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On the stability of a difference scheme for the three-phase non-isothermal flow problem

The article presents a study of stability of a finite difference scheme in terms of initial values and right-hand sides of the equations for the problem of three-phase non-isothermal flow in homogeneous isotropic porous media without capillary, gravitational forces and phase transitions. It is assumed that oil is homogeneous non-evaporable fluid, and phases are in local thermal equilibrium which means that fluids saturating the porous media and the rock have the same temperature in any elementary volume. The model describing this process consists of the mass conservation equation, equation of motion in the form of linear Darcy's law, equation of state, and phase balance equation. In the present work, so-called "global" formulation of the problem is used which is based on the introduction of a change of variables for pressure, called "global pressure". Using this approach, the original model equations reduce to a system of five partial differential equations with respect to pressure, temperature, velocity, and two saturations. The stability analysis of the scheme is carried out using the method of a priori estimates. A priori estimate for the solution of the difference problem is obtained with limitations on the value of the time step and the norm of the temperature derivative.
Key words: three-phase non-isothermal flow, finite difference method, stability, non-linear term, a priori estimate.

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Исследование устойчивости разностной схемы для задачи трехфазной неизотермической фильтрации

В работе проводится исследование устойчивости конечно-разностной схемы по начальным данным и правой части уравнений системы для задачи трехфазной неизотермической фильтрации в однородной изотропной среде без учета капиллярных, гравитационных сил и фазовых переходов. Предполагается, что нефть - однородная неиспаряемая жидкость и фазы находятся в локальном тепловом равновесии, при котором флюиды, насыщающие пористую среду, и порода имеют одинаковую температуру в любом элементарном объеме. Модель, описывающая данный процесс, включает в себя уравнение неразрывности, уравнение движения в виде линейного закона Дарси, уравнение состояния и уравнение баланса насыщенностей фаз. В работе используется так называемая «глобальная» постановка рассматриваемой задачи, в основу которой положена замена переменных для давления, названная «глобальным давлением». Используя данный подход, исходные уравнения модели сводятся к системе из пяти дифференциальных уравнений в частных производных относительно давления, температуры, скорости и двух насыщенностей. Анализ устойчивости схемы проводится методом априорных оценок. Получена априорная оценка для решения разностной задачи с ограничениями на величину временного шага и норму производной температуры.
Ключевые слова: трехфазный неизотермический поток, метод конечных разностей, устойчивость, нелинейное слагаемое, априорная оценка.

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Үш фазалы изотермалық емес фильтрация есебі үшін айырымдық сұлбаның орнықтылығын зерттеу

Бұл жұмыста біртекті изотропты ортада капиллярлық, гравитациялық күштерді және фазалық айналымдарды ескермейтін үш фазалы изотермалық емес фильтрация есебі үшін ақырлы айырымдық сұлбаның бастапқы шарттары мен жүйе теңдеулерінің оң жақтары бойынша орнықтылығы зерттеледі. Бұл жұмыста мұнай - біртекті булануға ұшырамайтын сұйықтық және фазалар локалды жылу тепе-теңдікте, яғни ортаның әрбір элементарлы аумағында мұнай қатпары және оны қанықтыратын флюидтердің температурасы бірдей болатыны ұйғарылады. Бұл үрдісті сипаттайтын модель үздіксіздік теңдеуі, сызықты Дарси заңы түріндегі қозғалыс теңдеуі, күй теңдеуі мен фазалардың қанықтық балансы теңдеулерінен тұрады. Бұл жұмыста қарастырылып отырған есептің "глобалды" қойылымы пайдаланады, ол "глобалды қысым" деп аталатын айнымалыны ауыстыруды енгізуге негізделген. Бұл тәсілді пайдаланып, модельдің бастапқы теңдеулері қысым, температура, жылдамдық және екі қанықтық үшін бес дербес туындылы дифференциалдық теңдеуден тұратын жүйеге келтіріледі. Ақырлы айырымдық сұлба орнықтылығының анализі априорлы бағалау әдісімен жүргізіледі. Ақырлы айырымдық есептің шешімі үшін уақыт қадамы мен температура туындысының нормасына қойылған шектеулермен априорлы бағалау алынды.
Түйін сөздер: үш фазалы изотермалық емес ағын, ақырлы айырымдар әдісі, орнықтылық, сызықты емес қосылғыш, априорлы бағалау.

