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Initial length scale estimate for the Schrödginer operator with a random fast oscillating potential in a multi-dimensional layer

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We consider the Dirichlet Laplacian in a multi-dimensional layer located between two parallel hyperplanes of codimension one. Such operator is perturbed by a fast oscillating random potential. Namely, the layer is partitioned into periodicity cells by a given periodic lattice and in each cell we consider a fast oscillating potential depending on a random variable multiplied by a global small parameters. All random variables associated with the periodicity cells are assumed to be independent and identically distributed. The fast oscillating potential introduced in the way standard for the homogenization theory. Namely, it depends on slow and fast variables, is compactly supported w.r.t. the slow variables and is periodic w.r.t. the fast ones. The main obtained result is the initial length scale estimate for the considered operator. Such estimate is the induction base for proving the spectral localization at the bottom of the spectrum by the multiscale analysis.
Key words: random Hamiltonian, fast oscillating potential, initial length scale estimate, small parameter, multi-dimensional layer.

Көп өлшемді қабаттағы кездейсоқ тез тербелуші шамамен Шрендингер операторына арналған бастауыш өлшемдерді бағалау

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Бір мөлшерлестік екі қатар гипержазықтығының арасында оналасқан көп өлшемді қабатта Дирихле шартымен Лапласиан қарастырылады. Аталмыш оператор кездейсақ тез тербелуші шамамен қозғалады. Атап айтқанда, қабат кейбір периодтық ұяшықтарына бөлінеді және әр ұяшықта ғаламдық кіші параметрге көбейтілген кездейсоқ тұрақсыз шамалар тәуелсіз және бірдей бөлінгендермен бағамдалады. Тез тербелуші шама орташалау теориясына әдеттегідей енгізіледі. Атап айтқанда, ол жылдам және баяу тұрақсыз шамасына тәуелді, баяу тұрақсыз шамасы бойынша финитен жылдам тұрақсыз шамасы бойыншада мерзімді. Қол жеткізілген негізгі нәтиже-қарастырылатын операторға бастауыш өлшемдерді бағалау. Осы баға көп масштабты талдау арқылы спектрдің төменгі қырындағы спектрлік оқшаулауды дәлелдеу мақсатындағы индукция базасы болып табылады.
Түйін сөздер: кездейсоқ гамильтониан, тез тербелуші шам, бастауыш өлшемдерді бағалау, кіші параметр, көп өлшемді қабат.

Оценка начальных масштабов для оператора Шредингера со случайным быстро осциллирующим потенциалом в многомерном слое

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Рассматривается Лапласиан с условием Дирихле в многомерном слое, заключенном между двумя параллельными гиперплоскостями коразмерности один. Такой оператор возмущается быстро осциллирующими случайным потенциалом. А именно, слой делится на периодические ячейки некоторой периодической решеткой и в каждой ячейке рассматривается быстро осциллирующий потенциал, зависящий от случайной переменной, умноженной на глобальный малый параметр. Все случайные переменные, соответствующие ячейкам периодичности, предполагаются независимыми и одинаково распределенными. Быстро осциллирующий потенциал вводится обычным для теории усреднении образом. А именно, он зависит от быстрых и медленных переменных, финитен по медленным переменным и периодичен по быстрым переменным. Главный полученный результат – оценка начальных масштабов для рассматриваемого оператора. Такая оценка является базой индукции для доказательства спектральной локализации на нижнем краю спектра с помощью многомасштабного анализа.
Ключевые слова: случайный гамильтониан, быстро осциллирующий потенциал, оценка начальных масштабов, малый параметр, многомерный слой

