alpha)$, $J(w) \in C^{1,1}(L_2(I_1, R^m))$ imply that

$$J(w_n) - J(w_n - \alpha J'(w_n)) \ge \alpha(1 - \frac{\alpha l}{2})\|J'(w_n)\|^2, \quad \alpha \ge 0, \quad n - 0, 1, 2, \dots,$$

where $l = const > 0$ is the Lipchitz constant from (13). Then

$$J(w_n) - J(w_{n+1}) \ge \frac{1}{2l}\|J'(w_n)\|^2 > 0.$$

This yields that $\lim\limits_{n\to\infty} J'(u_n) = 0$ and the numerical sequence $\{J(u_n)\}$ decreases monotonically. The first statement of the theorem is proved.

As the functional $J(w)$ is convex the set $M(w_0)$ is convex. Then

$$0 \le J(w_n) - J(w_*) \le < J'(w_n), w_n - w_* >_{L_2} \le \|J'(w_n)\| \, \|w_n - w_*\| \le D\|J'(w_n)\|,$$

here D is a diameter of set $M(w_0)$. Since $M(w_0)$ is weakly bicompact, the functional $J(w)$ is weakly lower semicontinuous, it follows that the set $W_* \ne \varnothing$, $W_* \subset M(w_0)$ and $\{w_n\} \subset M(w_0)$, $w_* \in M(w_0)$. Note that

$$0 \le \lim_{n\to\infty} J(w_n) - J(w_*) \le D \lim_{n\to\infty} \|J'(w_n)\| = 0, \quad \lim_{n\to\infty} J(w_n) = J(w_*) = J_*.$$

Consequently the sequence $\{w_n\} \subset M(w_0)$ is minimizing. Estimation (19), where $m_0 = 2D^2 l$, follows from the inequalities

$$J(w_n) - J(w_{n+1}) \le \frac{1}{2l}\|J'(w_n)\|^2, \quad 0 \le J(w_n) - J(w_*) \le D\|J'(w_n)\|,$$

$$w_n \overset{\text{weakly}}{\to} w_* \quad \text{as} \quad n \to \infty.$$

The second statement of the theorem is proved.

It follows from (10) that $J(w) \ge 0$, $\forall w$, $w \in L_2(I_1, R^m)$. The sequence $\{w_n\} \subset L_2(I_1, R^m)$ is minimizing for any initial guess $w_0 = w_0(\tau) \in L_2(I_1, R^m)$, i.e. $J(w_*) = \min\limits_{w \in L_2(I_1, R^m)} J(w) = J_* = \inf\limits_{w \in L_2(I_1, R^m)} J(w)$. If $J(w_*) = 0$, then

$$\beta = \int_a^b K(\tau) w_*(\tau) d\tau.$$

Therefore the integral equation (1) has solution if and only if $J(w_*) = 0$, where $w_* = w_*(\tau) \in L_2(I_1, R^m)$ is a solution to the integral equation (1). If $J(w_*) > 0$, then $w_* = w_*(\tau)$, $\tau \in I_1$ is not a solution of the integral equation (1). In other words, whenever $J(w_*) > 0$, the integral equation (1) hasn't a solution for the given $\beta \in R^n$. Thus the statements 3, 4 are proved.

If the inequality (16) holds, then the functional (10) under the condition (11) is strongly convex. Whence

$$J(w_n) - J(w_*) \leq < J'(w_n), w_n - w_* > -\frac{\mu}{2}\|w_n - w_*\|^2 \leq 2\mu\|J'(w_n)\|^2, \quad n = 0, 1, 2, \ldots,$$

$$J(w_n) - J(w_{n+1}) \geq \frac{1}{2l}\|J'(w_n)\|^2, \quad n = 0, 1, 2, \ldots.$$

Hence $a_n - a_{n+1} \geq \frac{\mu}{l} a_n$, where $a_n = J(w_n) - J(w_*)$. Consequently $0 \leq a_{n+1} \leq a_n(1 - \frac{\mu}{l}) = qa_n$. Then $a_n \leq qa_{n-1} \leq q^2 a_{n-2} \leq \ldots \leq q^n a_0$. This implies

$$0 \leq J(w_n) - J(w_*) \leq [J(w_0) - J(w_*)]q^n, \quad q = 1 - \frac{\mu}{l}, \quad 0 \leq q \leq 1, \quad \mu > 0.$$

It can be shown that the estimation

$$\|w_n - w_*\| \leq \left(\frac{2}{\mu}\right)[J(w_0) - J(w_*)]q^n, \quad n = 0, 1, 2, \ldots$$

holds for any strongly convex functional. Then $\|w_n - w_*\| \to 0$ as $n \to \infty$. Theorem is proved.

