3-бөлім | Раздел 3 | Section 3

Қолданылмалы математика | Прикладная математика | Applied Mathematics

IRSTI 27.35.33, 27.41.19, 06.52.13

Application of differential evolution algorithm for solving the Solow model with the addition of human capital

Kabanikhin S.I., Institute of Computational Mathematics and Mathematical Geophysics of SB RAS, Novosibirsk, Russian Federation, +7(383) 3308353, E-mail: kabanikhin@sscc.ru
Bektemessov M.A., Abai Kazakh National Pedagogycal University Almaty, Kazakhstan, +7(777) 2629301, E-mail: maktagali@mail.ru
Shishlenin M.A., Institute of Computational Mathematics and Mathematical Geophysics of SB RAS, Novosibirsk, Russian Federation, +7(383) 3308460, E-mail: mshishlenin@ngs.ru
Yang Xin-She, Middlesex University, London, United Kingdom, +44 (0)20 8411 5555, E-mail: x.yang@mdx.ac.uk
Bektemessov Zh.M., Al-Farabi Kazakh National University, Almaty, Kazakhstan, +7(702)9062830, E-mail: jolaman252@gmail.com

This paper is devoted to a numerical study of defining of parameters of dynamical systems arising in financial and economic problems. The importance of parameters that are difficult to measure is great, so defining them will help to make forecasts and a work plan for the future at the governmental level. An effective way to restore parameters is to solve the inverse problem. The method of coefficient recovery using the algorithm of differential evolution, which was proposed by Rainer Storn and Kenneth Price, is presented in this paper. On the example of solving the direct problem of the mathematical model of neoclassical economic growth of Robert Solow and the results obtained, the inverse problem was solved and unknown parameters were determined. The Solow model is based on the Cobb-Douglas production function, taking into account labor, capital and exogenous neutral technical progress. Also, for further calculations, the economic model proposed by Mankiw-Romer-Weil based on the Solow model was considered, but with the addition of human capital, where the number of variables and coefficients that need to be restored has already increasing. A direct problem was also solved, results were obtained that were applied in the algorithm of differential evolution for parameters recovery.

Key words: economical model, inverse problems, optimization, differential evolution, Solow model.

Адам капиталын қосу арқылы Солоу моделін шешу үшін дифференциалдық эволюция алгоритмін қолдануы

Кабанихин С.И., РҒА СБ есептеу математикасы және математикалық геофизика институты, Новосибирск қ., Ресей Федерациясы, +7(383) 3308353, E-mail: kabanikhin@sscc.ru
Бектемесов М.А., Абай атындағы қазақ Ұлттық педагогикалық университеті, Алматы қ., Қазақстан Республикасы, +7(777) 2629301, E-mail: maktagali@mail.ru
Шишленин М.А., РҒА СБ есептеу математикасы және математикалық геофизика институты, Новосибирск қ., Ресей Федерациясы, +7(383) 3308460, E-mail: mshishlenin@ngs.ru
Янг С.-Ш., Мидлсекс Университеті, Лондон қ., Біріккен Корольдігі, +44 (0)20 8411 5555, E-mail: x.yang@mdx.ac.uk
Бектемесов Ж.М., әл-Фараби атындағы қазақ Ұлттық университеті, Алматы қ., Қазақстан Республикасы, +7(702)9062830, E-mail: jolaman252@gmail.com

