UDC 517.968.2

Zhunussova Zh.Kh.[1*], Nugmanova G.N.[2**], Moldanazarova U.[2]

[1]Al-Farabi Kazakh National University, Republic of Kazakhstan, Almaty
[2]Eurasian National University named after L.N.Gumilev, Republic of Kazakhstan, Astana
E-mail: *zhzhkh@mail.ru, **nugmanovagn@gmail.com

About gauge equivalent of the generalized Landau-Lifshitz equation

In mathematics an inverse scattering transformation is a method for solving some nonlinear equations with partial derivatives. Discovering of the method became one of the crucial events in mathematical physics in the last 40 years [1]–[6]. The method presents a nonlinear analogue, in a sense generalized Fourier transformation, which is applied to solve a lot of linear equations with particular derivatives. Title "inverse scattering problem"is originated from key idea of recovery time evolution of the potential from time evolution its scattering data: inverse scattering is related to the problem about recovery of the potential from its scattering matrix, in difference from direct scattering the problem of finding a scattering matrix of potential. The inverse scattering transformation can be applied to many co-called exactly decided models, i.e. completely integrable infinite systems. Firstly it was presented by Clifford S. Gardner, John M. Greene and Martin D. Kruskal and others (1967, 1974) for Korteweg de Vries equation, and soon spread to nonlinear Scrodinger equation, sine-Gordon equation and Toda chain equation. Later the method was used to solve many another equations, such that Kadomtsev-Petviashvili equation, Ishimori equation, Dime equation etc. Characteristic property of solutions obtained by inverse scattering method is existence of solitons, solutions reminding as particles as waves, which do not have any analogues for linear equations with particular derivatives and are applied to nonlinear optics and plasma physics, and its quantum version describes many-particle system with б-shaped interaction.
Key words: spin system, soliton, inverse scattering transformation, integrable systems, compatibility condition, Lax pair.

Жунусова Ж.Х., Нугманова Г.Н., Молданазарова У.
О калибровочном эквиваленте обобщенного уравнения Ландау-Лифшица

В математике преобразованием обратного рассеяния является метод решения некоторых нелинейных уравнений с частными производными. Открытие данного метода стало одним из самых важных событий в математической физике в последние 40 лет [1]-[6]. Метод представляет собой нелинейный аналог, а в каком-то смысле обобщения преобразования Фурье, которое само по себе применяется для решения многих линейных уравнений в частных производных. Название "метод обратной задачи рассеяния"происходит от ключевой идеи восстановления временной эволюции потенциала от временной эволюции его данных рассеяния: обратное рассеяние относится к задаче о восстановлении потенциала от его матрицы рассеяния, в отличие от прямого рассеяния задача нахождения матрицы рассеяния от потенциала. Обратное преобразование рассеяния может быть применено ко многим из так называемых точно решаемых моделей, то есть вполне интегрируемых бесконечномерных систем. Впервые он был представлен Клиффорда С. Гарднер, Джон М. Грином, и Мартин Д. Крускала и др. (1967, 1974) для уравнения Кортевега-де Фриза, и вскоре распространяется на нелинейное уравнение Шредингера, уравнение синус-Гордона и уравнения цепочки Тода. Позднее данный метод был использован для решения многих других уравнений, таких как уравнение Кадомцева-Петвиашвили, уравнение Ишимори, уравнение Дим, и так далее. Характерным свойством решений, полученных методом обратного рассеяния является существование солитонов, решений, напоминающих как частицы и волны, которые не имеют аналогов для линейных уравнений с частными производными и применяются в нелинейной оптике и в физике плазмы, а его квантовый вариант описывает многочастичную систему с б-образным взаимодействием.
Ключевые слова: спиновая система, солитон, преобразование обратного рассеяния, интегрируемые системы, условие совместности, пара Лакса.

Жүнісова Ж.Х., Нұғманова Г.Н., Молданазарова У.