Consider problems 5, 6. In particular, the set $W(\tau)$ is defined by: either

$$W(\tau) = \{w(\cdot) \in L_2(I_1, R^m)/\alpha_i(\tau) \leq w_i(\tau) \leq \beta_i(\tau), \ i = \overline{1, m}, \ \text{a.e.} \ \tau \in I_1\},$$

or

$$W(\tau) = \{w(\cdot) \in L_2(I_1, R^m)/\|w\|^2 \leq R^2\}.$$

where $\alpha(\tau) = (\alpha_1(\tau), \ldots, \alpha_m(\tau))$, $\beta(\tau) = (\beta_1(\tau), \ldots, \beta_m(\tau))$, $\tau \in I_1$, are given continuous functions, $R > 0$ is a given number.

Solving problems 5, 6 is reduced to investigation the extremal problem:

$$J_1(w, u) = |\beta - \int_a^b K(\tau)w(\tau)d\tau| + \|w - u\|^2_{L_2} \to \inf \tag{20}$$

under the conditions

$$w(\cdot) \in L_2(I_1, R^m), \quad u(\tau) \in W(\tau), \quad \tau \in I_1. \tag{21}$$

Theorem 7. *Let a kernel of the operator $K(\tau)$ be mesurable and belong to L_2. Then:*

1) the functional (20) *under the condition* (21) *is continuously Frechet differentiable, the gradient*

$$J_1'(w, u) = (J_{1w}'(w, u), J_{1u}'(w, u)) \in L_2(I_1, R^m) \times L_2(I_1, R^m)$$

for any point $(w, u) \in L_2(I_1, R^m) \times W(\tau)$ is defined by

$$J_{1w}'(w, u) = -2K^*(\tau)\beta + 2\int_a^b K^*(\tau)K(\sigma)w(\sigma)d\sigma + 2(w - u) \in L_2(I_1, R^m), \tag{22}$$

$$J'_{1u}(w,u) = -2(w-u) \in L_2(I_1, R^m); \tag{23}$$

2) the gradient of the functional $J'_1(w,u)$ satisfies the Lipchitz condition

$$\begin{array}{c} \|J'_1(w+h,u+h_1) - J'_1(w,u)\| \le l_1(\|h\| + \|h_1\|), \\ \forall (w,u), \ \ (w+h,u+h_1) \in L_2(I_1,R^m) \times L_2(I_1,R^m); d\eta, \end{array} \tag{24}$$

3) the functional (20) *under the condition* (21) *is convex.*

A proof the theorem is similar to theorem 5's proof.

Theorem 8. *Let for extremal problem* (20), (21) *the sequences be constructed by*

$$w_{n+1}(\tau) = w_n(\tau) - \alpha_n J'_{1w}(w_n,u_n), \ \ n = 0,1,2,\dots,$$

$$u_{n+1}(\tau) = P_W[u_n(\tau) - \alpha_n J'_{1u}(w_n,u_n)], \ \ n = 0,1,2,\dots.$$

where $P_W[\cdot]$ is a projection of a point onto the set W,

$$\varepsilon_0 \le \alpha \le \frac{2}{l_2 + 2\varepsilon_1}, \ \ \varepsilon_0 > 0, \ \ \varepsilon_1 > 0, \ \ n = 0,1,2,\dots,$$

l_1 is the Lipchitz constant from (24)*, in the case $\varepsilon_1 = \frac{l_1}{2}$, $\varepsilon_0 = \alpha_n = \frac{1}{l_1}$, $J'_{1w}(w_n,u_n)$, $J'_{1u}(w_n,u_n)$ are defined by* (22), (23) *respectively. Then the numerical sequence $\{J_1(w_{n1},u_n)\}$ is monotone decreasing, the limits $\lim\limits_{n\to\infty}\|w_n - w_{n+1}\| = 0$, $\lim\limits_{n\to\infty}\|u_n - u_{n+1}\| = 0$.*