1 Introduction

The models describing non-isothermal multi-phase flow in porous media have received significant attention in applied mathematics recently. For example, different approaches to the numerical solution of the multi-phase non-isothermal flow problems have been studied in [1, 2, 3, 4, 5, 6, 7]. In [7], a new "global" formulation of the three-phase non-isothermal flow problem is proposed, which is based on the introduction of a change of variables for the pressure, called "global" pressure, to eliminate the gradients of capillary pressures from the equations for pressure and temperature. This approach was initially proposed in [8, 9] for two-phase and three-phase isothermal flows.

In this paper, we consider a three-phase non-isothermal flow of immiscible fluids in homogeneous, isotropic porous media without capillary and gravitational forces. The system of equations describing three-phase non-isothermal flow is considered under the assumptions that the movement of phases obeys a linear generalized Darcy law, the phases are in the local thermal equilibrium, which means that the fluids saturating the porous media and the rock have the same temperature in any elementary volume. Furthermore, oil is assumed to be a homogeneous non-evaporable fluid and the oil reservoir consists of a single rock. In this case, the model describing this process is usually consists of the mass conservation equation, Darcy law, energy equation, equation of state, and an equation for balance of saturations. Using the approach, proposed in [7], the original equations of the non-isothermal model are reduced to a system of partial differential equations with respect to pressure, temperature and two saturations. We study a finite difference scheme for the considered problem. A priori estimate, showing the stability of the difference scheme with respect to the initial data and right-hand sides of equations, is obtained.

2 Formulation of the problem

We consider the problem of three-phase non-isothermal flow in the domain $\overline{Q}_T = \overline{\Omega} \times [0,\, t_1]$, where $\overline{\Omega} = [0,\, l] \times [0,\, l]$ is a square with boundary Γ [7]:

$$c_T \frac{\partial T}{\partial t} + \vec{u} \cdot \nabla T - \nabla \cdot (k_h \nabla T) = f_T, \tag{1}$$

$$\frac{\partial p}{\partial t} - \nabla \cdot (k_p \nabla p) = \beta_T \frac{\partial T}{\partial t} + f_p, \tag{2}$$

$$\frac{\partial s_\alpha}{\partial t} - \nabla \cdot (\nu_\alpha \nabla p) = f_\alpha, \;\; \alpha = w,\, o, \tag{3}$$

$$\vec{u} = -\lambda \nabla p \tag{4}$$

with initial and boundary conditions

$$T\,(x, 0) = T_0, \;\; p\,(x, 0) = p_0, \;\;\; s_\alpha\,(x, 0) = s_{\alpha 0}, \tag{5}$$

$$-k_h \frac{\partial T}{\partial \vec{n}}\Big|_\Gamma = 0, \;\; -k_p \frac{\partial p}{\partial \vec{n}}\Big|_\Gamma = 0 \tag{6}$$

where subscripts w, o, g denote the phases of water, oil and steam; T is temperature; p is pressure; s_α is the saturation of the phase α; c_T, λ, k_h, k_p, β_T, ν_α are some functions of the space variable and time for which the following conditions hold:

$$c_0 \le (c_T,\, \lambda,\, k_h,\, k_p,\, \beta_T,\, \nu_\alpha) \le c_1. \tag{7}$$