1 Introduction

Random Hamiltonians are elliptic operators depending on countably many random variables. Such operators attract a lot of interest since they are often used for describing waves in disordered media and they possess many interesting mathematical properties. One of such properties is the spectral Anderson localization. It is known that the spectrum of many Random Hamiltonians is almost surely a fixed deterministic set. And spectral localization says that some part of this spectrum is pure point. Such property was found for many particular examples, see, for instance, (Martinelli, 1984 : 197-217), (Fröhlich, 1983 : 151-184), (Baker, 2008 : 397-415), (Borisov, 2011 : 58-77), (Borisov, 2013 : 2877-2909), (Bourgain, 2009 : 969-978), (Erdös, 2012a : 900-923), (Erdös, 2012b : 507-542), (Ghribi, 2007 : 123-138), (Ghribi, 2010 : 127-149), (Hislop, 2002 : 12-47), (Kleespies, 2000 : 1345-1365), (Klopp, 1993 : 810-841), (Klopp, 1995a : 265-316), (Klopp, 1995b : 553-569) , (Klopp, 2002 : 711-737), (Klopp, 2012 : 587-621), (Klopp, 2009 : 1133-1143), (Klopp, 2003 : 795-811), (Kostrykin, 2006 : 267-392), (Lenz, 2008 : 121-161), (Lenz, 2009 : 219-254), (Lenz, 2004 : 733-752), (Leonhardt, 2015), (Stolz, 2000 : 173-183), (Ueki, 1994 : 10) , (Ueki, 2000 : 473-498), (Ueki, 2008 : 565-608), (Veselić, 2002 : 199-214), and the references therein. One of main ways for proving the spectral localization is the multiscale analysis, (Martinelli, 1984 : 197-217), (Fröhlich, 1983 : 151-184). It is based on a certain induction whose basis is the initial length scale estimate.

In papers (Borisov, 2016 : 2341-2377), (Borisov, 2017), a general approach was developed for proving initial length scale estimate for operators with small random perturbations. The main advantage of the obtained result is that the perturbations were described by abstract symmetric operators covering many particular examples considered before. The core of the approach was an original technique based on a non-symmetric version of the Birman-Schwinger principle, which gave an opportunity to prove an important deterministic estimate. Such technique allowed the authors to consider perturbations non-monotone w.r.t. the random variables and this was an important step in studying random Hamiltonians.

The present paper can be considered as an example of random perturbation, to which general results of work (Borisov, 2017) can be applied. A non-trivial feature of this example is that it is described by a compactly supported fast oscillating potential. Such perturbation is singular and does not fit the assumptions made for the perturbation in (Borisov, 2017). However, here we employ a special transformation of the operator proposed in (Borisov, 2015 : 33-54) so that this transformation preserves the spectrum and reduces the considered operator to one fitting the assumptions of (Borisov, 2017). Finally, this allows us to prove the initial length scale estimate for the considered model.

2 Problem and main results

Let $x' = (x_1, \dots, x_n)$, $x = (x', x_{n+1})$ be Cartesian coordinates in $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$, respectively, $n \geqslant 1$, $\Pi := \{x : 0 < x_{n+1} < d\}$ be a multidimensional layer of width $d > 0$, Γ be a periodic lattice in $\mathbb{R}^n$ with a basis $e_1, \dots, e_n$ and a periodicity cell $\square' := \Big\{x : x' = \sum\limits_{j=1}^{n} a_j e_j,\ a_j \in (0,1)\Big\}$. Denote $\square := \square' \times (0, d)$.

By $W_0 = W_0(x, \xi)$, $\xi = (\xi_1, \dots, \xi_n)$, we denote a real-valued function defined on $\mathbb{R}^{2n+2}$. We assume that this function is 1-periodic w.r.t. each variable ξ_i, $i = 1, \dots, n+1$, has a zero mean

$$\int\limits_{(0,1)^{n+1}} W_0(x,\xi)\, d\xi = 0 \quad \text{for each} \quad x \in \mathbb{R}^{n+1}, \tag{1}$$

and is a compactly supported as a function of x:

$$\operatorname{supp} W_0(\cdot, \xi) \subseteq M \Subset \square \quad \text{for each} \quad \xi \in \mathbb{R}^{+1}, \tag{2}$$

where M is some fixed set independent of ξ. The function W_0 is supposed to have the following smoothness:

$$\frac{\partial^{|\alpha|+|\beta|} W_0}{\partial x^\alpha \partial \xi^\beta} \in C(\mathbb{R}^{2n+2}), \quad \alpha, \beta \in \mathbb{Z}_+^{n+1}, \quad |\alpha| \leqslant 3, \quad |\beta| \leqslant 1. \tag{3}$$

By ε we denote a small positive parameter and we introduce one more function:

$$\begin{aligned} &W(x,\varepsilon) := \varepsilon^{-b} W\left(x, \frac{x}{\varepsilon}\right), && \varepsilon > 0, \\ &W(x,0) := 0, && \varepsilon = 0, \end{aligned} \tag{4}$$

where $b < 1$ is a given fixed number.

Let $\zeta = (\zeta_k)_{k\in\Gamma}$ be a sequence of independent identically distributed random variables with the values in segment $[0,1]$; the associated distribution measure is denoted by μ. We assume that this measure is defined on $[0,1]$. By $\mathbb{P} := \bigotimes_{k\in\Gamma} \mu$ we denote the product of the measures on space $\Omega := \times_{k\in\Gamma}[0,1]$. The elements of the latter space are sequences $(\zeta_k)_{k\in\Gamma}$. By $\mathbb{E}(\cdot)$ we denote the expectation value of a random variable w.r.t. the probability $\mathbb{P}$.