If in addition the set $M(w_0,u_0) = \{(w,u) \in L_2 \times W / J_1(w,u) \le J(w_0,u_0)\}$ is bounded, then:

1) the sequence $\{w_n,u_n\} \subset M(w_0,u_0)$ is minimizing, i.e.

$$\lim_{n\to\infty} J_1(w_n,u_n) = J_* = \inf J(w,u), \ \ (w,u) \in L_2 \times W;$$

2) the sequence $\{w_n,u_n\} \subset M(w_0,u_0)$ weakly converges to the set

$$X_* = \{(w_*,u_*) \in L_2 \times W / J_1(w_*,u_*) = \min J_1(w,u) = J_* = \inf J_1(w,u), \ \ (w,u) \in L_2 \times W\};$$

3) a necessary and sufficient condition for integral equation (1) *under the condition $w(\tau) \in W$ to have a solution is that $J_1(w_*,u_*) = J_{1*} = 0$.*

A proof of theorem is similar to the proof of theorem 6.

4 Integral equation with parameter

Consider problems 7, 8 for the integral equation (2).

Theorem 9. *A necessary and sufficient condition for the integral equation* (2) *to have solution for any $\mu(t) \in L_2(I,R^n)$ is that the $n \times n$ matrix*

$$C(t) = \int_a^b K(t,\tau)K^*(t,\tau)d\tau \ \ t \in I \tag{25}$$

be positive definite for all $t \in I$, where $(*)$ means transposed.

A proof of theorem is similar to the proof of theorem 1.

Theorem 10. *Let the matrix $C(t)$, $\forall t$, $t \in I$ given by (25) be positive definite. Then a general solution of integral equation (2) for any $\mu(t) \in L_2(I, R^n)$ is given by*

$$v(t,\tau) = K^*(t,\tau)C^{-1}(t)\mu(t)+\gamma(t,\tau)-K^*(t,\tau)C^{-1}(t)\int\limits_a^b K(t,\tau)\gamma(t,\tau)d\tau, \ \ t \in I, \ \tau \in I_1, \quad (26)$$

where $\gamma(t,\tau) \in L_2(S_1, R^m)$ is an arbitrary function, $\mu(t) \in L_2(I, R^n)$.

The proof of the theorem is similar to the proof of theorem 2.

The main properties of solutions to the integral equation (2):

1. The function $v(t,\tau)$ from (26) can be represented in the form $v(t,\tau) = v_1(t,\tau)+v_2(t,\tau)$, where $v_1(t,\tau) = K^*(t,\tau)C^{-1}(t)\mu(t)$, $v_2(t,\tau) = \gamma(t,\tau) - K^*(t,\tau)C^{-1}(t)\int\limits_{t_0}^{t_1} K(t,\tau)\gamma(t,\tau)d\tau$, $\gamma(t,\tau) \in L_2(S_1, R^m)$ is an arbitrary function. The function $v_1(t,\tau)$ is a particular solution of the integral equation (3), and the function $v_2(t,\tau)$ is a solution to the homogeneous integral equation

$$\int\limits_a^b K(t,\tau)v_2(t,\tau)d\tau = 0.$$

Indeed,

$$\int\limits_a^b K(t,\tau)v_1(t,\tau)d\tau = \int\limits_a^b K(t,\tau)K^*(t,\tau)d\tau c^{-1}(t)\mu(t) = \mu(t), \ \ t \in I,$$

$$\int\limits_a^b K(t,\tau)v_2(t,\tau)d\tau = \int\limits_a^b K(t,\tau)\gamma(t,\tau)d\tau - \int\limits_a^b K(t,\tau)K^{t,\tau}d\tau C^{-1}(t)\int\limits_a^b K(t,\tau)\gamma(t,\tau)d\tau = 0;$$