Жалпылама Ландау-Лифшиц теңдеуінің калибрлі эквиваленттігі туралы

Кейбір сызықты емес дербес туындылы теңдеулерді шешудің әдісі математикада кері сейілу түрлендіруі деп аталады. Осы әдістің ашылуы соңғы 40 жылдағы математикалық физикадағы елеулі оқиғалардың бірі [1]-[6]. Бұл әдіс өздігінен сызықты емес аналог, ал кей мағынада дербес туындылы сызықты теңдеулерді шешу үшін қолданылатын Фурье түрлендіруінің жалпылауы болып табылады. "Кері сейілу есебінің әдісі"атауы сейілу берілгендерінің уақыт эволюциясынан уақыт эволюциясының потенциалының қайта құрылуының негізгі идеясын құрайды: тікелей сейілуге қарағанда, потенциалдан сейілу матрицасының табылу есебі кері сейілудің сейілу матрицасынан потенциалды қайта құру есебіне қатысты қолданылады. Көптеген нақты шешілетін моделдерге немесе интегралданатын ақырсыз жүйелерге кері сейілу теңдеулерді қолдануға болады. Алғаш рет ол Клиффорд С. Гарднер, Джон М. Грин, Мартин Д. Крускал және басқаларымен (1967, 1974) Кортевег де Фриз теңдеуі үшін ұсынылды, кейін Шредингердің сызықты емес теңдеуіне, синус-Гордонның теңдеуіне және Тода шынжырына таралды. Кейін осы әдіс Кадомцев-Петвиашвили теңдеуі, Ишимори теңдеуі, Дим теңдеуі, т.с.с көптеген теңдеулерді шешу үшін қолданылды. Кері сейілу әдісі арқылы алынған шешімдердің характеристикалық қасиеттері болып солитондардың, дербес туындылы сызықты теңдеулерге аналогы жоқ бөлшектер мен толқын секілді шешімдердің бар болуы саналады және сызықты емес оптика мен плазма физикасында қолданылады, ал оның кванттық түрі б-түрлі өзара әсерлі көпбөлшекті жүйені сипаттайды.
Түйін сөздер: спиндік жүйе, солитон, кері сейілу турлендіруі, интегралданатын жүйе, үйлесімділік шарты, Лакс жұбы.

1 Derivative nonlinear Schrodinger equation

Derivative nonlinear Schrodinger equation-I;

$$\phi_t + i\phi_{xx} + (|\phi|^2 \phi)_x = 0 \tag{1}$$

compatibility condition of the linear differential equations

$$\Phi_x = U\Phi \tag{2}$$

$$\Phi_t = V\Phi, \tag{3}$$

where U, V can be expressed in the form

$$U = -i\lambda^2\sigma_3 + \lambda \begin{pmatrix} 0 & \phi \\ -\phi^* & 0 \end{pmatrix} \tag{3a}$$

$$V = -(2i\lambda^4 - i|\phi|^2 \lambda^2)\sigma_3 + 2\lambda^3 \begin{pmatrix} 0 & \phi \\ -\phi^* & 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 & i\phi_x - |\phi|^2 \phi \\ i\phi_x^* + |\phi|^2 \phi^* & 0 \end{pmatrix}. \tag{3b}$$

Derivative nonlinear Schrodinger equation-II;

$$q_t - iq_{xx} + |q|^2 q_x = 0 \tag{4}$$

compatibility condition of the linear differential equations;

$$Q_x = WQ \tag{5a}$$

$$Q_t = ZQ, \tag{5b}$$

where W and Z the following matrixes

$$W = (-i\lambda^2 + \frac{i}{4}|q|^2)\sigma_3 + \lambda \begin{pmatrix} 0 & q \\ -q^* & 0 \end{pmatrix} \tag{6a}$$

