Numerical Investigation one System Reaction-Diffusion with Double Nonlinearity

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Abstract. In this paper we demonstrate the possibilities of the self-similar and approximately self-similar approaches for studying solutions of a nonlinear mutual reaction-diffusion system. The asymptotic behaviour of compactly supported solutions and free boundary is studied. Based on established qualitative properties of solutions numerical computation is carried out. The solutions are presented in visualization form with animation, which allows observing evolution of the studied process in time.

Keywords: reaction, diffusion, self-similar, asymptotic, compactly supported solutions, free boundary, numerical computation visualization.

1 Introduction

Let's consider properties of the Cauchy problem for following system of mutual reaction-diffusion equations with double nonlinearity in the domain $Q = \left\{(t,x) : t > 0,\, x \in R^N\right\}$

$$\begin{array}{l} \frac{\partial u}{\partial t} = div\left(|x|^k u^{m_1-1}|\nabla u|^{p-2}\nabla u\right) + \gamma(t)u^{b_1}v^{q_1}, \\ \frac{\partial v}{\partial t} = div\left(|x|^k v^{m_2-1}|\nabla v|^{p-2}\nabla v\right) + \gamma(t)v^{b_2}u^{q_2}, \end{array} \tag{1}$$

$$\begin{array}{c} u\left(0,x\right) = u_0\left(x\right) \geq 0, \\ v\left(0,x\right) = v_0\left(x\right) \geq 0, \quad x \in R^N, \end{array} \tag{2}$$

where $k \in R$, $m_1, m_2 > 1$, $p \geq 2$ b_1, b_2, q_1, q_2- positive numbers, $\nabla(\cdot) = \underset{x}{grad}(\cdot)$, and $u_0\left(x\right) \geq 0$, $v_0\left(x\right) \geq 0$, $0 < \gamma(t) \in C(0,+\infty)$.

System (1) describes different physical process in two componential inhomogeneous nonlinear environment, for example the processes of the reaction-diffusion, heat conductivity, polytrophic filtration of liquid and gas with a source power of which is equal $\gamma(t)u^{b_1}v^{q_1}$, $\gamma(t)v^{b_2}u^{q_2}$. Particular cases, when $k = 0$, $p = 2$, $m_1 = m_2 = 0$ were considered in works [1,3]. In the work [4] was investigated estimates of the different type solution, the asymptotes of the self-similar and approximately self-similar weak solutions and the front, of the problem Cauchy to the one equation of system (1).

Since the system (1) in the domain where $u = v = 0$ is degenerate, then it is in the area of degeneration may not have classical solutions. Therefore, we study the generalized solution of the system has the properties: $0 \leq u, v \in C\left(Q\right)$ и $|x|^k u^{m_1-1}|\nabla u|^{p-2}\nabla u$, $|x|^k v^{m_2-1}|\nabla v|^{p-2}\nabla v \in C(Q)$.

Constructed on the basis of self-similar approaches system self-similar solutions, estimates for the free boundary, properties of solutions of the system are obtained. On the basis of qualitative research tasks developed a set of programs for reaction- diffusion systems with double nonlinearity and numerical calculations and visualization of the reaction-diffusion system describing (1). For the numerical solution of the problem have been applied by the racing method, the method of alternating directions in the multidimensional case [1,3].

2 Approximate self-similar and self-similar equations

For construction of the self-similar and approximately self-similar system for the system (1) the solutions of the system $u(t,x)$, $v(t,x)$ we will search in the form

$$\begin{cases} u(t,x) = \bar{u}(t) w\left(\tau(t), \varphi(|x|)\right), \\ v(t,x) = \bar{v}(t) z\left(\tau(t), \varphi(|x|)\right), \end{cases} \tag{3}$$

where $\bar{u}(t)$, $\bar{v}(t)$ are the solutions of following equations

$$\frac{d\bar{u}}{dt} = \gamma(t)\bar{u}^{b_1}\bar{v}^{q_1},$$
$$\frac{d\bar{v}}{dt} = \gamma(t)\bar{v}^{b_2}\bar{u}^{q_2}.$$

Integrating the equations, when $\gamma(t) = 1$ we got

$$\bar{u} = A(T+t)^{\frac{1}{-b_1 - lq_1 + 1}},$$

$$\bar{v} = B(T+t)^{\frac{k}{-b_1 - lq_1 + 1}}.$$

where $A = \left[(-b_1 - lq_1 + 1)\left(\frac{1}{l}\right)^{\frac{q_1}{-b_2+q_1+1}}\right]^{\frac{1}{-b_1 - lq_1 + 1}}$, $B = A^l \left(\frac{1}{l}\right)^{\frac{1}{-b_2+q_1+1}}$, $l = \frac{-b_1+q_2+1}{-b_2+q_1+1}$.

