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The numerical solution of the initial and boundary value problem for one-dimensional nonstationary nonlinear Boltzmann's six-moment system equations with the Vladimirov-Marshak boundary conditions

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The Boltzmann equation is a complex integral-differential equation and the basis of the kinetic theory of gases. It describes the behavior of a rarefied gas in space of time and velocity. It is used to the study of electron transport in solids and plasmas, neutron transport in nuclear reactors, in the tasks of remote sensing of the Earth from Space. The moment method is one of effective methods for solution of the Boltzmann equation. The system of Boltzmann's moment equations is intermediate between kinetic and hydrodynamic levels of description of state of the rarefied gas and form class of nonlinear partial differential equations. If particle distribution function will be decomposed into an Fourier series on complete orthogonal system of functions, then Boltzmann's equation will be equivalent to an infinite system of partial differential equations relative to the moments of the particle distribution function in the complete system of eigenfunctions of linearized operator. But solving infinite system of differential equations impossible. Therefore, an approximate solution of the initial and boundary value problem for the Boltzmann equation can be determined by the moment method. This article describes the one-dimensional nonlinear nonstationary Boltzmann's moment system equations in the third approximation, which a hyperbolic system of partial differential equations and contains six equations for the moments of the particle distribution function. And formulates the statement of the initial and boundary value problem for the Boltzmann's moment system equations in the third approximation and shows the results of the numerical solution with the Vladimirov-Marshak boundary conditions.
Key words: Boltzmann's six moment system equations, Vladimirov-Marshak boundary conditions, particle distribution function.

Владимиров-Маршак шеттік шарттарымен Больцманның бір өлшемді сызықсыз алты-моменттік теңдеулер жүйесі үшін алғашқы-шеттік есептің сандық шешімі

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Больцман теңдеуі - күрделі интеграл-дифференциал теңдеу және газдардың кинетикалық теориясының негізі. Уақыт және жылдамдық бойынша кеңістіктегі сиретілген газдың күйін сипаттайды. Қатты дене және плазмадағы электрондардың тасымалдануын, ядролық реакторлардағы нейтрондардың тасымалдануын және ғарыштан Жерді қашықтықтан зерттеу жұмыстарында қолданылады. Больцман теңдеуін шешудің тиімді әдістерінің бірі моменттік әдіс болып табылады. Больцман моменттік теңдеулерінің жүйесі сиретілген газдың күйін сипаттауда кинетикалық және гидродинамикалық деңгейлердің аралығында жатқан дербес туындылы сызықты емес теңдеулер класын құрайды.

Егер бөлшектердің үлестірім функциясы толық ортогонал функциялар жүйесі бойынша Фурье қатарына жіктелсе, онда Больцман теңдеуі жіктеу коэффициенттеріне сай сызықты оператордың өзіндік функцияларының толықтығынан дербес туындылы дифференциалдық теңдеулердің шексіз жүйесінің баламасы болады. Бірақ дифференциалдық теңдеулердің шексіз жүйесін шешу мүмкін емес. Сондықтан Больцман теңдеуі үшін бастапқы-шеттік есептің жуықталған шешімін табу үшін моменттік әдісті қолданамыз. Бұл мақалада дербес туындылы сызықсыз гиперболалық дифференциалдық теңдеулер класына жататын және алты дербес туындылы сызықсыз дифференциалдық теңдеуден тұратын Больцманның алты-моменттік теңдеулердің жүйесі қарастырылған. Больцманның бір өлшемді сызықсыз алты-моменттік теңдеулер жүйесі үшін алғашқы-шеттік есептің қойылымы келтіріліп, Владимиров-Маршак жалпылынған шарттарымен осы жүйенің сандық шешімінің нәтижелері көрсетілген.
Түйін сөздер: Больцманның алты-моменттік теңдеулер жүйесі, Владимиров-Маршак жалпыланған шарттары, бөлшектердің үлестірім функциясы.