$$Z = \left(-2i\lambda^4 + i|q|^2\lambda^2 + \frac{1}{4}(qq_x^* - q^*q_x) - \frac{i}{8}|q|^4\right)\sigma_3 + 2\lambda^3 \begin{pmatrix} 0 & q \\ -q^* & 0 \end{pmatrix} +$$

$$+\lambda \begin{pmatrix} 0 & iq_x - |q|^2 q/2 \\ iq_x^* + |q|^2 q^*/2 & 0 \end{pmatrix}, \tag{6b}$$

here $\phi(t,x)$ and $q(t,x)$ is a complex function (classic charged field), and $|\phi|^2 = \phi\phi^*$, $|q|^2 = qq^*$, where $*$ mens complex conjugation.

2 Spin system which is gauge equivalent the derivative nonlinear Schrodinger equation

In the chapter we propose a new spin model which is gauge equivalent the derivative nonlinear Schrodinger equation.
We present the notion of gauge equivalent introduced by L.Takhtadzhyan and V.Zakharov as the following definitions [1]:
Definition 1. Equations allowed the Lax pair

$$U_{jt} - V_{jx} + [U_j, V_j] = 0, \qquad j = 1, 2$$

or

$$\Phi_x = U_j\Phi$$

$$\Phi_t = V_j\Phi$$

are called by integrable.

Definition 2. Integrable equations are called gauge equivalent, if they are connected by transformation $\Phi_1 = g^{-1}\Phi_2$, $U_1 = gU_2g^{-1} + g_xg^{-1}$, $V_1 = gV_2g^{-1} + g_tg^{-1}$ with matrix function g, not dependents on symbols pseudo-characters by other nondependable values operators entering to U и V.

We formulate the theorem.

Theorem 1. Spin system

$$S_t - iSS_{xx} - \frac{i}{2}tr(S_x^2) - 4\lambda_0^2\left(1 - \frac{1}{32\lambda_0^4}tr(S_x^2)\right)S_x = 0 \tag{7}$$

with boundary condition

$$S^2 = I \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is gage equivalent the derivative nonlinear Schrodinger equation-I (1).

Proof. We consider gauge transformation

$$\psi = g^{-1}\Phi, \tag{8}$$

where $\Phi(x,t)$ is a solution of the system equation (2), and $\psi(x,t)$ is a function of a class continuous functions which is solution of origin Lax pair corresponding equivalent spin system. Let $g(x,t)$ be a solution of the system (2) at $\lambda = \lambda_0, \lambda_0 = const$: $g = \Phi|_{\lambda=\lambda_0}$.

Derivatives by x from (8) give the following

$$\psi_x = -g^{-1}g_xg^{-1}\Phi + g^{-1}\Phi_x, \tag{9}$$

where $\Phi = g\psi$ и Φ_x is taken from system (2).

We introduce the notation [1]

$$S = g^{-1}\sigma_3 g \tag{10}$$

where σ_3 is a Pauli matrix

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then from (9a) with take into account (10) we obtain

$$\psi_x = \left(-i(\lambda^2 - \lambda_0^2)S + \frac{\lambda - \lambda_0}{2\lambda_0}SS_x\right)\psi. \tag{11}$$

Now we calculate the derivative by t from (8)

$$\psi_t = -g^{-1}g_tg^{-1}\Phi + g^{-1}\Phi_t. \tag{12}$$

This equation is transformed with consideration (2) and (10)

$$\psi_t = \left(-\left[2i(\lambda^4 - \lambda_0^4) - \frac{i}{8\lambda_0^2}(\lambda - \lambda_0)^2 tr(S_x^2)\right]S + \right.$$
$$\left. + \left[\frac{\lambda^3 - 2\lambda_0^3 + \lambda_0^2\lambda}{\lambda_0} - \frac{(\lambda - \lambda_0)}{16\lambda_0^3}tr(S_x^2)\right]SS_x + \frac{i(\lambda - \lambda_0)}{2\lambda_0}S_{xx}\right)\psi. \tag{13}$$