Substituting of the (3) to the equation (1) reduced it to the following system of the equations

$$\begin{aligned} \frac{\partial w}{\partial \tau} &= \varphi^{1-s}\frac{\partial}{\partial \varphi}\left(\varphi^{s-1} z^{m-1}\left|\frac{\partial w}{\partial \varphi}\right|^{p-2}\frac{\partial w}{\partial \varphi}\right) - \bar{u}^{b_1-(m_1+p-2)}\bar{v}^{q_1}\left(w - w^{b_1}z^{q_1}\right) \\ \frac{\partial z}{\partial \tau} &= \varphi^{1-s}\frac{\partial}{\partial \varphi}\left(\varphi^{s-1} w^{m-1}\left|\frac{\partial z}{\partial \varphi}\right|^{p-2}\frac{\partial z}{\partial \varphi}\right) - \bar{v}^{b_2-(m_2+p-2)}\bar{u}^{q_2}\left(z - z^{b_2}w^{q_2}\right) \end{aligned} \tag{4}$$

The functions $\tau(t)$, $\varphi(x)$ are chosen as following

$$\begin{aligned} \tau(t) &= \int_0^t \bar{u}^{m_1+p-3}(\eta)\,d\eta = \int_0^t \bar{v}^{m_2+p-3}(\eta)\,d\eta, \\ &\qquad \texttt{if}\ \ m_i + p - 3 \neq 0, \quad i = 1,2, \\ \tau(t) &= T + t, \ \ \texttt{if}\ \ m_i + p - 3 = 0, \quad i = 1,2, \end{aligned} \tag{5}$$

$\varphi(r) = \frac{1}{p_1}|r|^{p_1}$, $|r| = \sqrt{\sum_{i=1}^{N} x_i^2}$, $p_1 = \frac{p-k}{p}$, $s = p\frac{N}{p-k}$, $k < p$.

The system (4) we will make a replacement a kind

$$w(\tau,\varphi) = f(\xi), \ \ z(\tau,\varphi) = \psi(\xi), \tag{6}$$

where $\xi = \frac{\varphi(r)}{\tau^{1/p}}$ and the functions $f(\xi), \psi(\xi)$ satisfied to the approximately self - similar system equations

$$\xi^{1-s}\frac{d}{d\xi}\left(\xi^{s-1} f^{m_1-1}\left|\frac{df}{d\xi}\right|^{p-2}\frac{df}{d\xi}\right) + \frac{\xi}{p}\frac{df}{d\xi} - \tau\bar{u}^{b_1-(m_1+p-2)}\bar{v}^{q_1}\left(f - f^{b_1}\psi^{q_1}\right) = 0,$$

$$\xi^{1-s}\frac{d}{d\xi}\left(\xi^{s-1} \psi^{m_2-1}\left|\frac{d\psi}{d\xi}\right|^{p-2}\frac{d\psi}{d\xi}\right) + \frac{\xi}{p}\frac{d\psi}{d\xi} - \tau\bar{v}^{b_2-(m_2+p-2)}\bar{u}^{q_2}\left(\psi - \psi^{b_2}f^{q_2}\right) = 0.$$

It is easy to calculate that

$$\tau \bar{u}^{b_1-(m_1+p-2)}\bar{v}^{q_1} \equiv 1,$$
$$\tau \bar{v}^{b_2-(m_2+p-2)}\bar{u}^{q_2} \equiv 1.$$

In this case for the functions $f(\xi)$, $\psi(\xi)$ we have the following self-similar system of equation in "radial"form

$$\begin{aligned} &\xi^{1-s}\frac{d}{d\xi}\left(\xi^{s-1}f^{m_1-1}\left|\frac{df}{d\xi}\right|^{p-2}\frac{df}{d\xi}\right)+\frac{\xi}{p}\frac{df}{d\xi}-a_1\left(f-f^{b_1}\psi^{q_1}\right)=0,\\ &\xi^{1-s}\frac{d}{d\xi}\left(\xi^{s-1}\psi^{m_2-1}\left|\frac{d\psi}{d\xi}\right|^{p-2}\frac{d\psi}{d\xi}\right)+\frac{\xi}{p}\frac{d\psi}{d\xi}-a_2\left(\psi-\psi^{b_2}f^{q_2}\right)=0,\end{aligned} \tag{7}$$

where

$$a_1 = A^{b_1-(m_1+p-2)}B^{q_1},$$
$$a_2 = B^{b_2-(m_2+p-2)}A^{q_2}.$$

Next we will study asymptotes of the finite solutions of the system (7).