Численное решение начально-краевой задачи для одномерной нестационарной нелинейной шестимоментной системы уравнений Больцмана при граничных условиях Владимирова-Маршака

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Уравнение Больцмана - сложное интегро-дифференциальное уравнение и основа кинетической теории газов. Оно описывает состояние разряженного газа в пространстве по времени и скоростям. Применяется для изучения переноса электронов в твердых телах и плазме, переносе нейтронов в ядерных реакторах, в задачах дистанционного зондирования Земли из космоса. Одним из эффективных методов решения уравнения Больцмана является моментный метод. Система моментных уравнений Больцмана является промежуточной между кинетическим и гидродинамическими уровнями описания состояния разряженного газа и образует класс нелинейных уравнений в частных производных. Если функция распределения частиц будет разложена в ряд Фурье по полной ортогональной системе функций, то уравнение Больцмана окажется эквивалентным бесконечной системе дифференциальных уравнений в частных производных относительно коэффициентов разложения в силу полноты системы собственных функций линеаризованного оператора. Но решить бесконечную систему дифференциальных уравнений невозможно. Поэтому для нахождения приближенного решения начально-краевой задачи для уравнения Больцмана применяем моментный метод. В данной статье рассмотрена одномерная нелинейная нестационарная система моментных уравнений Больцмана в третьем приближении, которая является гиперболической системой дифференциальных уравнений в частных производных и содержит 6 уравнений относительно моментов функции распределения частиц. Сформулирована постановка начально-краевой задачи для системы моментных уравнений Больцмана в третьем приближении и приведены результаты численного решения данной системы при обобщенных условиях Владимирова-Маршака.
Ключевые слова: шестимоментная система уравнений Больцмана, граничные условия Владимирова-Маршака, функция распределения частиц.

1 Introduction

An approximate solution of the initial and boundary value problem for the Boltzmann equation can be determined by the moment method.According to the moment method particle distribution function decomposed into an infinity series of complete orthogonal system of functions.Boltzmann's equation is equivalent to an infinite system of differential equations relative to the moments of the particle distribution function in the complete system of eigenfunctions of linearized operator. As a rule we limit study by finite moment system of equations because solving infinite system of equations does not seem to be possible. There arises the problem of boundary conditions for a finite system of equations that approximate the microscopic boundary conditions for Boltzmann's equation. We consider the third approximation of one-dimensional nonlinear nonstationary Boltzmann's moment system equations and give the approximation of a homogeneous microscopic boundary condition for a nonlinear onedimensional Boltzmann equation.

Moment methods differs from each other by choosing different systems of basis functions. For instance, Grad [1,2] obtained the moment system by expanding the particle distribution function in Hermite polynomials near the local Maxwell distribution. Grad used Cartesian coordinates of velocities and Grad's moment system contained as coefficients such unknown hydrodynamic characteristics like density, temperature, average speed, and so forth. In work [3] we have obtained the moment system which differs from Grad's system of equations. We used spherical velocity coordinates and decomposed the distribution function into a series of eigenfunctions of the linearized collision operator[4,5], which is the product of Sonine polynomials and spherical functions. The expansion coefficients, moments of the distribution function is defined differently than in the Grad. The resulting system of equations, which correspond to the partial sum of series and which we called Boltzmann's moment system of equations, is a nonlinear hyperbolic system in relation to the moments of the particles distribution function. The differential part of the resulting system is linear in relation to the moments of the distribution function and nonlinearity is included as moments of collision integral [6]. The moments of a nonlinear collision operator are expressed through coefficients of Talmi and Klebsh-Gordon[7,8].

2 Literature review

Note that Boltzmann's moment equations are intermediate between Boltzmann(kinetic theory) and hydrodynamic levels of description of state of the rarefied gas and form class of nonlinear partial differential equations. Existence of such class was noticed by Grad in his articles (Grad,1949:331-407) and (Grad,1958:205-294). He obtained the moment system of equations by expanding the particle distribution function in Hermite polynomials near the local Maxwell distribution. Grad used Cartesian coordinates of velocities and Grad's moment system contained as coefficients such unknown hydrodynamic characteristics like density,temperature,average speed,and so forth. Formulation of boundary conditions for Grad's system is almost impossible, as the characteristic equations for various approximations of Grad's hyperbolic system contain unknown parameters like density,temperature, and average speed. However, 13- and 20-moment Grad equations are widely used in solving many problems of the kinetic theory of gases and plasma. In work (Sakabekov,2002) we have obtained the moment system which differs from Grad's system of equations. A homogeneous boundary condition for particles for particles distribution function was approximated and proved the correctness of initial and boundary value problem for nonlinear nonstationary Boltzmann's moment system of equations in three-dimensional space.