Негізгі жұмыс қаржы-экономикалық мәселелерде туындайтын динамикалық жүйелердің параметрлерін анық таудың сандық зерттеуіне арналған. Өлшеуі қиын параметрлердің маңыздылығы соншалық , олардың анық тамалары болашақ та мемлекеттік деңгейде болжам жасау және жұмыс жоспарын жасауға көмектеседі. Параметрлерін қалпына келтірудің тиімді жолы - кері есептерді шешу. Осы мақалада Райнер Сторн мен Кеннет Прайс ұсынған дифференциалдық эволюция алгоритмін пайдалану арқылы коэффициенттерді қалпына келтіру әдісі ұсынылған. Роберт Солоудың неоклассикалық экономикалық өсуінің математикалық моделін тікелей шешіп және алынған нәтижелерді мысалға келтіріп, кері проблема шешілді және белгісіз параметрлер анықталды. Солоу моделі Кобб-Дугластың өндіріс функциясына негізделеді, еңбек, капиталды және экзогендік бейтарап техникалық прогресті ескере оты-
рып. Сондай-ақ , келесі есептеулер үшін Мэнкью-Ромер-Уэйл ұсынған, бірақ адам капиталын қосу арқ ылы, Солоу моделіне негізделген экономикалық модельді қарастырамыз, бірақ қалпына келтіруді қажет ететін ауыспалылар мен коэффициенттердің саны өседі. Сонымен бірге тікелей есеп шешілді, параметрлерін қалпына келтіру үшін дифференциалдық эволюция алгоритмінде қолданылған нәтижелер алынды.
Түйін сөздер: экономикалық модель, кері есептер, оңтайландыру, дифференциалдық эволюция, Солоу моделі.

Применение алгоритма дифференциальной эволюции для решения модели Солоу с добавлением человеческого капитала

Кабанихин С.И., Институт вычислительной математики и математической геофизики СО РАН, г. Новосибирск, Российская Федерация, +7(383) 3308353, E-mail: kabanikhin@sscc.ru
Бектемесов М.А., Казахский национальный педагогический университет им. Абая, г. Алматы, Казахстан, +7(777) 2629301, E-mail: maktagali@mail.ru
Шишленин М.А., Институт вычислительной математики и математической геофизики СО РАН, г. Новосибирск, Российская Федерация, +7(383) 3308460, E-mail: mshishlenin@ngs.ru
Янг С.-Ш., Университет Мидлсекс, г. Лондон, Объединенное Королевство, +44 (0)20 8411 5555, E-mail: x.yang@mdx.ac.uk
Бектемесов Ж.М., Казахский национальный университет имени аль-Фараби, г. Алматы, Республика Казахстан, +7(702)9062830, E-mail: jolaman252@gmail.com

Данная работа посвящена численному исследованию определения параметров динамических систем, возникающих в финансовых и экономических задачах. Значимость параметров, труднодоступных для измерения, велика, поэтому их определение поможет на государственном уровне составлять прогнозы и план работы на будущее. Эффективным способом восстановления параметров является решение обратной задачи. В работе приведен метод восстановления коэффициентов с помощью алгоритма дифференциальной эволюции, которая была предложена Райнером Сторном и Кеннетом Прайсом. На примере решения прямой задачи математической модели неоклассического экономического роста Роберта Солоу и полученных результатов, была решена обратная задача и были определены неизвестные параметры. Модель Солоу основана на производственной функции Кобба-Дугласа, с учетом труда, капитала и экзогенного нейтрального технического прогресса. Также, для дальнейших расчетов рассматривается экономическая модель, предложенная Мэнкью-Ромер-Уэйлом (Mankiw - Romer - Weil), основанная на модели Солоу, но с добавлением человеческого капитала, где уже увеличивается количество переменных и коэффициентов которые надо восстановить. Также решалась прямая задача, были получены результаты, которые применялись в алгоритме дифференциальной эволюции для восстановления параметров.
Ключевые слова: экономическая модель, обратные задачи, оптимизация, дифференциальная эволюция, модель Солоу.

1 Introduction

Nowadays economic processes plays a huge role in human's life, because it is in every part of our daily routine: food, clothes, home, entertainment and many others. Peoples' economic independence is the one of the most valuable things that is why we go to work and earn money. Also no one wants to waste hard earned money, so we try to optimize it by mathematical tools. It is not only the personal issue, but also of the countries and government, where interesting roles play the population growth and capital accumulation. There is a mathematical model of neoclassic economical growth of Robert Solow (Solow 1956: 65 - 94) which will help to predict economic situations by using differential evolution algorithm. There will be used methods for solving inverse problems to identify the coefficients by known data at the fixed time moments.