3 Asymptotes of the self - similar solutions

Now we will study asymptotes of weak compactly supported solutions (c.s.) of system (7) when $\gamma(t) = 1$.

Consider system of equations (7) with the following boundary condition

$$\begin{aligned} f(0) = c_1 \geqslant 0,\ f(d_1) = 0,\\ \psi(0) = c_2 \geqslant 0,\ \psi(d_2) = 0,\end{aligned} \tag{8}$$

where $d_1, d_2 < +\infty$.

The existence of a self-similar weak solution to problem (7)-(8) for one equation, in the case $\gamma(t) = 0$, $n = l = 0$, $p = 2$ was studied in [5] and conditions for the existence of c.s. solutions were obtained. In the [4] was studied the principal terms of the asymptotic behavior of solutions of self and self-approximation equations, the behavior of the front (free boundary), depending on the setting, the condition of the global solvability of the Cauchy problem for one equation.

We will transform the system (7) to convenient for investigating form using replacement

$$\begin{aligned} f(\xi) &= \bar{f}(\xi)y_1(\eta),\\ \psi(\xi) &= \bar{\psi}(\xi)y_2(\eta), \qquad \eta = -\ln\left(c-\xi^{\frac{p}{p-1}}\right), \end{aligned} \tag{9}$$

,

and $\bar{f}(\xi) = A_1\left(c-\xi^{\frac{p}{p-1}}\right)_+$, $\bar{\psi}(\xi) = A_2\left(c-b\xi^{\frac{p}{p-1}}\right)_+$ $c > 0$, $A_i > 0$, $i = 1, 2$.

Here we have following theorem

Theorem Let $m_i = 3$, $b_i + q_i = 1$ $i = 1, 2$. Then the finite solution of the system (7) at $\xi \to c^{\frac{p-1}{p}}$ have asymptotes

$$\begin{aligned} f(\xi) &= \bar{f}(\xi)y_1^0(1+o(1)),\\ \psi(\xi) &= \bar{\psi}(\xi)y_2^0(1+o(1)),\end{aligned} \tag{10}$$

where $0 < y_i^0 < +\infty$ $(i = 1, 2)$,

$$\begin{aligned} y_1^0 &= \left[a_1^{\frac{b_2-q_1}{q_1}}a_2 s^{q_1}c^{\gamma g_1}\gamma^{-(p-1)g_1}A_1^{k_1(1-b_2)-q_2}A_2^{b_2\left(\frac{q_2}{q_1}-1\right)+(m_2+p-3)}\right]^{\frac{1}{k_1(b_2-1)+q_2}},\\ y_2^0 &= \left[a_2^{\frac{b_1-q_2}{q_2}}a_1 s^{q_2}c^{\gamma g_2}\gamma^{-(p-1)g_2}A_2^{k_2(1-b_1)-q_1}A_1^{b_1\left(\frac{q_1}{q_2}-1\right)+(m_1+p-3)}\right]^{\frac{1}{k_2(b_1-1)+q_1}},\end{aligned} \tag{11}$$

where $g_i = \frac{1+q_i-b_3-i}{q_i}$, $k_i = \frac{-b_i+(m_i+p-2)}{q_i}$, $\gamma = \frac{p-1}{p}$.