In works (Cergignani,1975) and (Kogan,1967) used spherical velocity coordinates and decomposed the distribution function into a series of eigenfunctions of the linearized collision operator, which is the product of Sonine polynomials and spherical functions. The structure of Boltzmann's moment system of equations corresponds to the structure of Boltzmann's equation;namely, the differential part of resulting system is linear in relation to the moments of the distribution function and nonlinearity is included as moments of collision operator (Kumar,1966:113-141).In work (Sakabekov,2014) the initial and boundary value problem for one-dimensional nonstationary Boltzmann's equation with boundary conditions of Maxwell was approximated by a corresponding problem for Boltzmann's moment system of equations. The boundary conditions for Boltzmann's moment system of equations were called Maxwell-Auzhan conditions. In work (Levelmore,1996:1021-1065) has been presented a systematic nonperturbative derivation of a hierarchy of closed systems of moment equations corresponding to any classical theory,it is a fundamental work where closed systems of moment equations describe a transition regyme.

3 Material and methods

We write in an expanded form system of one-dimensional Boltzmann's moment equations in the k-th approximation:

$$\frac{\partial f_{nl}}{\partial}x+\frac{1}{\alpha}\frac{\partial}{\partial x}[l(\sqrt{\frac{2(n+l+\frac{1}{2})}{(2l-1)(2l+1)}}f_{n,l-1}-\sqrt{\frac{2(n+l)}{(2l-1)(2l+1)}}f_{n+1,l-1})+$$
$$+(l+1)(\sqrt{\frac{2(n+l+\frac{1}{2})}{(2l+1)(2l+3)}}f_{n,l+1}-\sqrt{\frac{2n}{(2l+1)(2l+3)}}f_{n-1,l+1})]=I_{nl},2n+1=0,1,2,...k, \quad (1)$$

where the moments I_{nl} of a nonlinear collision operator are expressed through coefficients of Talmi and Klebsh-Gordon [7,8] and have the next form:

$$I_{nl}=\sum\langle N_3L_3n_3l_3:l\mid nl00:l\rangle\langle N_3L_3n_3l_3:l\mid n_1l_1n_2l_2:l\rangle(\frac{l_10l_20}{l0})(\sigma_{l_3}-\sigma_0)f_{n_1l_1}f_{n_2l_2} \quad (2)$$

where $\langle N_3L_3n_3l_3:l\mid n_1l_1n_2l_2:l\rangle$ are generalized coefficients of Talmi, $(l_10l_20/l0)$ are Klebsh-Gordon coefficients, $\alpha=\frac{1}{\sqrt{R\theta}}$ is the constant , R is Boltzmann's constant, and θ is the ideal gas temperature. This system corresponds to decomposition of the particle distribution function by eigenfunctions of the linearized collision operator.

If in (1) $2n+1$ takes the values from 0 to 3 , we get Boltzmann's moment system equations in the third approximation:

$$\frac{\partial f_{00}}{\partial t}+\frac{1}{\alpha}\frac{\partial f_{01}}{\partial x}=0, \quad (3)$$

$$\frac{\partial f_{02}}{\partial t}+\frac{1}{\alpha}\frac{\partial}{\partial x}(\frac{2}{\sqrt{3}}f_{01}+\frac{3}{\sqrt{5}}f_{03}-\frac{2\sqrt{2}}{\sqrt{15}}f_{11})=J_{02}, \quad (4)$$

$$\frac{\partial f_{10}}{\partial t}+\frac{1}{\alpha}\frac{\partial}{\partial x}(-\sqrt{\frac{2}{3}}f_{01}+\sqrt{\frac{5}{3}}f_{11})=0, \quad (5)$$