In this paper we consider the following perturbed random operator:

$$\mathcal{H}^\varepsilon(\zeta) := -\Delta + \sum_{k\in\Gamma} W(x' - k, x_{n+1}, \varepsilon\zeta_k) \tag{5}$$

in $L_2(\Pi)$ subject to the Dirichlet boundary condition. The domain of the operators $\mathcal{H}^\varepsilon(\zeta)$ is the space $\mathring{W}_2^2(\Pi, \partial\Pi)$, where $\mathring{W}_2^2(Q, S)$ is the Sobolev space of the functions in $W_2^2(Q)$ vanishing on a surface $S \subset Q$, Q is a domain. This operator is self-adjoint.

The main aim of this paper is to prove the initial length scale estimate for $\mathcal{H}^\varepsilon(\zeta)$. In order to formulate this estimate, we need to introduce additional notations. First we introduce an auxiliary operator

$$\mathcal{H}^\varepsilon_\square := -\Delta + W(x,\delta), \quad \delta\in[\delta_0,\delta_0], \tag{6}$$

in $L_2(\square)$ subject to the Dirichlet boundary condition on $\partial\square\cap\partial\Pi$ and to the periodic boundary condition on the lateral boundaries γ of $\square$, and δ_0 is a fixed number. The domain of $\mathcal{H}^\varepsilon_\square$ consists of the functions in $\mathring{W}_2^2(\square,\partial\square\cap\partial\Pi)$ satisfying periodic boundary conditions on γ. Operator $\mathcal{H}^\delta_\square$ is self-adjoint.

Let Λ_δ be the smallest eigenvalue of $\mathcal{H}^\delta_\square$, $\Psi_\delta=\Psi_\delta(x)$ be the associated eigenfunction normalized in $L_2(\square)$. On the lateral surface of the cell $\square$ we define the function

$$\rho_\delta := \frac{1}{\Psi_\delta}\frac{\partial\Psi_\delta}{\partial\nu}, \tag{7}$$

where ν is the outward normal.

Given $\alpha\in\Gamma$, $N\in\mathbb{N}$, the symbols $\Pi_{\alpha,N}$ and $\Gamma_{\alpha,N}$ denote a piece of Π and a piece of Γ:

$$\Pi_{\alpha,N} := \left\{x:\ x'=\alpha+\sum_{i=1}^{n} a_i e_i,\ a_i\in(0,N),\ 0<x_{n+1}<d\right\},$$

$$\Gamma_{\alpha,N} := \left\{x'\in\Gamma:\ x'=\alpha+\sum_{i=1}^{n} a_i e_i,\ a_i=0,1,\ldots,N-1\right\}.$$

By $\mathcal{H}^\varepsilon_{\alpha,N}(\zeta)$ we denote a kind of "restriction" of the operator $\mathcal{H}^\varepsilon(\zeta)$ on $\Pi_{\alpha,N}$. Namely,

$$\mathcal{H}^\varepsilon_{\alpha,N}(\zeta) := -\Delta + \sum_{k\in\Gamma_{\alpha,N}} W(x'-k,x_{n+1},\varepsilon\zeta_k) \tag{8}$$

is the operator in $L_2(\Pi_{\alpha,N})$ subject to the Dirichlet condition on $\partial\Pi_{\alpha,N}\cap\partial\Pi$ and to the Robin condition

$$\left(\frac{\partial}{\partial\nu}-\rho_\varepsilon\right)u=0 \quad \text{on} \quad \gamma_{\alpha,N}, \tag{9}$$

where $\gamma_{\alpha,N}$ is the lateral boundary of $\Pi_{\alpha,N}$. The domain of the operator $\mathcal{H}^\varepsilon_{\alpha,N}(\zeta)$ consists of the functions in $\mathring{W}_2^2(\Pi_{\alpha,N},\partial\Pi_{\alpha,N}\cap\partial\Pi)$ satisfying condition (9). The operator $\mathcal{H}^\varepsilon_{\alpha,N}(\zeta)$ is self-adjoint.

Let $\Lambda^\varepsilon_{\alpha,N}$ be the smallest eigenvalue of the operator $\mathcal{H}^\varepsilon_{\alpha,N}(\zeta)$.