2 Literature review

The study of economic models was developed in the 1930s XX century. The first publications on production functions include the work of Cobb and Douglas (Cobb 1928: 139-165), which gave impetus to subsequent studies in the field of economics as the work of Douglas (Douglas 1976: 903-916), Houthhakker (Houthhakker 1955: 27-31), Berndt (Berndt 1973: 81-113), and further based on the work of Cobb-Douglas and their production function, the work of Solow (Solow 1956: 65-94) and Swan (Swan 1954: 334-361) was published. These works served to develop the proposed models described in the articles of Mankiw-Romer-Weil (Mankiw 1992: 407-437), Nazrul (Nazrul 1995: 1127-1170), Benhabib (Benhabib 1994: 143-173), Temple (Temple 1999: 112-156), Durlauf (Durlauf 2005: 555-677) and many others.

As for the identification of the parameters, the work was based on the publications of Yang (Yang 2014: 1-300), Storn and Price (Storn 1995: 1-12), (Storn 1997: 341-359), and later Lampinen (Storn, Lampinen 2005: 1-539).

In his 1956s article Solow proposed that the study of economic growth starts by assuming a standard neoclassical production function with decreasing returns on capital (Solow 1956: 65 - 94). He showed that two variables such as the population growth and rates of saving determine the steady-state level of income per capita. Because these variables illustrates different steady-states of various countries. Solow's model gives simple testable predictions about how these variables influence the steady-state level of income. The higher the rate of saving, the richer the country. The higher the rate of population growth, the poorer the country.

Solow's model takes the rates of saving, population growth, and technological progress as exogenous. There are two inputs, capital and labor, which are paid their marginal products. The Cobb-Douglas production function was assumed, so production at time t is given as

$$Y(t) = K(t)^{\alpha}(A(t)L(t))^{1-\alpha} \tag{1}$$

The notation is as following: Y is output, K capital, L labor and A the level of technology. L and A are assumed to grow exogenously at rates $\boldsymbol{n}$ and $\boldsymbol{g}$:

$$L(t) = L(0)e^{nt} \tag{2}$$

$$A(t) = A(0)e^{gt} \tag{3}$$

The number of effective units of labor, *A(t)L(t)*, growth at rate $\boldsymbol{n}+\boldsymbol{g}$. The model assumes that a constant fraction of output, s_k, is invested. Defining $\boldsymbol{k}$ as the stock of capital per effective unit of labor, $\boldsymbol{k} = K/AL$ and $\boldsymbol{y}$, and y as the level of output per effective unit of labor, $\boldsymbol{y} = Y/AL$, where we substite $y(t)$ as $k^\alpha(t)$, so the evolution of $\boldsymbol{k}$ is governed by

$$\dot{k}(t) = s_k k^\alpha(t) - (n + g + \delta)k(t) \tag{4}$$

where δ is the rate of depreciation. The sum of rates of population growth, technological progress and depreciation can be rewriten as s. So the direct problem that we solve has the following form

$$\begin{cases} \dot{k}(t) = s_k k^\alpha(t) - sk(t) \\ k(0) = 1 \end{cases} \tag{5}$$

with the time period $t \in [0, 0.5]$ i.e. half of the year with the time step $h = 0.1$ and the given coefficients as $s_k = 0.5, \alpha = 0.3$ and $s = -0.5$.

Then for the inverse problem, we assume that the coefficients are unknown and to identify them, we can use, the additional data from the direct problem, the obtained solution at the time steps $t = 0.0, 0.1, ..., 0.5$ for the differential evolution algorithm.

3 Materials and methods

Differential Evolution (DE) is a vector-based metaheuristic algorithm. The method of differential evolution was developed by Rayner Storn and Kenneth Price in 1995 and published in 1997 (Storn 1995: 1-12), (Storn 1997: 341-359). The main example of the application of this algorithm is search engine Yandex uses the differential evolution method to improve its ranking algorithms.

Differential evolution is a method for solving multidimensional mathematical optimization related to the class of stochastic optimization algorithms (that it uses some random numbers) and has some mutation with some similarity to genetic algorithms, but, unlike genetic algorithm, DE does not require working with variables in a binary code. It is a gradient-free optimization method because it requires only the ability to calculate the values of the objective functions, but not its derivatives. The method of differential evolution is designed to find a global minimum (or maximum) of non-differentiable, nonlinear, multimodal functions with many design variables. The method is easy to implement and use (it contains three control parameters that require selection), it can also be easily parallelized (Storn, Lampinen 2005: 1-539).