$$\frac{\partial f_{01}}{\partial t}+\frac{1}{\alpha}\frac{\partial}{\partial x}(f_{00}+\frac{2}{\sqrt{3}}f_{02}-\sqrt{\frac{2}{3}}f_{10})=0, \quad (6)$$

$$\frac{\partial f_{03}}{\partial t}+\frac{1}{\alpha}\frac{\partial}{\partial x}\frac{3}{\sqrt{5}}f_{02}=J_{03}, \tag{7}$$

$$\frac{\partial f_{11}}{\partial t}+\frac{1}{\alpha}\frac{\partial}{\partial x}(-\frac{2\sqrt{2}}{\sqrt{15}}f_{02}+\sqrt{\frac{5}{3}}f_{10})=J_{11} \tag{8}$$

$$x\in(-a,a),t>0,$$

where $f_{00}=f_{00}(t,x), f_{01}=f_{01}(t,x), f_{02}=f_{02}(t,x), f_{03}=f_{03}(t,x), f_{10}=f_{10}(t,x), f_{11}=f_{11}(t,x)$, are the moments of the particle distribution function; $J_{02}=(\sigma_2-\sigma_0)(f_{00}f_{02}-f_{01}f_{01}/\sqrt{3})/2, J_{03}=(1/4)(\sigma_3+3\sigma_1-4\sigma_0)f_{00}f_{03}+(1/4\sqrt{5})(2\sigma_1+\sigma_0-3\sigma_3)f_{01}f_{02}, J_{11}=(\sigma_1-\sigma_0)(f_{00}f_{11}+(1/2)(\sqrt{5}/\sqrt{3})f_{10}f_{01}-(\sqrt{2}/\sqrt{15})(f_{01}f_{02})$ are moments of the collision integral, where $\sigma_0,\sigma_1,\sigma_2 and \sigma_3$ are the Fourier coefficients. We consider the Vladimirov-Marshak boundary conditions:

$$\frac{1}{2\alpha}[\frac{1}{\sqrt{\pi}}(-\sqrt{2}f_{00}-\sqrt{\frac{2}{3}}f_{02}+\frac{1}{\sqrt{3}}f_{10})+f_{01}]\mid_{x=-a}=0 \tag{9}$$

$$\frac{1}{2\alpha}[\frac{1}{\sqrt{\pi}}(-\sqrt{\frac{2}{3}}f_{00}-2\sqrt{2}f_{02}+f_{10})+\frac{2}{\sqrt{3}}f_{01}+\frac{3}{\sqrt{5}}f_{03}-\frac{2\sqrt{2}}{\sqrt{15}}f_{11}]\mid_{x=-a}=0 \tag{10}$$

$$\frac{1}{2\alpha}[\frac{1}{\sqrt{\pi}}(\sqrt{\frac{1}{3}}f_{00}+f_{02}-\frac{3}{\sqrt{2}}f_{10})-\sqrt{\frac{2}{3}}f_{01}+\sqrt{\frac{5}{3}}]\mid_{x=-a}=0 \tag{11}$$

$$\frac{1}{2\alpha}[\frac{1}{\sqrt{\pi}}(-\sqrt{2}f_{00}-\sqrt{\frac{2}{3}}f_{02}+\frac{1}{\sqrt{3}}f_{10})-f_{01}]\mid_{x=a}=0 \tag{12}$$

$$\frac{1}{2\alpha}[\frac{1}{\sqrt{\pi}}(-\sqrt{\frac{2}{3}}f_{00}-2\sqrt{2}f_{02}+f_{10})-\frac{2}{\sqrt{3}}f_{01}-\frac{3}{\sqrt{5}}f_{03}+\frac{2\sqrt{2}}{\sqrt{15}}f_{11}]\mid_{x=a}=0 \tag{13}$$

$$\frac{1}{2\alpha}[\frac{1}{\sqrt{\pi}}(\frac{1}{\sqrt{3}}f_{00}+f_{02}-\frac{3}{\sqrt{2}}f_{10})+\sqrt{\frac{2}{3}}f_{01}-\sqrt{\frac{5}{3}}f_{11}]\mid_{x=a}=0, \tag{14}$$

and introduce the following vectors and matrix :