Our first result provides a lower bound for the gap between $\Lambda^\varepsilon_{\alpha,N}(\zeta)$ and Λ_ε.

Theorem 1 *There exist constants $N_1\in\mathbb{N}$ and $c_0>0$, $\eta>0$ such that for all $\alpha\in\Gamma$, $N\geqslant N_1$, $\varepsilon\leqslant c_0N^{-4}$ the inequality*

$$\Lambda^\varepsilon_{\alpha,N}(\zeta)-\Lambda_\varepsilon \geqslant \frac{\varepsilon^{2-2b}}{\eta N^n}\sum_{k\in\Gamma_{\alpha,N}}(1-\zeta_k^{1-b}) \tag{10}$$

holds true for all ζ.

The next result is the Combes-Thomas estimate.

Theorem 2 *Let α, β_1, $\beta_2 \in \Gamma$, m_1, $m_2 \in \mathbb{N}$ be such that $B_1 := \Pi_{\beta_1,m_1} \subset \Pi_{\alpha,N}$, $B_2 := \Pi_{\beta_2,m_2} \in \Pi_{\alpha,N}$. There exists $N_2 \in \mathbb{N}$ such that for $N \geqslant N_2$ the estimate*

$$\|\chi_{B_1}(\mathcal{H}^{\varepsilon}_{\alpha,N}(\zeta) - \lambda)^{-1}\chi_{B_2}\| \leqslant \frac{C_1}{\delta} e^{-C_2\delta \operatorname{dist}(B_1,B_2)}$$

holds true, where C_1, C_2 are positive constants independent of ε, α, N, B_1, B_2, m_1, m_2, λ and $\delta := \operatorname{dist}(\lambda, \sigma(\mathcal{H}^{\varepsilon}_{\alpha,N}(\zeta))) > 0$, $\sigma(\cdot)$ is the spectrum of an operator, $\chi_{\cdot}$ is the characteristic function of a set, $\|\cdot\|$ is the norm of an unbounded operator in $L_2(\Pi_{\alpha,N})$.

Two above theorems are in fact deterministic results since they are valid for each ζ. The next two theorems provide probabilistic results. The first of them reads as follows.

Theorem 3 *Let $\gamma \in \mathbb{N}$, $\gamma \geqslant 17$. Then for $N \geqslant N_1$, where N_1 was defined in Theorem 1, the interval*

$$J_N := \left[\frac{2^{\frac{2}{1-b}}}{\eta^{\frac{1}{1-b}}\big(\mathbb{E}(\zeta_0^{\frac{1-b}{2}})\big)^{\frac{2}{1-b}} N^{\frac{8}{1-b}}}, \frac{c_0}{N^{\frac{8}{\gamma(1-b)}}}\right]$$

is non-empty and for $N \geqslant N_3$, where N_3 is a some constant depending on γ, N_1, N_2, η, c_0, $\mathbb{E}(\zeta_0^{\frac{1-b}{2}})$, and for $\varepsilon \in J_N$ the estimate

$$\mathbb{P}\big(\xi \in \Omega : \Lambda^{\varepsilon}_{\alpha,N}(\xi) - \Lambda^{\varepsilon} \leqslant N^{-\frac{1}{2}}\big) \leqslant N^{n\left(1-\frac{1}{\gamma}\right)} e^{-c_1 N^{\frac{n}{\gamma}}}$$

holds true, where constant c_1 depends on measure μ only.

And the second probabilistic result is the initial length scale estimate.

Theorem 4 *Let $\alpha \in \Gamma$, $\gamma \geqslant 17$, β_1, $\beta_2 \in \Gamma_{\alpha,N}$, m_1, $m_2 > 0$ be such that $B_1 := \Pi_{\beta_1,m_1} \in \Pi_{\alpha,N}$, $B_2 := \Pi_{\beta_2,m_2} \in \Pi_{\alpha,N}$. There exists a constant $c_2 > 0$ independent of ε, α, N, β_1, β_2, m_1, m_2 such that for $N \geqslant N_3$, $\varepsilon \in J_N$*

$$\begin{aligned}&\mathbb{P}\left(\forall \lambda \leqslant \Lambda_{\varepsilon} + \frac{1}{2\sqrt{N}} : \|\chi_{B_1}(\mathcal{H}^{\varepsilon}_{\alpha,N}(\xi) - \lambda)^{-1}\chi_{B_2}\| \leqslant 2\sqrt{N} e^{-c_2 \frac{\operatorname{dist}(B_1,B_2)}{\sqrt{N}}}\right) \\ &\quad \geqslant 1 - N^{n\left(1-\frac{1}{\gamma}\right)} e^{-c_1 N^{\frac{n}{\gamma}}},\end{aligned}$$

where the same notations were used as in Theorem 3.