Initially, a certain set of vectors ($n \geq 4$), called the generation t, representing possible solutions of the optimization problem, is generated. By vectors we mean points of d-dimensional space in which the objective function $f(x)$ is defined, which is required to be minimized. And then the vector on each generation is written as

$$x_i^t = (x_{1,i}^t, x_{2,i}^t, ..., x_{d,i}^t) \tag{6}$$

and has d-parameters in d-dimensional space.

The algorithm of differential evolution is divided into three stages: mutation, crossover and selection. At each iteration, the algorithm generates a new generation of vectors, randomly combining vectors from the previous generation. For each new generation $(t+1)$ of a vector from each vector x_i from the old generation (t), we randomly select three vectors x_p, x_q and x_r, with the exception of the vector itself, and generate so-called a mutant vector according to the following formula:

$$v_i^{t+1} = x_p^t + F(x_q^t - x_r^t) \tag{7}$$

where $F \in [0,2]$ is a constant called the differential weight. Although in theory it is assumed that $F \in [0,2]$, but in practice $F \in [0,1]$ gives greater efficiency and stability. In fact, almost all literature studies use $F \in (0,1)$.

Crossover operation is performed on the mutant vector, during which the coordinates of the mutant vector are replaced by the corresponding coordinates from the base vector. Each coordinate is replaced with some probability ($r_i \in [0,1]$) and the crossover coefficient $C_r \in [0,1]$ is used, which is also part of the differential evolution setting controlling the rate or probability of crossing. All this is presented as

$$u_{j,i}^{t+1} = \begin{cases} v_{j,i} & \text{if } r_i \leq C_r \\ x_i & \text{otherwise} \end{cases} j = 1, 2, .., d \tag{8}$$

This way, it can be decided randomly whether to exchange each component with a donor vector or not.

The vector obtained after crossover is called the test vector. If it is better than the base vector (that is, the value of the objective function has improved), then in the new generation the base vector is replaced by a trial one, otherwise the base vector is stored in the new generation.

$$x_i^{t+1} = \begin{cases} u_i^{t+1} & \text{if } f(u_i^{t+1} \leq f(x_i^t) \\ x_i^t & \text{otherwise} \end{cases} \tag{9}$$

It is worth noting that the overall search efficiency is controlled by two parameters: the differential weight F and the probability of crossing Cr (Yang 2014: 1-300).

4 Results and discussion

In the first experiment, the direct problem of the Solow model with given coefficients and initial conditions, $k(0) = 1$, was solved in the time period t from 0 to 0.5 measured in years (that is, six months) and the following were obtained results (Figure 1):

$$\begin{cases} \dot{k}(t) = s_k k^{\alpha}(t) - sk(t) \\ \qquad k(0) = 1 \end{cases} \tag{10}$$

