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On the probabilistic solution of the Cauchy problem for parabolic equations

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The questions about finding (conditional) mathematical expectations, the joint and marginal distribution of different functionals from the trajectories of random processes, expressed through the process itself, the ordinary stochastic integral and the stochastic Ito integral (stochastic integrals are understood as integrals in the mean square sense) are among the important issues of both the theory itself random processes and its numerous applications. But it is not always possible to find (joint) distributions of the indicated functionals by direct computations, therefore, they usually resort to some methods of finding the required characteristics. One of such methods is the so-called method of differential equations, which reduces the problem of finding joint distributions of functionals from random processes to solving (connected with these functionals) partial differential equations.
The aim of the present paper is to find a joint distribution of the above-mentioned types of functionals and the integrands present in the definitions of these functionals depend both on the time and on the spatial coordinates.
To do this, we first derive an equation for the joint characteristic function of the functionals under consideration and show that in some special cases the determination of the Laplace transformation of the solution of this equation can be reduced to the solution of an ordinary differential equation with constant coefficients. As an application, explicit distributions of some functionals of the Wiener process are found and some of their possible applications are discussed.
Key words: Wiener process, stochastic integral, conditional mathematical expectation along the trajectories of the process, joint distribution, joint characteristic function, Laplace transformation.

Параболалык теңдеулер үшін Коши есебінің ыктималдыктық шешуі туралы

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Кездейсоқ процестер теориясында және оның көптеген қолданылымдарындағы маңызды сұрақтардың бірі – ол кездейсоқ процестердің өздері, әдеттегі стохастикалық интегралдық және Итоның стохастикалық интегралдары (стохастикалық интегралдары орташа квадраттық мағынадағы интегралдар ретінде түсініледі) өрнектелген функционалдарының процестердің траекториялары бойынша алынған (шартты) математикалық күтімдерін, бірлескен және маргиналды үлестірімдерін табу туралы сұрақтар. Бірақ берілген функционалдардың (бірлескен) үлестірімдерін тікелей есептеулер арқылы табу үнемі мүмкін бола бермейді, сондықтан керек сипаттамаларды табу үшін басқа қандай да бір әдістерді қолдануға тура келеді. Мұндай әдістердің бірі – дифференциалдық теңдеулер әдісі деп аталады. Бұл әдіс бойынша кездейсоқ процесстердің функционалдарының бірлескен үлестірімін табу есебі (берілген функционалдарға байланысты) дербес туындылы дифференциалдық теңдеулерді шешуге келтіріледі.

Бұл жұмыстың мақсаты – функционалдар анықтамаларындағы интеграл астындағы функциялар уақыттан және кеңістіктегі кординаттан да тәуелді болатын жағдай үшін жоғарыда аталған функционалдардың бірлескен үлестірімін табу. Ол үшін алдымен қарастырылып отырған функционалдардың бірлескен сипаттамалық функциясы үшін сәйкес теңдеу алынады да кейбір дербес жағдайларда бұл теңдеу шешімінің Лаплас түрлендіруі коэффициенттері тұрақты әдеттегі дифференциалдық теңдеулерді шешуге келтіруге болатыны көрсетіледі. Қолданылу мысалы ретінде Винер процесінің функционалы ретінде анықталған кейбір функционалдардың үлестірімдерінің айқын түрлері табылған және оның қолданылуларының мүмкін болатын кейбір мәселелері талқыланған.
Түйін сөздер: Винер процесі, стохастикалық интеграл, процесс траекториялары бойынша шартты математикалық күтім, бірлескен үлестірім, бірлескен сипаттамалық функция, Лаплас түрлендіруі.

О вероятностном решений задачи Коши для параболических уравнений

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Вопросы о нахождении (условных) математических ожидании, совместных и маргинальных распределении различных функционалов от траекторий случайных процессов, выражаемых через самого процесса, обычного стохастического интеграла и стохастического интеграла Ито (стохастические интегралы понимаются как интегралы в среднеквадратичном смысле) относятся к числу важных вопросов как самой теории случайных процессов, так и ее многочисленных приложений. Но не всегда удается найти (совместных) распределений указанных функционалов прямыми вычислениями, поэтому обычно прибегают к тем или иным способам нахождения нужных характеристик. Одним из таких методов является так называемый метод дифференциальных уравнений, который сводит задачу о нахождении совместных распределений функционалов от случайных процессов к решению (связанных с данными функционалами) дифференциальных уравнений в частных производных.
Целью настоящей работы является нахождение совместного распределения указанных выше видов функционалов, причем присутствующие в определениях этих функционалов подинтегральные функции зависят как от временной, так и от пространственной координат. Для этого сначала выводится уравнение для совместной характеристической функции рассматриваемых функционалов и показывается, что в некоторых частных случаях нахождение преобразования Лапласа решения этого уравнения можно свести к решению обыкновенного дифференциального уравнения с постоянными коэффициентами. В качестве приложения найдены явные виды распределения некоторого функционалов от винеровского процесса и обсуждены некоторые их возможные приложения.
Ключевые слова: Винеровский процесс, стохастический интеграл, условное математическое ожидание по траекториям процесса, совместное распределение, совместная характеристическая функция, преобразование Лапласа.