Let us discuss briefly the main results. Theorem 1 says that the minimum of the eigenvalue $\Lambda^{\varepsilon}_{\alpha,N}(\zeta)$ w.r.t. ζ is attained when all ζ_k takes their maximal values $\zeta_k = 1$. Moreover, it provides an effective lower bound for the distance from $\Lambda^{\varepsilon}_{\alpha,N}(\zeta)$ to its minimum. The second theorem gives a standard Combes-Thomas estimate, which states in fact the exponential decay of the Green function for the operator $\mathcal{H}^{\varepsilon}_{\alpha,N}$. These two theorems are main tools in proving further deterministic results. The first of them, Theorem 3 states that the probability of close location of $\Lambda^{\varepsilon}_{\alpha,N}(\zeta)$ is exponentially small as N grows. And this allows us to prove then the initial length scale estimate in Theorem 4. As it was mentioned in the Introduction, such estimate is the induction base for proving Anderson localization by the multiscale analysis.

3 Proof of the main result

The proof of Theorems 1, 2, 3, 4 is based on the general results obtained in (Borisov, 2017). First we describe briefly these results.

Let $\mathcal{L}(t)$, $t \in [0, t_0]$, be a family of linear operators

$$\mathcal{L}(t) = t\mathcal{L}_1 + t^2\mathcal{L}_2 + t^3\mathcal{L}_3(t), \tag{11}$$

where $\mathcal{L}_i$ are bounded symmetric operators from $W_2^2(\square)$ into $L_2(\square)$. The operator $\mathcal{L}_3(t)$ is assumed to be bounded uniformly in t and Lipschitz continuous:

$$\left\|\big(\mathcal{L}_3(t_2) - \mathcal{L}_3(t_1)\big)u\right\|_{L_2(\square)} \leqslant C|t_2 - t_1|\|u\|_{W_2^2(\square)} \tag{12}$$

for all $t_1, t_2 \in [0, t_0]$, $u \in W_2^2(\square)$ with a constant C independent of t_1, t_2, u.

Let $\mathcal{H}_\square^\delta$ be the operator

$$\mathcal{H}_\square^\delta := -\Delta + \mathcal{L}(\delta)$$

in $L_2(\square)$ subject to the Dirichlet condition on $\partial\square\cap\partial\Pi$ and to the periodic boundary condition on γ. As $\delta = 0$, the operator $\mathcal{H}_\square^\delta$ becomes just a pure Laplacian subject to the above described boundary conditions. The domain of $\mathcal{H}_\square^\delta$ consists of the functions in $\mathring{W}_2^2(\square, \partial\square\cap\partial\Pi)$ satisfying periodic boundary condition on γ.

The smallest eigenvalue Λ_0 of $\mathcal{H}_\square^\delta$ is the smallest eigenvalue of the operator $-\frac{d^2}{dx_{n+1}^2}$ on $(0, d)$ subject to the Dirichlet condition. The associated eigenfunction $\hat{\Psi}_0$ is supposed to be normalized as

$$\|\Psi_0\|_{L_2(0,d)} = \frac{1}{\sqrt{|\square'|}}. \tag{13}$$

The first assumption made in (Borisov, 2017) was as follows.

(A1). The identity $(\mathcal{L}_1\Psi_0, \Psi_0)_{L_2(\square)} = 0$ holds true.

This assumption implies that the equation

$$(\mathcal{H}_\square^0 - \Lambda_0)\Psi_1 = -\mathcal{L}_1\Psi_0 \tag{14}$$

is solvable and has a unique solution orthogonal to Ψ_0 in $L_2(\square)$. Hereafter by the symbol Ψ_1 we denote exactly such solution.

The second assumption was

(A2). The inequality $\Lambda_2 := (\Psi_1, \mathcal{L}_1\Psi_0)_{L_2(\square)} + (\mathcal{L}_2\Psi_0, \Psi_0)_{L_2(\square)} < 0$ holds true.