To solve the problem, we use the finite difference method. Let us introduce a uniform grid $\overline{\omega}_{h\tau}$ with the steps in the spatial variable $h_1 = h_2 = h$ and time τ in $\overline{Q}_T$:

$$\overline{\omega}_{h\tau} = \overline{\omega} \times \overline{\omega}_\tau = \left\{(x,\, t): \;\; x \in \overline{\omega}, \;\; t \in \overline{\omega}_\tau\right\},$$

$$\overline{\omega} = \overline{\omega}_1 \times \overline{\omega}_2, \;\; \overline{\omega}_m = \left\{x_m^{i_m} = i_m h: \;\; i_m = 0,\, 1, ...,\, N_m, \;\; N_m h = l\right\},$$

$$\overline{\omega}_\tau = \left\{t_n = n\tau: \;\; n = 0,\, 1, ...,\, N_t, \;\; N_t \tau = t_1\right\}.$$

We associate the following difference scheme with the differential problem (1)-(6) in $\overline{\omega}_{h\tau}$:

$$c_T T_t + \Lambda_1 \hat{T} = f_T, \tag{8}$$

$$p_t + \Lambda_2 \hat{p} = \beta_T T_t + f_p, \tag{9}$$

$$s_{\alpha,t} + \Lambda_{3\alpha} \hat{p} = f_\alpha, \;\;\; \alpha = w,\, o \tag{10}$$

with the initial conditions

$$T_i^0 = T_0, \;\; p_i^0 = p_0, \;\; s_{\alpha,i}^0 = s_{\alpha 0} \tag{11}$$

where $\Lambda_1 = \Lambda_1^{(1)} + \Lambda_1^{(2)}, \;\; \Lambda_2 = \Lambda_2^{(1)} + \Lambda_2^{(2)}, \;\; \Lambda_{3\alpha} = \Lambda_{3\alpha}^{(1)} + \Lambda_{3\alpha}^{(2)}$,

$$\Lambda_1^{(m)} w = \begin{cases} -\frac{1}{2}\left(\lambda \check{p}_{\overline{x}_m}\right)^{(+1_m)} w_{x_m} - \frac{2}{h} \check{k}_h^{(+1_m)} w_{x_m}, & x_m = 0, \\ -\frac{1}{2}\left(\lambda \check{p}_{\overline{x}_m}\right)^{(+1_m)} w_{x_m} - \frac{1}{2}\left(\lambda \check{p}_{\overline{x}_m}\right) w_{\overline{x}_m} - \left(\check{k}_h w_{\overline{x}_m}\right)_{x_m}, & x_m \in \omega_m, \\ -\frac{1}{2}\left(\lambda \check{p}_{\overline{x}_m}\right) w_{\overline{x}_m} + \frac{2}{h} \check{k}_h w_{\overline{x}_m}, & x_m = l, \end{cases}$$

$$\Lambda_2^{(m)} w = \begin{cases} -\frac{2}{h}\breve{k}_p^{(+1_m)} w_{x_m}, & x_m = 0, \\ -\left(\breve{k}_p w_{\overline{x}_m}\right)_{x_m}, & x_m \in \omega_m, \\ \frac{2}{h}\breve{k}_p w_{\overline{x}_m}, & x_m = l, \end{cases} \qquad \Lambda_{3\alpha}^{(m)} w = \begin{cases} -\frac{2}{h}\breve{\nu}_\alpha^{(+1_m)} w_{x_m}, & x_m = 0, \\ -\left(\breve{\nu}_\alpha w_{\overline{x}_m}\right)_{x_m}, & x_m \in \omega_m, \\ \frac{2}{h}\breve{\nu}_\alpha w_{\overline{x}_m}, & x_m = l, \end{cases}$$

and ω_m is a set of internal nodes of the grid $\overline{\omega}_m$. Here and below we use the notation accepted in [10]. Let us define norms and a dot product as follows:

$$\|w\|_0^2 = (w,\, w)_{\overline{\omega}}\,, \quad \|\nabla w\|_0^2 = \|w_{\overline{x}_1}\|_0^2 + \|w_{\overline{x}_2}\|_0^2\,, \quad \|w\|_1^2 = \|\nabla w\|_0^2 + \|w\|_0^2\,,$$