2. The functions $v_1(t,\tau) \in L_2(S_1, R^m)$, $v_2(t,\tau) \in L_2(S_1, R^m)$ are orthogonal, i.e. $v_1 \perp v_2$. Indeed,

$$< v_1, v_2 >_{L_2} = \int\limits_a^b v_1^*(t,\tau)v_2(t,\tau)d\tau = \int\limits_a^b \mu^*(t)C^{-1}(t)K(t,\tau)[\gamma(t,\tau)-$$

$$-K^*(t,\tau)C^{-1}(\tau)\int\limits_a^b K(t,\tau)\gamma(t,\tau)\tau]d\tau = \mu^*(t)C^{-1}(t)\int\limits_a^b K(t,\tau)\gamma(t,\tau)d\tau-$$

$$-\mu^*(t)C^{-1}(t)\int\limits_a^b K(t,\tau)K^*(t,\tau)d\tau C^{-1}(t)\int\limits_a^b K(t,\tau)\gamma(t,\tau)d\tau \equiv 0, \ \ (t,\tau) \in S_1;$$

3. The function $v_1(t,\tau) = K^*(t,\tau)C^{-1}(t)\mu(t)$, $(t,\tau) \in S_1$ is a solution of the integral equation (3) with minimal norm in $L_2(S_1, R^m)$. Indeed, $\|v(t,\tau)\|^2 = \|v_1(t,\tau)\|^2 + \|v_2(t,\tau)\|^2$.

This implies that $\|v(t,\tau)\|^2 \geq \|v_1(t,\tau)\|^2$. If the function $\gamma(t,\tau) \equiv 0$, $(t,\tau) \in S_1$, then the function $v_2(t,\tau) \equiv 0$, $(t,\tau) \in S_1$. Hence $v(t,\tau) = v_1(t,\tau)$, $\|v\| = \|v_1\|$;

4. A solution set for the integral equation (3) is convex.

Example 2. The integral equation

$$K_1 v = \int_a^b e^{t\tau} v(t,\tau) d\tau = \sin t, \quad t \in [1;2], \quad \tau \in [0,1],$$

is given. For this example $K(t,\tau) = e^{t\tau}$. Then

$$C(t) = \int_0^1 e^{2t\tau} d\tau = \frac{1}{2t}[e^{2t} - 1] > 0, \ \forall t, \ t \in [1,1].$$

This yields that this integral equation has solution

$$v(t,\tau) = \frac{2t}{e^{2t}-1} e^{t\tau} \sin t + \gamma(t,\tau) - \frac{2t}{e^{2t}-1} e^{t\tau} \int_0^1 e^{t\tau} \gamma(t,\tau) d\tau,$$

where $C^{-1}(t) = \dfrac{2t}{e^{2t}-1}$, $t \in [1,2]$, $\gamma(t,\tau) \in L_2(S_1, R^1)$, $S_1 = \{(t,\tau) \ / \ 1 \leq t \leq 2, \ 0 \leq \tau \leq 1\}$ is an arbitrary function.

The function $v(t,\tau) = v_1(t,\tau) + v_2(t,\tau)$, where $v_1(t,\tau) = \dfrac{2t}{e^{2t}-1} e^{t\tau} \sin t$, $(t,\tau) \in S_1$ is a particular solution, $v_2(t,\tau) = \gamma(t,\tau) - \dfrac{2t}{e^{2t}-1} e^{t\tau} \int_0^1 e^{t\tau} \gamma(t,\tau) d\tau$ is a solution of the homogeneous integral equation $\int_0^1 e^{t\tau} v_2(t,\tau) d\tau = 0$. It's easily shown that $< v_1, v_2 >_{L_2} = 0$, $\forall t$, $t \in [1,2]$.

This work was partially supported by the grant of the Committee of Science of the Ministry of Education and Science of the Republic of Kazakhstan (project 3311/GF4 MON RK)

5 Conclusion

A necessary and sufficient condition for existence of a solution to an integral equation with an arbitrary right-hand side has been obtained, and a general solution to the equation has been constructed. A solvability criterion in the form of requirement for an infimum of a specified functional has been formulated and proved, and a method for the solution construction has been developed. An integral equation with constrained unknown function is reduced to an extremal problem which allows to construct a solution satisfying a given constraint. A test method for existence of a solution to an integral equation with a parameter and a method for solution construction are described in detail and their correctness has been proved.

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