By $\Phi_1 \in \mathring{W}_2^2(\square, \partial\square \cap \partial\Pi)$ we denote the solution to the problem

$$(-\Delta - \Lambda_0)\Phi_1 = -\mathcal{L}_1\Psi_0 \quad \text{in} \quad \square, \qquad \Phi_1 = 0 \quad \text{on} \quad \partial\square\cap\partial\Pi, \qquad \frac{\partial\Phi_1}{\partial\nu} = 0 \quad \text{on} \quad \gamma, \tag{15}$$

orthogonal to Ψ_0 in $L_2(\square)$. It was shown in (Borisov, 2017) that this problem is uniquely solvable.

The next assumptions made in (Borisov, 2017) was

(A3). The inequality $\eta := -\Lambda_2 + \big(\mathrm{Re}\,(\Phi_1 - \Psi_1), \mathrm{Re}\,\mathcal{L}_1\Psi_0\big)_{L_2(\square)} \geqslant 0$ holds true.

There was one more assumption in (Borisov, 2017). It was shown that this fourth assumption is satisfied provided the operator $\mathcal{L}(t)$ is real-valued in the sense that $\mathcal{L}(t)u$ is a real-valued function for real-valued u.

In our case the multiplication by the potential $W(x,t)$ can not be represented as some operator $\mathcal{L}(t)$ defined by (11). This is why we employ the approach proposed in (Borisov, 2015 : 21).

Let $W_1 = W_1(x,\xi)$ be the solution to the equation

$$\Delta_\xi W_1(x,\xi) = W_0(x,\xi), \quad \xi \in (0,1)^{n+1} \tag{16}$$

subject to periodic boundary conditions obeying the orthogonality condition:

$$\int\limits_{(0,1)^{n+1}} W_1(x,\xi)\,d\xi = 0, \quad x \in \mathbb{R}^{n+1}. \tag{17}$$

It was shown in (Borisov, 2015 : 21) that this problem is uniquely solvable. We introduce extra two functions:

$$W_*(x,\varepsilon) := \varepsilon^{-b} W_1\left(x, \frac{x}{\varepsilon}\right), \quad \varepsilon > 0, \qquad W_*(x,0) := 0, \quad \varepsilon = 0, \tag{18}$$

and

$$Q_{\alpha,N}(x,\varepsilon,\zeta) := 1 + \sum_{k\in\Gamma_{\alpha,N}} (\varepsilon\zeta_k)^{2-b} W_*(x'-k, x_{n+1}, \varepsilon\zeta_k). \tag{19}$$

By $\mathcal{V}^\varepsilon_{\alpha,N}(\zeta)$ we denote the operator of multiplication by $Q_{\alpha,N}(x,\varepsilon,\zeta)$. It was shown in (Borisov, 2015 : 21) that

$$\begin{aligned}\big(\mathcal{V}^\varepsilon_{\alpha,N}(\zeta)\big)^{-1}\mathcal{H}^\varepsilon_{\alpha,N}(\zeta)\mathcal{V}^\varepsilon_{\alpha,N}(\zeta) = -\Delta + \sum_{k\in\Gamma_{\alpha,N}} (\varepsilon\zeta_k)^{1-b}\Bigg(\sum_{j=1}^{n+1} A_j(x'-k, x_{n+1}, \varepsilon\zeta_k)\frac{\partial}{\partial x_j} + \\ + A_0(x'-k, x_{n+1}, \varepsilon\zeta_k)\Bigg).\end{aligned} \tag{20}$$

The operator in the right hand side of this identity is considered in $L_2(\Pi_{\alpha,N})$ subject to the same boundary conditions as $\mathcal{H}^\varepsilon_{\alpha,N}(\zeta)$. The coefficients A_j, A_0 are given by the formulae

$$\begin{aligned}A_j(x,\varepsilon) &:= -\frac{\varepsilon}{1+\varepsilon^{2-b}W_*(x,\varepsilon)}\frac{\partial W_*}{\partial x_j}(x,\varepsilon),\\ A_0(x,\varepsilon) &:= -\frac{1}{1+\varepsilon^{2-b}W_*(x,\varepsilon)}\Bigg(2\sum_{j=1}^{n+1}\frac{\partial^2 W_1}{\partial x_j \partial\xi_j}\left(x,\frac{x}{\varepsilon}\right) + \varepsilon(\Delta_x W_1)\left(x,\frac{x}{\varepsilon}\right) \\ &\qquad + \varepsilon^{1-b} W_1\left(x,\frac{x}{\varepsilon}\right) W_*(x,\varepsilon)\Bigg).\end{aligned}$$

The first two terms in the brackets in the right hand side of the formula for A_0 should be treated in the sense of the partial derivatives w.r.t. x and ξ for function $W_1(x,\xi)$ followed by the substitution $\xi=\frac{x}{\varepsilon}$. The functions $A_j(x,\varepsilon)$, $A_0(x,\varepsilon)$ are bounded uniformly in x, ε and ζ_k.