$$\|w_{\overline{x}_1}\|_0^2 = \sum_{\omega_1^+}\sum_{\overline{\omega}_2} w_{\overline{x}_1}^2\,(x)\,h^2, \quad \|w_{\overline{x}_2}\|_0^2 = \sum_{\overline{\omega}_1}\sum_{\omega_2^+} w_{\overline{x}_2}^2\,(x)\,h^2,$$

$$\|\varphi\|_{-1} = \sup_{w\neq 0}\frac{|(\varphi,\, w)_{\overline{\omega}}|}{\|w\|_1}, \quad \|w\|_C = \max_{x\in\overline{\omega}}|w\,(x)|\,, \quad (w,\, \tilde{w})_{\overline{\omega}} = \sum_{\overline{\omega}_1}\sum_{\overline{\omega}_2} w\,(x)\,\tilde{w}\,(x)\,h^2,$$

$$\omega_m^+ = \left\{x_m^{i_m} = i_m h,\ \ i_m = 1, ..., N_m,\ \ N_m h = l\right\}.$$

For simplicity, we also assume that the following conditions hold:

$$(f_T,\, 1)_{\overline{\omega}} = (f_p,\, 1)_{\overline{\omega}} = 0,\ \ (p,\, 1)_{\overline{\omega}} = \tilde{p},\ \ (T,\, 1)_{\overline{\omega}} = \tilde{T}. \tag{12}$$

3 Investigation of the stability of the difference scheme

Lemma 1. The following estimate is valid:

$$(\Lambda_2 v,\, v)_{\overline{\omega}} \geq \frac{2c_0}{3l^2}\left(l^2\,\|v\|_1^2 - (v,\, 1)_{\overline{\omega}}^2\right). \tag{13}$$

Proof. Using the difference analogue of Green's formula and conditions (7), we obtain:

$$(\Lambda_2 v,\, v)_{\overline{\omega}} = \left(\breve{k}_p v_{\overline{x}_1},\, v_{\overline{x}_1}\right)_{\omega_1^+\times\overline{\omega}_2} + \left(\breve{k}_p v_{\overline{x}_2},\, v_{\overline{x}_2}\right)_{\overline{\omega}_1\times\omega_2^+} \geq c_0\,\|\nabla v\|_0^2\,. \tag{14}$$

Further, using the Poincare inequality and the definition of the norm $\|v\|_1$, we obtain:

$$\|\nabla v\|_0^2 \geq 2\left(\|v\|_1^2 - \|\nabla v\|_0^2\right) - \frac{2}{l^2}\,(v,\, 1)_{\overline{\omega}}^2\,,$$

which implies

$$\|\nabla v\|_0^2 \geq \frac{2}{3}\,\|v\|_1^2 - \frac{2}{3l^2}\,(v,\, 1)_{\overline{\omega}}^2\,. \tag{15}$$

Substitution of (15) into (14) gives the lemma.

Lemma 2. The following estimate holds:

$$(\Lambda_{3\alpha} w,\, \tilde{w}) \leq \frac{1}{2}c_1 M\varepsilon\,\|\nabla w\|_0^2 + \frac{c_1}{2\varepsilon}\,\|\nabla\tilde{w}\|_0^2\,, \quad \varepsilon > 0.$$