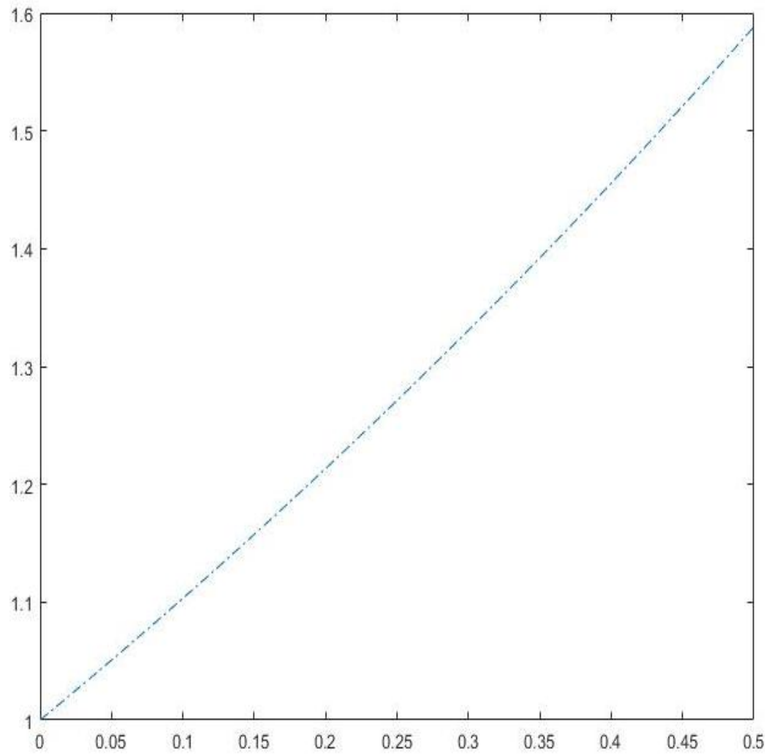

Figure 1 - Growth of capital intensity with given coefficients

To the results obtained, a 5% error was added to restore the parameters, so using them in the differential evolution algorithm gave the results of the identified coefficients (Table 1).

Table 1– Comparison of restored values with true

Coefficients	True solution	Restored values	Difference
s_k	0.5	0.4749	0.0251
α	0.3	0.3487	0.0487
s	-0.5	-0.1344	0.3656

As it is shown in the table 4, the restored unknown coefficients are way too similar to the true ones. The value of the identified parameter s_k is 0.4749, which slightly differs from the true solution of 0.5, and the value of the parameter α is 0.3487, which diverges from the true solution 0.3 only by 0.0487. However, compared with the previous coefficients, the last parameter s, while restoring, showed not so good results and has a difference with the true value of about 0.37, which is very much for such small coefficients.

In the second experiment, the model proposed by Mankiw-Romer-Weil based on the Solow model (Mankiw 1992: 407-437) was used. Where they changed the production function as

$$\begin{cases} \dot{k}(t) = s_k k(t)^\alpha h(t)^\beta - sk(t) \\ \dot{h}(t) = s_h k(t)^\alpha h(t)^\beta - sh(t) \end{cases} \tag{11}$$

where $k = K/AL$ and $h = H/AL$ are quantities per effective unit of labor. They assumed that the same production function applies to human capital, physical capital and consumption. It means the one unit of consumption can be transformed costlessly into either one unit of physical capital or one unit of human capital. In addition, it is assumed that human capital depreciates at the same rate as physical capital (Temple 1999: 112-156), (Durlauf 2005: 555-677).

To solve the direct problem of the second experiment, we used the following initial conditions and coefficients (true): $k(0) = 10, h(0) = 0.1, s_k = 0.3, s_h = 0.2, s = 0.042, \alpha = 0.6, \beta = 0.5$; in a time interval of up to 10 years, and the following results of solving a direct problem were obtained (Figure 2):

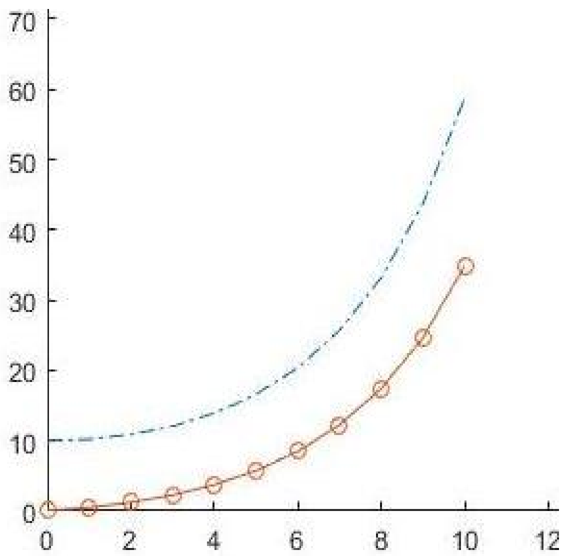

Figure 2 - The graph of direct solutions for k is the stock of physical capital (blue dotted line) and h is the stock of human capital (red line with circles)

There was illustrated that both stocks of physical and human capital experienced exponential growth at the hole period $T \in [0, 10]$ and repeating the curve of each other and doing no crossover. The graph of physical capital started from 10 billion and finished at nearly 60 billion, while the human capital began from 0.1 billion and in the end of period was at about 36 billion. The obtained results, where we took only 10 points, helped us to recreate the graph (Figure 3) using only data of stock of physical capital and to identify the unknown 5 coefficients (Table 2).