$u=(f_{00},f_{02},f_{10})'$,
$w=(f_{01},f_{03},f_{11})'$,
$J_1=(0,J_{02},0)'$,
$J_2=(0,J_{03},J_{01})'$,

$$A=\frac{1}{\alpha}\begin{pmatrix}1&0&0\\ \frac{2}{\sqrt{3}}&\frac{3}{\sqrt{5}}&-\frac{2\sqrt{2}}{\sqrt{15}}\\ -\sqrt{\frac{2}{3}}&0&\sqrt{\frac{5}{3}}\end{pmatrix}, B=-\frac{1}{\alpha\sqrt{\pi}}\begin{pmatrix}\sqrt{2}&\sqrt{\frac{2}{3}}&-\frac{1}{\sqrt{3}}\\ \sqrt{\frac{2}{3}}&2\sqrt{2}&-1\\ -\frac{1}{\sqrt{3}}&-1&\frac{3}{\sqrt{2}}\end{pmatrix}.$$

where matrix A и B are nonsingular.

We write the initial and boundary value problem for six-moment Boltzmann's system equations with boundary conditions (9)-(14) in a vector-matrix form:

$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = J_1, \tag{15}$$

$$\frac{\partial u}{\partial t} + A'\frac{\partial u}{\partial x} = J_2, t \in (0,T], x \in (-a,a), \tag{16}$$

$$u\mid_{t=0} = u_0(x), w\mid_{t=0} = w_0(x), x \in [-a,a], \tag{17}$$

$$(Aw \mp Bu)\mid_{x=\pm a} = 0, t \in (0,T]), \tag{18}$$

where $u_0(x)$ и $w_0(x)$ are given initial vector functions. It requires to find the solution of the system (15,16) with the initial conditions (17) and boundary conditions (18).

4 Results and discussion

We use the explicit method for solving the initial and boundary value problem numerically. According to the explicit method we approximate the system (15,16):

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} + A\frac{W_{i+1}^n - W_i^n}{\Delta x} = J_1(U_i^n, W_i^n),$$

$$\frac{W_i^{n+1} - W_i^n}{\Delta t} + A'\frac{U_i^n - U_{i-1}^n}{\Delta x} = J_2(U_i^n, W_i^n).$$

And the values of U_i^{n+1} and W_i^{n+1} are equal to:

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}A(W_{i+1}^n - W_i^n) + \Delta t J_1(U_i^n, W_i^n),$$

$$W_i^{n+1} = W_i^n - \frac{\Delta t}{\Delta x}A'(Ui^n - U_{i-1}^n) + \Delta t J_2(U_i^n, W_i^n).$$

We use the next initial values:

$f_{00}^0(x) = 1 - x$, $f_{02}^0(x) = x$, $f_{10}^0(x) = x(1-x)$, $f_{01}^0(x) = 1 - x^2$, $f_{03}^0(x) = x^2$, $f_{11}^0(x) = x^2(1-x^2)$,

$$U_0(x) = \begin{pmatrix} 1-x \\ x \\ x(1-x) \end{pmatrix}, W_0(x) = \begin{pmatrix} 1-x^2 \\ x^2 \\ x^2(1-x^2) \end{pmatrix},$$

$\sigma_0 = 0.3, \sigma_1 = 0.5, \sigma_2 = 0.7, \sigma_3 = 0.9, \alpha = 0.2, n \in [1,10], i \in [1,9]$.

The numerical results (the values of U(x,t) and W(x,t), corresponding to the values of $f_{00}, f_{01}, f_{02}, f_{03}, f_{10}, f_{11}$) were obtained on C++ program and shown in this 6 figures.