Proof. Indeed,

$$(\Lambda_{3\alpha} w,\, \tilde{w}) = (\nu_\alpha w_{\overline{x}_1},\, \tilde{w}_{\overline{x}_1})_{\omega_1^+\times\overline{\omega}_2} + (\nu_\alpha w_{\overline{x}_2},\, \tilde{w}_{\overline{x}_2})_{\overline{\omega}_1\times\omega_2^+} \leq$$

$$\leq c_1 \left(\frac{\varepsilon'}{2} \|w_{\overline{x}_1}\|_0^2 + \frac{1}{2\varepsilon'} \|\tilde{w}_{\overline{x}_1}\|_0^2 + \frac{\varepsilon''}{2} \|w_{\overline{x}_2}\|_0^2 + \frac{1}{2\varepsilon''} \|\tilde{w}_{\overline{x}_2}\|_0^2 \right) \leq \frac{c_1 M \varepsilon}{2} \|\nabla w\|_0^2 + \frac{c_1}{2\varepsilon} \|\nabla \tilde{w}\|_0^2 .$$

We now obtain an a priori estimate for the solution of the difference problem (8)-(11). Let us multiply the equation (9) by $2\tau\hat{p}$:

$$\|\hat{p}\|_0^2 - \|p\|_0^2 + \tau^2 \|p_t\|_0^2 + 2\tau \left(\Lambda_2 \hat{p},\, \hat{p}\right)_{\overline{\omega}} = \left(\beta_T T_t,\, 2\tau\hat{p}\right)_{\overline{\omega}} + \left(f_p,\, 2\tau\hat{p}\right)_{\overline{\omega}} \tag{16}$$

and evaluate the inner products using Lemma 1, Cauchy inequality, and conditions (7):

$$2\tau \left(\Lambda_2 \hat{p},\, \hat{p}\right)_{\overline{\omega}} \geq \frac{4\tau c_0}{3l^2} \left(l^2 \|\hat{p}\|_1^2 - (\hat{p},\, 1)_{\overline{\omega}}^2 \right),$$

$$\left|\left(\beta_T T_t,\, 2\tau\hat{p}\right)_{\overline{\omega}}\right| \leq 2\tau c_1 \|T_t\|_0 \|\hat{p}\|_0 \leq \frac{\tau}{\varepsilon_1} \|T_t\|_0^2 + M_1 \tau c_1^2 \varepsilon_1 \|\hat{p}\|_1^2 ,$$

$$\left|\left(f_p,\, 2\tau\hat{p}\right)_{\overline{\omega}}\right| \leq 2\tau \|f_p\|_{-1} \|\hat{p}\|_0 \leq \frac{\tau}{\varepsilon_2} \|f_p\|_{-1}^2 + M_1 \tau \varepsilon_2 \|\hat{p}\|_1^2 .$$

Here and below M_i denote some positive numbers not depending on h and τ. Applying these inequalities to (16), we obtain:

$$\|\hat{p}\|_0^2 - \|p\|_0^2 + \tau^2 \|p_t\|_0^2 + M_1' \tau \|\hat{p}\|_1^2 \leq \varepsilon_1^{-1} \tau \|T_t\|_0^2 + \varepsilon_2^{-1} \tau \|f_p\|_{-1}^2 + \frac{4\tau c_0}{3l^2} (\hat{p},\, 1)_{\overline{\omega}}^2 \tag{17}$$

with $M_1' = (4/3)c_0 - M_1 c_1^2 \varepsilon_1 - M_1 \varepsilon_2$.

Let us multiply the equation (10) by $2\tau\hat{s}_\alpha$ for $\alpha = w,\, o$:

$$\|\hat{s}_\alpha\|_0^2 - \|s_\alpha\|_0^2 + \tau^2 \|s_{\alpha,t}\|_0^2 + \left(\Lambda_{3\alpha} \hat{p},\, 2\tau\hat{s}_\alpha\right)_{\overline{\omega}} = \left(f_\alpha,\, 2\tau\hat{s}_\alpha\right)_{\overline{\omega}} \tag{18}$$

and evaluate the inner products in (18) using Lemma 2, Cauchy inequality and conditions (7):

$$2\tau \left|\left(\Lambda_{3\alpha} \hat{p},\, \hat{s}_\alpha\right)_{\overline{\omega}}\right| \leq \tau c_1 M_2 \varepsilon_3 \|\hat{p}\|_1^2 + \frac{\tau c_1}{\varepsilon_3} \|\nabla \hat{s}_\alpha\|_0^2 ,$$

$$2\tau \left|\left(f_\alpha,\, \hat{s}_\alpha\right)_{\overline{\omega}}\right| \leq 2\tau \|f_\alpha\|_{-1} \|\hat{s}_\alpha\|_0 \leq \tau \|f_\alpha\|_{-1}^2 + \tau \|\hat{s}_\alpha\|_0^2 .$$