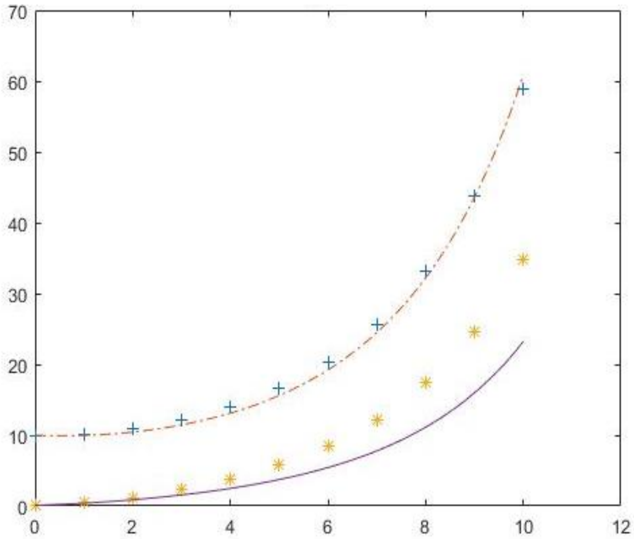

Figure 3 - Restoration of two graphs (red and purple lines) using only the obtained data on physical capital (10 blue crosses)

As it can be seen the recreated graph line goes through the given known data, which means that inverse problem solved correct enough. Also the graph repeated the exponential way of growth and did not over rose the last measured point at $T = 10$, whereas the graph line of the stock of human capital showed the similar trade and repeated the graph of direct solution (yellow stars), but from the 4th year there is started the residual and it was getting bigger with following years.

Table 2– Comparison of recovered values (using data only on physical capital) with true

Coefficients	True solution	Restored values	Difference
s_k	0.3	0.3576	0.0576
s_h	0.2	0.1469	0.0531
α	0.6	0.6599	0.0599
β	0.5	0.5031	0.0031
s	0.042	0.0806	0.0386

There clearly showed that two coefficients out five are identified almost correct with small differences as 0.0599 and 0.0576 in coefficients α and s_k, also the coefficient showed a great result with residual from true solution equal to 0.0031, while other two parameters s_h и s namely have errors and difference in values from true solution equal to 0.0531 and 0.0386 respectively.

After the quite success of restoring parameters only using the data of one variable, we tried to do it with data set of both variables and got sufficient results (Figure 4 and Table 3).

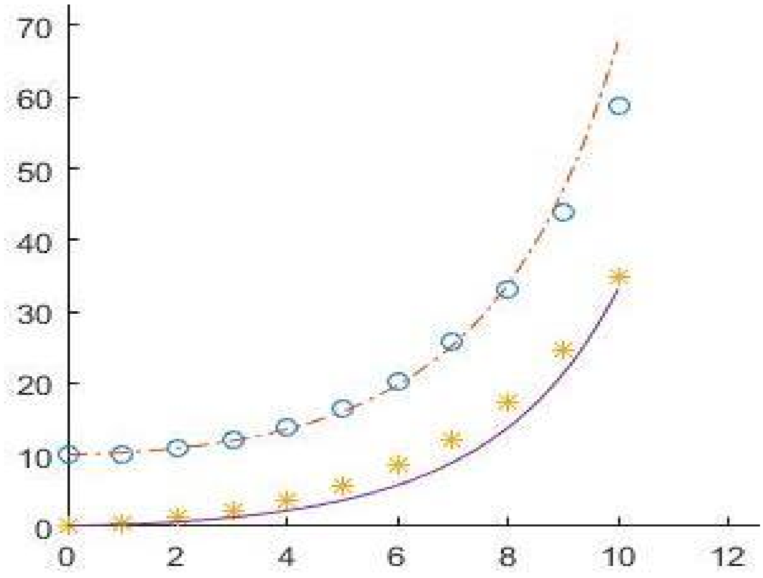

Figure 4 - Restoration of two graphs (red dashed and purple lines), using the obtained data on both physical and human capital (10 blue circles and stars)

Even though there are some residuals of physical capital with known data and over rising in the end of period and of human capital with its known data in the middle of period, the coefficients identified much better than in previous experiment (Table 3).