The right hand side of identity (20) can be represented as (11) satisfying at the same time Assumptions (A1), (A2), (A3). Namely, the operators $\mathcal{L}_i$ are supposed to be depending on t, $\mathcal{L}_i(t)=\mathcal{L}_i(t)$, and are introduced as

$$\begin{aligned}&\mathcal{L}_1(t):=\sum_{j=1}^{n+1}K_j^{(1)}(x,t)\frac{\partial}{\partial x_j}+K_0^{(1)}(x,t),\\&\mathcal{L}_2(t):=K_0^{(2)}(x,t),\quad \mathcal{L}_3(t):=0,\end{aligned}\tag{21}$$

as $t>0$, where

$$\begin{aligned}K_j^{(1)}(x,t):=&\frac{t^{\frac{1}{1-b}}}{1+t^{\frac{2-b}{1-b}}W_*(x,t^{\frac{1}{1-b}})}\frac{\partial}{\partial x_j}W_*(x,t^{\frac{1}{1-b}}),\\K_0^{(1)}(x,t):=&-\frac{1}{1+t^{\frac{2-b}{1-b}}W_*(x,t^{\frac{1}{1-b}})}\Bigg(2\sum_{j=1}^{n+1}\frac{\partial^2 W_1}{\partial x_j\partial\xi_j}\left(x,\frac{x}{t^{\frac{1}{1-b}}}\right)\\&+t^{\frac{1}{1-b}}(\Delta_x W_1)\left(x,\frac{x}{t^{\frac{1}{1-b}}}\right)\Bigg)+t\,P(t^{\frac{1}{1-b}}),\\K_0^{(2)}(x,t):=&-P(t^{\frac{1}{1-b}})+\frac{W_1\left(x,\frac{x}{t^{\frac{1}{1-b}}}\right)W_*\left(x,\frac{x}{t^{\frac{1}{1-b}}}\right)}{1+t^{\frac{2-b}{1-b}}W_*(x,t^{\frac{1}{1-b}})},\\P(t):=&\frac{t^{a-1}}{|\square'|}\int\limits_{\square}\frac{\psi_0^2(x_{n+1})}{1+t^{2-b}W_*(x,\varepsilon)}\Bigg(2\sum_{j=1}^{n+1}\frac{\partial^2 W_1}{\partial x_j\partial\xi_j}\left(x,\frac{x}{t}\right)+\varepsilon(\Delta_x W_1)\left(x,\frac{x}{t}\right)\Bigg)\,dx,\end{aligned}$$

and for $t=0$, operators $\mathcal{L}_i$ are determined by the formulae

$$\begin{aligned}&\mathcal{L}_1(0):=0,\quad \mathcal{L}_3(0):=0,\\&\mathcal{L}_2(0):=-\frac{1}{2}\int\limits_{\square}dx\,\psi_0^2(x_{n+1})\int\limits_{(0,1)^{n+1}}|\nabla_\xi W_1(x,\xi)|^2\,d\xi.\end{aligned}\tag{22}$$

It is easy to make sure that under such choice of operator $\mathcal{L}(t)$, the perturbation in right hand side of (20) becomes (11), if as a new small parameter we choose ε^{1-b}, and as new random variables we take ζ_k^{1-b}.

The above obtained operator $\mathcal{L}(t)$ is not symmetric, since the operator $\mathcal{L}_i$ are not. This cause no big troubles since the symmetricity of these operators was employed in (Borisov, 2015 : 21) just in certain steps, which can be verified independently. The next point is that the operators $\mathcal{L}_1$, $\mathcal{L}_2$ depend on t. In this case the technique of (Borisov, 2017) works as well, just Assumptions (A1), (A2), (A3) are to be satisfied uniformly in t.

In (Borisov, 2017), the symmetricity of $\mathcal{L}(t)$ was employed to ensure the realness of eigenvalues Λ_0 and $\Lambda_{\alpha,N}^{\varepsilon}(\zeta)$. This is obviously true in our case. The next point, where the symmetricity was used, is the bracketing at several steps. In our case the bracketing can be done for the original operators $\mathcal{H}^{\delta}_{\square}$ and $\mathcal{H}^{\varepsilon}_{\alpha,N}(\zeta)$ since they are self-adjoint and then transformation (20) should be applied. Then all the proofs from (Borisov, 2017) can be reproduced literally. This is why in our case we need just to check Assumptions (A1), (A2),(A3) uniformly in t.