Using these inequalities, we obtain from (18):

$$\|\hat{s}_\alpha\|_0^2 - \|s_\alpha\|_0^2 + \tau^2 \|s_{\alpha,t}\|_0^2 \leq \tau c_1 M_2 \varepsilon_3 \|\hat{p}\|_1^2 + \frac{\tau c_1}{\varepsilon_3} \|\nabla \hat{s}_\alpha\|_0^2 + \tau \|f_\alpha\|_{-1}^2 + \tau \|\hat{s}_\alpha\|_0^2 .$$

Using the inequality $\|\nabla v\|_0^2 \leq \frac{8}{h^2} \|v\|_0^2$, valid for any grid function defined on a uniform grid, we obtain:

$$\left(1 - \tau - \frac{8\tau c_1}{\varepsilon_3 h^2}\right) \|\hat{s}_\alpha\|_0^2 - \|s_\alpha\|_0^2 + \tau^2 \|s_{\alpha,t}\|_0^2 \leq \tau c_1 M_2 \varepsilon_3 \|\hat{p}\|_1^2 + \tau \|f_\alpha\|_{-1}^2 .$$

Choosing ε_3 and $\tau \leq \tau_0$,

$$\tau_0 = \frac{\varepsilon_3 h^2}{8c_1 + \varepsilon_3 h^2}, \tag{19}$$

we obtain the inequality

$$\|\hat{s}_\alpha\|_0^2 - M_3 \|s_\alpha\|_0^2 + M_3\tau^2 \|s_{\alpha,t}\|_0^2 \le \tau M_4 \left(\varepsilon_3 \|\hat{p}\|_1^2 + \|f_\alpha\|_{-1}^2\right). \tag{20}$$

Similarly, we multiply the equation (8) by $2\tau\hat{T}$:

$$c_0\left(\left\|\hat{T}\right\|_0^2 - \|T\|_0^2 + \tau^2\|T_t\|_0^2\right) - 2\tau\sum_{m=1}^{2}\left(\left(k_h\hat{T}_{\overline{x}_m}\right)_{x_m}, \hat{T}\right)_{\overline{\omega}} \le$$

$$\le \tau\sum_{m=1}^{2}\left(\lambda^{(+1_m)}p_{\overline{x}_m}^{(+1_m)}\hat{T}_{x_m}, \hat{T}\right)_{\overline{\omega}} + \tau\sum_{m=1}^{2}\left(\lambda p_{\overline{x}_m}\hat{T}_{\overline{x}_m}, \hat{T}\right)_{\overline{\omega}} + \left(f_T, \hat{T}\right)_{\overline{\omega}} \tag{21}$$

and evaluate the inner products in (21) using Poincare inequality, Cauchy inequality and conditions (7):

$$-2\tau\sum_{m=1}^{2}\left(\left(k_h\hat{T}_{\overline{x}_m}\right)_{x_m}, \hat{T}\right)_{\overline{\omega}} \ge 2\tau c_0\left\|\nabla\hat{T}\right\|_0^2 \ge \frac{4\tau c_0}{3l^2}\left(l^2\left\|\hat{T}\right\|_1^2 - \left(\hat{T}, 1\right)_{\overline{\omega}}^2\right),$$