Thus, we solved the problem of finding the Lax pair which is gauge equivalent to linear system (2), it is Lax pair of the derivative nonlinear Schrodinger equation-I. The next step is reveal a spin system corresponding to Lax pair. We consider compatibility condition of the system (11), i.e.$\psi_{xt} = \psi_{tx}$.

$$\psi_{xt} = \left(-i(\lambda^2 - \lambda_0^2)S_t + \frac{\lambda - \lambda_0}{2\lambda_0}(SS_x)_t\right)\psi + \left(-i(\lambda^2 - \lambda_0^2)S + \frac{\lambda - \lambda_0}{2\lambda_0}SS_x\right)\psi_t \tag{14}$$

$$\psi_{tx} = \left(-2i(\lambda^4 - \lambda_0^4)S_x - \frac{i}{8\lambda_0^2}(\lambda - \lambda_0)^2(Str(S_x^2))_x + \right.$$
$$\left. + \frac{\lambda^3 - 2\lambda_0^3 + \lambda_0^2\lambda}{\lambda_0}(SS_x)_x - \frac{(\lambda - \lambda_0)}{16\lambda_0^3}(SS_xtr(S_x^2))_x + \frac{i(\lambda - \lambda_0)}{2\lambda_0}S_{xxx}\right)\psi +$$

$$+\left(-\left[2i(\lambda^4-\lambda_0^4)-\frac{i}{8\lambda_0^2}(\lambda-\lambda_0)^2tr(S_x^2)\right]S+\right.$$

$$\left.+\left[\frac{\lambda^3-2\lambda_0^3+\lambda_0^2\lambda}{\lambda_0}-\frac{(\lambda-\lambda_0)}{16\lambda_0^3}tr(S_x^2)\right]SS_x+\frac{i(\lambda-\lambda_0)}{2\lambda_0}S_{xx}\right)\psi_x \tag{15}$$

Equating (14) and (15) to each other with take into account the system (11) and (13), as well as expansion in powers λ we obtain the equation (7). Since the equations at $\lambda^5, \lambda^4, \lambda^3$ are equal to zero and we have combined the equations at λ, λ^0, then the obtained equation and equation at λ^2 are the same.

Theorem 1 is proved.

Theorem 2. Spin system which gauge equivalent the derivative nonlinear Schrodinger equation-II is equation (7).

Proof. We consider the gauge transformation

$$r = g^{-1}Q \tag{16}$$

where $Q(x,t)$ is a solution of the system equation (5), and $r(x,t)$ is a function of a class continuous functions which is solution of origin Lax pair corresponding equivalent spin system. Let $g(x,t)$ be a solution of the system (5) at $\lambda=\lambda_0, \lambda_0 = const$: $g = \Phi|_{\lambda=\lambda_0}$.

Derivative by x of (16) gives the following

$$r_x = -g^{-1}g_xg^{-1}Q + g^{-1}Q_x, \tag{17}$$

where $Q = gr$ и Q_x is taken of the system (5).

Then from (17) with consideration (10) we get

$$r_x = \left(-i(\lambda^2-\lambda_0^2)S + \frac{\lambda-\lambda_0}{2\lambda_0}SS_x\right)r. \tag{18}$$

Now we calculate the derivative by t from (16)

$$r_t = -g^{-1}g_tg^{-1}Q + g^{-1}Q_t. \tag{19}$$

This equation is transformed with taking into account (2) and (10)

$$r_t = \left(\left[-2i(\lambda^4-\lambda_0^4)+\frac{i}{8\lambda_0^2}(\lambda^2-3\lambda_0^2+2\lambda\lambda_0)tr(S_x^2)\right]S+\right.$$

$$\left.+\left[\frac{\lambda^3-2\lambda_0^3+\lambda\lambda_0^2}{\lambda_0}-\frac{(\lambda-\lambda_0)}{16\lambda_0^3}tr(S_x^2)\right]SS_x+\frac{i(\lambda-\lambda_0)}{2\lambda_0}S_{xx}\right)r. \tag{20}$$