Proof. In order to prove Theorem we use transformation (9). Transformations (9) reduces self-similar system (7) to the following form

$$\begin{gathered}\frac{d}{d\eta}L_1(y_1,y_2) + \left(\frac{p}{p-1}\right)^{-1} L_1(y_1,y_2) + A_1^{-(p+1)}\frac{1}{p-1}\left(\frac{p}{p-1}\right)^{-p} e^{-\eta}(y_1-y_1')+\\ +A_1^{-p}\frac{1}{p-1}\left(\frac{p}{p-1}\right)^{-p} e^{-\eta}(y_1-y_1') + a_1A_1^{-p}\left(\frac{p}{p-1}\right)^{-p}\frac{e^{-\eta}}{(a-e^{-\eta})}y_1+\\ +a_1A_1^{-(p+1)+b_1}A_2^{q_1}\left(\frac{p}{p-1}\right)^{-p}\frac{e^{-\eta}}{(c-e^{-\eta})}y_1^{b_1}y_2^{q_1}=0,\\ \frac{d}{d\eta}L_2(y_1,y_2) + \left(\frac{p}{p-1}\right)^{-1} L_2(y_1,y_2) + A_2^{-(p+1)}\frac{1}{p-1}\left(\frac{p}{p-1}\right)^{-p} e^{-\eta}(y_1-y_1')+\\ +A_2^{-p}\frac{1}{p-1}\left(\frac{p}{p-1}\right)^{-p} e^{-\eta}(y_2-y_2') + a_2A_2^{-p}\left(\frac{p}{p-1}\right)^{-p}\frac{e^{-\eta}}{(a-e^{-\eta})}y_2+\\ +a_2A_2^{-(p+1)+b_2}A_2^{q_2}\left(\frac{p}{p-1}\right)^{-p}\frac{e^{-\eta}}{(c-e^{-\eta})}y_2^{b_2}y_1^{q_2}=0,\end{gathered} \tag{12}$$

where $L_1(y_1,y_2) = y_1^{m_1-1}(|y_1-y_1'|)^{p-2}(y_1-y_1')$,
$L_2(y_1,y_2) = y_2^{m_2-1}(|y_2-y_2'|)^{p-2}(y_2-y_2')$,

Such transformation (9) allows us to reduce studying of the asymptotes of solutions to system (7) as $\eta \to \infty$ to studying those solutions of system (12), which in some neighborhood of $+\infty$ satisfy the inequalities

$$y_i - \frac{dy_i}{d\eta} \neq 0, \quad y_i(\eta) > 0,\ i = 1,2.$$

The system (12) when $\eta \to \infty$ have following system of algebraic equations

$$\begin{gathered}\frac{p}{p-1}sy_1^{p+1} - a_1A_1^{b_1-(p+1)}A_2^{q_1}\left(\frac{p}{p-1}\right)^{-p}c^{-\frac{p-1}{p}}y_1^{b_1}y_2^{q_1}=0,\\ \frac{p}{p-1}sy_2^{p+1} - a_2A_2^{b_2-(p+1)}A_1^{q_2}\left(\frac{p}{p-1}\right)^{-p}c^{-\frac{p-1}{p}}y_2^{b_2}y_1^{q_2}=0,\end{gathered} \tag{13}$$

We have solved the system of algebraic equations and obtain

$$y_1^0 = \left[a_1^{\frac{b_2-q_1}{q_1}}a_2s^{q_1}c^{\gamma g_1}\gamma^{-(p-1)g_1}A_1^{k_1(1-b_2)-q_2}A_2^{b_2\left(\frac{q_2}{q_1}-1\right)+(m_2+p-3)}\right]^{\frac{1}{k_1(b_2-1)+q_2}},$$

$$y_2^0 = \left[a_2^{\frac{b_1-q_2}{q_2}}a_1s^{q_2}c^{\gamma g_2}\gamma^{-(p-1)g_2}A_2^{k_2(1-b_1)-q_1}A_1^{b_1\left(\frac{q_1}{q_2}-1\right)+(m_1+p-3)}\right]^{\frac{1}{k_2(b_1-1)+q_1}}.$$

On the basis of qualitative research tasks developed a set of programs for reaction-diffusion systems with double nonlinearity and numerical calculations and visualization of the reaction-diffusion system describing (1). For the numerical solution of the problem have been applied by the racing method, the method of alternating directions in the multidimensional case. Iterative processes are based on Picard's method, Newton and special method. Many experiments were carried out in different values of the parameters of the system. The results of computational experiments show that all of these iterative methods are effective for solving nonlinear problems and give non-linear effects, if selected as the initial approximation approximate self-similar solutions of the equation, constructed above method of nonlinear splitting method and standard equations [1,2,3]. As expected, the application of Newton's method was with the smallest number of iterations than the methods of Picard and a special method, the correct choice of the initial approximation.

The results of numerical experiments are presented in visual form and with animation. In the multidimensional case for the approximation of the problem, the method of alternating directions.

Were held set of numerical experiments based on the asymptotic behavior of solutions obtained above.

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