5 Conclusion

The study of various problems for Boltzmann's moment system of equations is an important and actual task in the theory of a rarefied gas and applications of the moment system

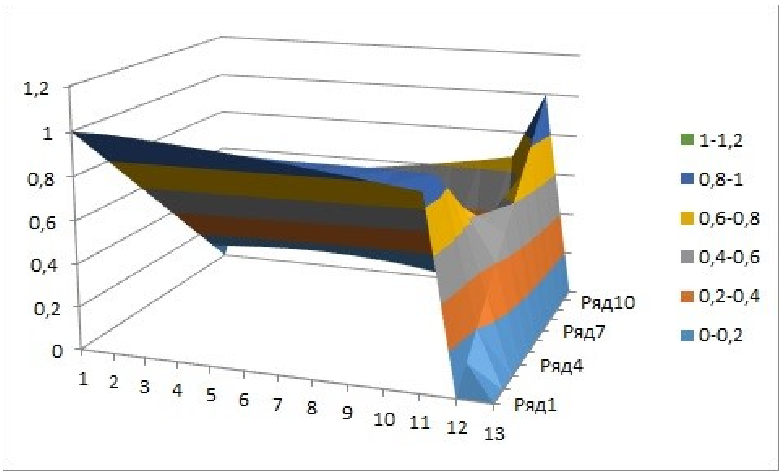

Figure 1 – The behavior of the moment f_{00} in space of time and variable x (gas density)