Table 3 – Comparison of recovered values (using both capital data) with true

Coefficients	True solution	Restored values	Difference
s_k	0.3	0.3134	0.0134
s_h	0.2	0.1783	0.0217
α	0.6	0.5212	0.0788
β	0.5	0.6402	0.1402
s	0.042	0.0038	0.0382

As it can be seen, the α and β coefficients got worse, by 0.0788 and 0.1402 namely, than in experiment written above, but also are close to the true solution. What about the parameters s_k and s_h, they showed significant progress in identifying, becoming really close to true values and decreasing the difference down to 0.0134 and 0.0217 namely. Also the s coefficient identified the way too bad and was far for 0.0382 from true value. Worse identifying of α and β parameters could be explained by similarity of equations and little difference of coefficients values.

This kind of problem with big residuals in identifying unknown parameters refers to identifiability analysis problem that could give us extra confidence in designed program, chosen method and algorithm, which helps to find unknowns by indicating identifiable and non-identifiable parameters, that some of our coefficients could be.

5 Conclusion

The work was carried out in the ICMaMG SB RAS, Novosibirsk, where an algorithm of differential evolution was considered and applied. As a result of all three experiments, it can be said that although the third coefficient (s) was incorrectly identified in all experiments and the difference varied from large to small, but still was significant than in s_k and s_h, in the second experiment. This can be defined as the non-identifiability of the coefficients or because of too big of their nonlinearity. Maybe changing the model or using more known data will give better results, as it was in the third experiment.

6 Gratitude

This work was supported by the grant of the Ministry of Education and Science of the Republic of Kazakhstan (project № AP05134121 «Numerical methods of identifiability of inverse and ill-posed problems of natural science»)

References

[1] Benhabib J. and Spiegel M. M., "The role of human capital in economic development evidence from aggregate cross-country data." *Journal of Monetary Economics* 43 (1994): 143 - 173.

[2] Berndt E. R. and Christensen L. R., "The Translog Function and the Substitution of Equipment, Structures, and Labor in U.S. manufacturing 1929–68." *Journal of Econometrics.* 1 (1973): 81–113.

[3] Cobb C. W. and Douglas P. H., "A Theory of Production." *American Economic Review.* 18 (1928): 139–165.

[4] Douglas P. H., "The Cobb-Douglas Production Function Once Again: Its History, Its Testing, and Some New Empirical Values." *Journal of Political Economy.* 84 (1976): 903–916.

[5] Durlauf S. N., Johnson P. A. and Temple J.R.W., "Chapter 8 Growth Econometrics" *Handbook of Economic Growth.* 1 (2005): 555–677.

[6] Houthakker H. S., "The Pareto Distribution and the Cobb–Douglas Production Function in Activity Analysis" *The Review of Economic Studies.* 23 (1955): 27–31.

[7] Storn R., Price K. and Lampinen R., *Differential Evolution: A Practical Approach to Global Optimization.* (Springer, 2005), 539 p.

[8] Mankiw G. N., Romer D. and Weil D. N., "A Contribution to the Empirics of Economic Growth" *The Quarterly Journal of Economics.* 107 (1992): 407–437.

[9] Nazrul I., "Growth Empirics: A Panel Data Approach" *The Quarterly Journal of Economics.* 110 (1995): 1127-1170.

[10] Solow R. M., "A contribution to the theory of economic growth" *Quarterly Journal of Economics. Oxford Journals.* 70 (1956): 65–94.

[11] Storn R. and Price K., "Differential Evolution — A Simple and Efficient Adaptive Scheme for Global Optimization over Continuous Spaces." *Technical Report TR-95-012, ICSI.* 95 (1995): 1–12.

[12] Storn R. and Price K., "Differential Evolution — A Simple and Efficient Differential Evolution — A Simple and Efficient Heuristic for Global Optimization over Continuous Spaces." *Journal of Global Optimization, ICSI.* 11 (1997): 341–359.

[13] Swan T. W., "Economic growth and capital accumulation." *Economic Record. Wiley.* 32 (1956): 334–361.

[14] Temple J., "The New Growth Evidence." *Journal of Economic Literature* 37 (1999): 112–156.

[15] Yang XS., *Nature-Inspired Optimization Algorithms.* (Elsevier, 2014), 300 p.