We consider the compatibility condition of the system (18) и (20), т.е.$r_{xt} = r_{tx}$.

$$r_{xt} = \left(-i(\lambda^2-\lambda_0^2)S_t + \frac{\lambda-\lambda_0}{2\lambda_0}(SS_x)_t\right)r + \left(-i(\lambda^2-\lambda_0^2)S + \frac{\lambda-\lambda_0}{2\lambda_0}SS_x\right)r_t \tag{21}$$

$$r_{tx} = \Bigg(-2i(\lambda^4 - \lambda_0^4)S_x + \frac{i}{8\lambda_0^2}(\lambda^2 - 3\lambda_0^2 + 2\lambda\lambda_0)(Str(S_x^2))_x +$$
$$+\frac{\lambda^3 - 2\lambda_0^3 + \lambda\lambda_0^2}{\lambda_0}(SS_x)_x - \frac{(\lambda - \lambda_0)}{16\lambda_0^3}(SS_x tr(S_x^2))_x + \frac{i(\lambda - \lambda_0)}{2\lambda_0}S_{xxx}\Bigg)r+$$
$$+\Bigg(\left[-2i(\lambda^4 - \lambda_0^4) + \frac{i}{8\lambda_0^2}(\lambda^2 - 3\lambda_0^2 + 2\lambda\lambda_0)tr(S_x^2)\right]S+$$
$$+\left[\frac{\lambda^3 - 2\lambda_0^3 + \lambda\lambda_0^2}{\lambda_0} - \frac{(\lambda - \lambda_0)}{16\lambda_0^3}tr(S_x^2)\right]SS_x + \frac{i(\lambda - \lambda_0)}{2\lambda_0}S_{xx}\Bigg)r_x \qquad (22)$$

By equating (21) and (22) to each other and substituting the expressions r_t, r_x, expansion by powers λ we get equation (7). Since the equations at $\lambda^5, \lambda^4, \lambda^3$ are equal to zero and we have composed the equations at λ, λ^0, the the obtained equation and equation at λ^2 are the same.

Theorem 2 is proved.

3 Conclusion

Derivative Heisenberg models which are equivalent two type derivative nonlinear Schrodinger equation obtained in this work. Estimated results have shown, that two type derivative nonlinear Schrodinger equation, considered in the work, corresponds the same derivative spin model.

This work was supported by the grant of the Committee of Science of the Ministry of Education and Science of the Republic of Kazakhstan (project 0893/GF4 MES RK)

References

[1] *Takhtadzhyan L., Fadeev L.* Hamiltonian approach in the theory of solitons -M.: Nauka, 1986, –528 p.

[2] *R. Myrzakulov, G.K. Mamyrbekova, G.N. Nugmanova, M. Lakshmanan.* // Phys. Lett A. - 2014. - V. 378, -P. 2118.

[3] *Myrzakulov R., Mamyrbekova G. K., Nugmanova G. N., Lakshmanan M.* Integrable (2+1)-dimensional spin models with self-consistent potentials // Symmetry. - 2015. - V. 7(3), -P. 1352-1375.

[4] *Zh. Zhunussova.* Nonlinear PDE as Immersions // Proceedings of the 9th ISAAC Congress, Springer, Series: Trends in Mathematics, ISBN 978-3-319-12576-3, - 2015. -P. 289–297.

[5] *Zh. Zhunussova.* About domain wall solution of the integrated spin system // KazNU Bulletin, ser. math., mech., inf. - 2014. - № 2(81). - P.46-51.

[6] *Yersultanova Z.S., Zhassybayeva M., Yesmakhanova K., Nugmanova G., Myrzakulov R.* Darboux Transformation and Exact Solutions of the integrable Heisenberg ferromagnetic equation with self-consistent potentials // International Journal of Geometric Methods in Modern Physics. - 2016. - V. 13. -P. 1550134.