It was shown in (Borisov, 2015 : 21) that Assumption (A1) holds true for the operator $\mathcal{L}_1$ defined by (21). As it was proved in (Borisov, 2015 : 21, Eq. (5.15)), we have the inequality

$$\big(\mathcal{L}_2(t)\Psi_0,\Psi_0\big)_{L_2(\square)} = -\int\limits_{\square} dx\Psi_0^2(x_{n+1}) \int\limits_{(0,1)^{n+1}} |\nabla_\xi W_1(x,\xi)|^2\,d\xi + O\big(t^{\frac{1}{1-b}}\big),\quad t\to+0. \tag{23}$$

It was also proved that

$$\begin{aligned}\big(\mathcal{L}_1(t)\Psi_0\big)(x,t) &= t\,P(t^{\frac{1}{1-b}})\Psi_0(x_{n+1})\\ &-\frac{\Psi_0(x_{n+1})}{1+t^{\frac{2-b}{1-b}}W_*(x,t^{\frac{1}{1-b}})}\left(2\sum_{j=1}^{n+1}\frac{\partial^2 W_1}{\partial x_j\partial\xi_j}\left(x,\frac{x}{t^{\frac{1}{1-b}}}\right)+t^{\frac{1}{1-b}}(\Delta_x W_1)\left(x,\frac{x}{t^{\frac{1}{1-b}}}\right)\right).\end{aligned}$$

Given such right hand in equation (14), one can construct the asymptotic expansion for its solution Ψ_1 by the multiscale method (Bakhvalov, 1984). This expansion holds true at least in $L_2(\square)$-norm and it implies that $\Psi_1 = O\big(t^{\frac{2}{1-b}}\big)$. Hence,

$$(\Psi_1,\mathcal{L}_1\Psi_0)_{L_2(\square)} = O\big(t^{\frac{2}{1-b}}\big),\quad t\to 0. \tag{24}$$

This identity and (23) yield

$$\big(\mathcal{L}_2(t)\Psi_0,\Psi_0\big)_{L_2(\square)} - \big(\Psi_1,\mathcal{L}_1(t)\Psi_0\big)_{L_2(\square)} \leqslant -\frac{1}{2}\int\limits_{\square} dx\Psi_0^2(x_{n+1}) \int\limits_{(0,1)^{n+1}} |\nabla_\xi W_1(x,\xi)|^2\,d\xi$$

that proves Assumption (A2).

As in the case of equation (14), one can construct the asymptotics for the solution to problem (15) by the multiscale method. It gives that $\Phi_1 = O(t^{\frac{2}{1-b}})$ in $L_2(\square)$-norm. This identity and (24) lead us to the formula

$$\begin{aligned}-\Lambda_2 + \big(\mathrm{Re}\,(\Phi_1-\Psi_1),\mathrm{Re}\,\mathcal{L}_1\Psi_0\big)_{L_2(\square)} &= \int\limits_{\square} dx\Psi_0^2(x_{n+1}) \int\limits_{(0,1)^{n+1}} |\nabla_\xi W_1(x,\xi)|^2\,d\xi + O\big(t^{\frac{1}{1-b}}\big)\\ &\geqslant \frac{1}{2}\int\limits_{\square} dx\Psi_0^2(x_{n+1}) \int\limits_{(0,1)^{n+1}} |\nabla_\xi W_1(x,\xi)|^2\,d\xi\end{aligned}$$

for sufficiently small t. This, Assumption (A3) is satisfied.

Since the functions W_0, W_1 are real-valued, for each real-valued function $u\in W_2^2(\square)$ the functions $\mathcal{L}_1(t)u$, $\mathcal{L}_2(t)u$ are real-valued. This ensures the fourth assumption in (Borisov, 2017).

Thus, we can apply the general results in (Borisov, 2017) to our model and it leads us to Theorems 1, 2, 3, 4.

4 Conclusion

One of main aims of the present paper is to show that the general results obtained previously in [1] can be also applied to certain singular perturbations. The main tool is to employ a non-unitary transformation of the operator, in fact, a certain mutliplication operator. This operator is to be introduced in such a way to remove the singular perturbation from the operator and replace it by a regular one. This approach was succesfully realized in [2] and we show that it works perfectly for our case as well. And as our main results show, the considered perturbation by a fast oscillating potential is negative in the sense that it shifts the spectrum down.

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