$$\tau\sum_{m=1}^{2}\left(\lambda^{(+1_m)}p_{\overline{x}_m}^{(+1_m)}\hat{T}_{x_m}, \hat{T}\right)_{\overline{\omega}} + \tau\sum_{m=1}^{2}\left(\lambda p_{\overline{x}_m}\hat{T}_{\overline{x}_m}, \hat{T}\right)_{\overline{\omega}} \le 2\tau c_1 M_1\sum_{m=1}^{2}\|p_{\overline{x}_m}\|_0\left\|\nabla\hat{T}\right\|_0\left\|\hat{T}\right\|_0 \le$$

$$\le \tau c_1\left(\varepsilon_4\|\nabla p\|_0^2 + \frac{M_1}{\varepsilon_4}\left\|\nabla\hat{T}\right\|_0^2\left\|\hat{T}\right\|_1^2\right) \le \tau c_1 M_1\varepsilon_4\|p\|_1^2 + \frac{\tau c_1 M_1^2}{\varepsilon_4}\left\|\hat{T}\right\|_1^4,$$

$$\left|\left(f_T, 2\tau\hat{T}\right)_{\overline{\omega}}\right| \le 2\tau\|f_T\|_{-1}\left\|\hat{T}\right\|_0 \le \varepsilon_5\tau\|f_T\|_{-1}^2 + \frac{M_1\tau}{\varepsilon_5}\left\|\hat{T}\right\|_1^2.$$

Applying these inequalities to (21), we have:

$$\left\|\hat{T}\right\|_0^2 - \|T\|_0^2 + \tau^2\|T_t\|_0^2 + \tau\left(\frac{4c_0}{3} - \frac{M_1}{c_0\varepsilon_5} - \frac{c_1M_1^2}{c_0\varepsilon_4}\left\|\hat{T}\right\|_1^2\right)\left\|\hat{T}\right\|_1^2 \le$$

$$\le \frac{\tau c_1 M_1\varepsilon_4}{c_0}\|p\|_1^2 + \frac{\varepsilon_5\tau}{c_0}\|f_T\|_{-1}^2 + \frac{4\tau}{3l^2}\left(\hat{T}, 1\right)_{\overline{\omega}}^2. \tag{22}$$

Combining the estimates (17), (20), (22), we obtain:

$$\left\|\hat{T}\right\|_0^2 - \|T\|_0^2 + \tau^2\left(1 - \frac{1}{\varepsilon_1\tau}\right)\|T_t\|_0^2 + \tau\left(\frac{4c_0}{3} - \frac{M_1}{c_0\varepsilon_5} - \frac{c_1M_1^2}{c_0\varepsilon_4}\left\|\hat{T}\right\|_1^2\right)\left\|\hat{T}\right\|_1^2 +$$

$$+ \|\hat{p}\|_0^2 - \|p\|_0^2 + \tau^2\|p_t\|_0^2 + \tau\left(\frac{4c_0}{3} - M_1c_1^2\varepsilon_1 - M_1\varepsilon_2 - 2M_4\varepsilon_3\right)\|\hat{p}\|_1^2 +$$

$$+ \sum_{\alpha=w,o}\|\hat{s}_\alpha\|_0^2 - M_3\sum_{\alpha=w,o}\|s_\alpha\|_0^2 + M_3\tau^2\sum_{\alpha=w,o}\|s_{\alpha,t}\|_0^2 \le$$

$$\le \frac{\tau c_1 M_1\varepsilon_4}{c_0}\|p\|_1^2 + \frac{4\tau}{3l^2}\left(\hat{T}, 1\right)_{\overline{\omega}}^2 + \frac{4\tau c_0}{3l^2}(\hat{p}, 1)_{\overline{\omega}}^2 + \frac{\varepsilon_5\tau}{c_0}\|f_T\|_{-1}^2 +$$

$$+\frac{\tau}{\varepsilon_2}\left\|f_p\right\|_{-1}^2+M_3\varepsilon_4\tau\sum_{\alpha=w,o}\left\|f_\alpha\right\|_{-1}^2. \quad (23)$$