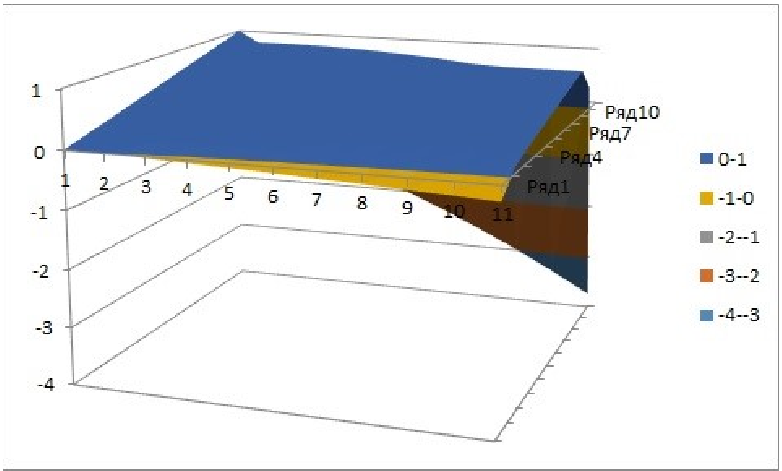

Figure 2 – The behavior of the moment f_{02} in space of time and variable x

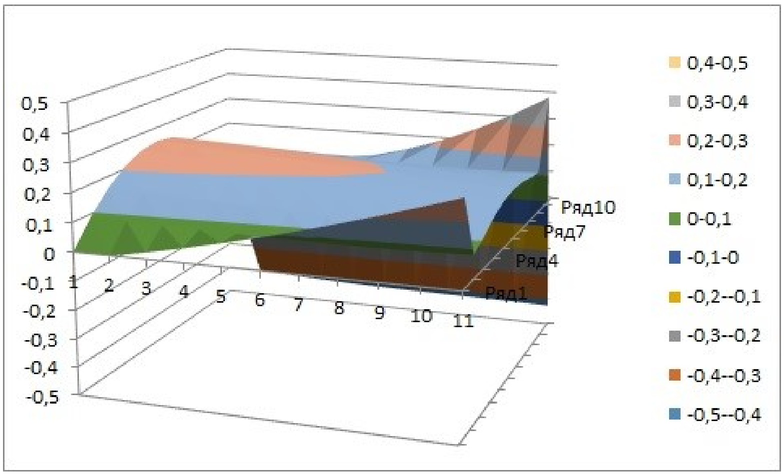

Figure 3 – The behavior of the moment f_{10} in space of time and variable x

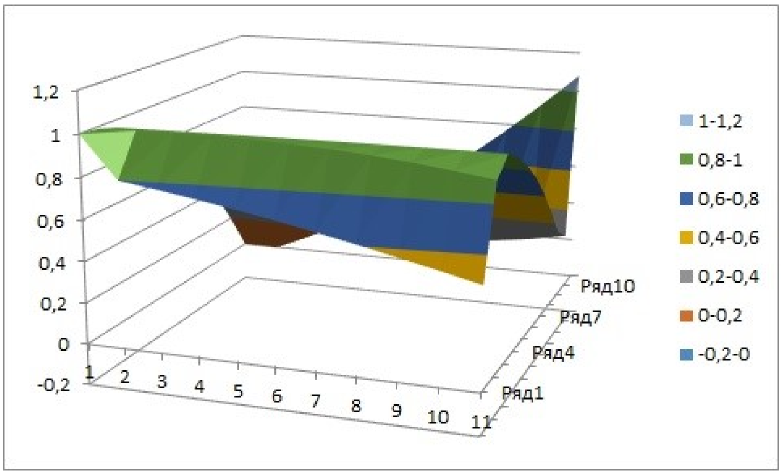

Figure 4 – The behavior of the moment f_{01} in space of time and variable x

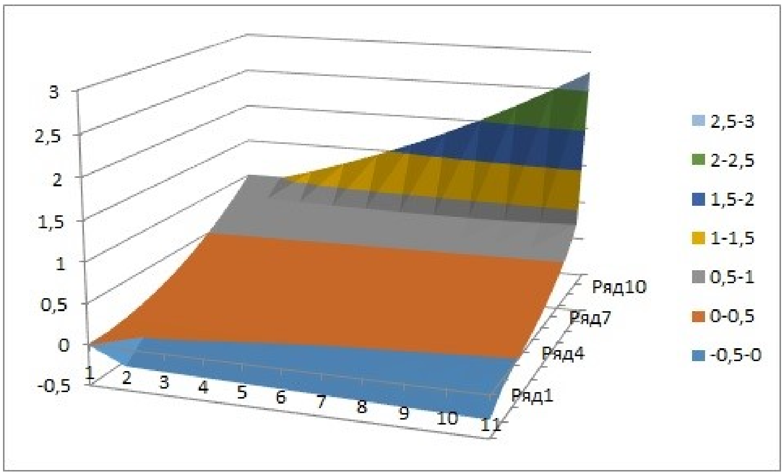

Figure 5 – The behavior of the moment f_{03} in space of time and variable x

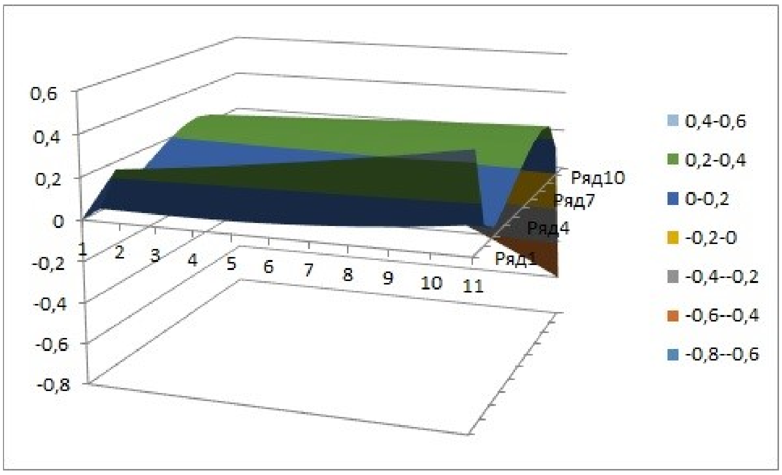

Figure 6 – The behavior of the moment f_{11} in space of time and variable x

equations.The Boltzmann equation describes the behavior of a rarefied gas in space of time and velocity and the numerical solution of the initial and boundary value problem for Boltzmann's six-moment system equations allows us to see the behavior of the moments of the particle distribution function $f_{00}, f_{01}, f_{02}, f_{03}, f_{10}, f_{11}$ in space of time and variable x. Obtained results showed that the parameter α (α depends on gas temperature) has influence on the behavior of the moments of the particle distribution function $f_{00}, f_{01}, f_{02}, f_{03}, f_{10}, f_{11}$.

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