Choosing τ, ε_1 and ε_2 from the conditions

$$c_1^2\varepsilon_1+\varepsilon_2\le 2M_1^{-1}\left((2/3)c_0-M_4\varepsilon_3\right), \quad (24)$$

$$\varepsilon_1^{-1}\le\tau\le\tau_0 \quad (25)$$

with τ_0, defined in (19), and assuming that the condition

$$\frac{4c_0}{3}-\frac{M_1}{c_0}\left(\frac{c_1M_1}{\varepsilon_4}+\frac{1}{\varepsilon_5}\right)\left\|\hat{T}\right\|_1^2\ge 0 \quad (26)$$

holds for some $\varepsilon_4,\varepsilon_5$, we obtain from (23):

$$\left\|\hat{T}\right\|_0^2+\left\|\hat{p}\right\|_0^2+\sum_{\alpha=w,o}\left\|\hat{s}_\alpha\right\|_0^2+\tau M_5\left\|\hat{T}\right\|_1^2+\tau M_6\left\|\hat{p}\right\|_1^2\le$$

$$\le\left\|T\right\|_0^2+\left\|p\right\|_0^2+M_3\sum_{\alpha=w,o}\left\|s_\alpha\right\|_0^2+\frac{\tau c_1M_1\varepsilon_4}{c_0}\left\|p\right\|_1^2+\frac{\varepsilon_5\tau}{c_0}\left\|f_T\right\|_{-1}^2+$$

$$+\frac{\tau}{\varepsilon_2}\left\|f_p\right\|_{-1}^2+\gamma_0+M_3\varepsilon_4\tau\sum_{\alpha=w,o}\left\|f_\alpha\right\|_{-1}^2 \quad (27)$$

where $\gamma_0=\frac{4\tau}{3l^2}\left(\tilde{T}+c_0\tilde{p}\right)$. Summing the inequality (27) over n' from 0 to n, we obtain:

$$\left\|T^{n+1}\right\|_0^2+\left\|p^{n+1}\right\|_0^2+\sum_{\alpha=w,o}\left\|s_\alpha^{n+1}\right\|_0^2+\left\|T^{n+1}\right\|_1^2+\left\|p^{n+1}\right\|_1^2\le$$

$$\le M\left(\gamma_0+\left\|T_0\right\|_0^2+\left\|p_0\right\|_0^2+\sum_{\alpha=w,o}\left(\left\|s_{\alpha0}\right\|_0^2+\left\|f_\alpha\right\|_{-1}^2\right)+\left\|f_T\right\|_{-1}^2+\left\|f_p\right\|_{-1}^2\right). \quad (28)$$

Thus, the following theorem is proved.

Theorem. Under the conditions (7), (12), (24)-(26) the following inequality holds for the solution of the difference problem (8)-(11):

$$\left\|T^{n+1}\right\|_1^2+\left\|p^{n+1}\right\|_1^2+\left\|s_w^{n+1}\right\|_0^2+\left\|s_o^{n+1}\right\|_0^2\le M\left(\left\|T_0\right\|_0^2+\left\|p_0\right\|_0^2+\left\|s_{w0}\right\|_0^2+\left\|s_{o0}\right\|_0^2\right)+$$

$$+M\left(\gamma_0+\left\|f_w\right\|_{-1}^2+\left\|f_o\right\|_{-1}^2+\left\|f_T\right\|_{-1}^2+\left\|f_p\right\|_{-1}^2\right).$$

The obtained estimate implies the stability of the difference scheme with respect to the initial data and right-hand sides of equations (8)-(11).

4 Conclusion

Thus, the stability of the considered finite difference scheme is studied in the present paper. A priori estimate for the solution of the difference problem is obtained under assumptions on the value of the time step and the norm of the temperature derivative. These restrictions result from the presence of the nonlinear term in the temperature equation, and the term with the time derivative of the temperature in the pressure equation. The obtained results can be used to study the stability of the difference scheme for a model taking into account capillary and gravitational forces.

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