1 Introduction

In the theory of random processes it is well known that the solution of the Cauchy problem for parabolic partial differential equations is closely related to the finding of conditional mathematical expectations on the trajectories (associated with the initial equation) of diffusion processes. Thus, if A – is an infinitesimal (generating) operator (Gikhman, 1977) of the diffusion process ξ_t, $t \geq 0$ – then this operator is definite on all finite twice continuously differentiable functions differential operator of not higher than second order and this operator can contain only derivatives of the first and second orders (for example, for multidimensional Wiener process $W_t = (W_t^1, W_t^2, \ldots, W_t^n)$ its generating operator is defined on all bounded and uniformly continuous with its frequent derivatives of the first and second order functions f and on them it is equal to $\frac{1}{2}\Delta f$, where Δ – the Laplace transformer). Then the probabilistic solution of the Cauchy problem

$$\frac{\partial u(t,x)}{\partial t} = Au(t,x) + C(x)u(t,x) + h(x), u(0,x) = u_0(x), \quad (1)$$

can be written in the form (Gikhman, 1977)

$$u(t,x) = M_x\left[\exp\left\{\int_0^t c(\xi_s)ds\right\}u_0(\xi_t) + \int_0^t \exp\left\{\int_0^s c(\xi_u)du\right\}h(\xi_s)ds\right], \quad (2)$$

where the sign M_x means taking a conditional mathematical expectation over all output at the initial time $t = 0$ from the point $x(\xi_0 = x)$ trajectories of the process ξ_t (it is clear that in this case it will be necessary to impose certain conditions on the functions c, g, u_0 and on the domain of the operator A).

Further, the representation (2) can be used to find the joint characteristic functions, hence the joint distribution, of the functionals of the form $h(\xi_t)$, $\int_0^t g(\xi_s)ds$, $h(\xi_\tau)$, $\int_0^\tau g(\xi_s)ds$, where τ – is the moment of the first attainment of some set.

In (Gikhman, 1977), (Wentzel, 1996) this method (the method of differential equations) was applied to find the joint characteristic function of the functionals $I_1(t) = \Phi(W_t)$, $I_2(t) = \int_0^t g(W_s)dW_s$, $I_3(t) = \int_0^t f(W_s)ds$, where W_t, $t \geq 0$ – (one-dimensional) Wiener process, $I_2(t)$ – is a stochastic integral of Ito, $I_3(t)$ – is a stochastic integral of a random function (integrals are understood as mean-square integrals).

In this paper we generalize the results of [(Skorokhod, 1970), (Akanbai, 2014 : 8)] and refine and correct some results of (Akanbai, 2014 : 8) on the cases when the functions g and f also depend on time: $g = g(t,x)$, $f = f(t,x)$, $t \geq 0$, $x \in R'$. Namely, first we obtain a parabolic equation for the function $u(t,x) = M_x\left[I_1(t)\exp\left\{I_2(t) + I_3(t)\right\}\right]$, where now $I_2(t) = \int_0^t g(t-s,W_s)dW_s$, $I_3(t) = \int_0^t f(t-s,W_s)ds$. Furthermore we will use the this equation to find the joint characteristic function of functionals $I_1(t)$, $I_2(t)$, $I_3(t)$. In the particular case, when $g = g(t,x)$, $f = f(t,x)$ we arrive to the results of (Skorokhod, 1970), and passing to the Laplace transformation of the found joint characteristic function, we reduce the problem to the solution of an ordinary differential equation with constant coefficients. After this, by making an inverse Laplace transformation, we obtain the final result (the joint distribution of the considered functionals). In conclusion, as an example, the distribution of the functional $I_2(t)$ is found in the case when the function $g(t,x) = signx$.

2 Literature Review

Finding the distribution of functionals from random processes, expressed by ordinary (stochastic) integrals and Ito stochastic integrals, are very important problems in the theory of random processes (Gikhman, 1977 : 568). It is also well known that obtaining a probabilistic solution of the Cauchy problem for parabolic equations reduces to finding conditional mathematical expectations along trajectories with appropriately constructed random processes. In particular, in Wentzel (1996: 320), in terms of infinitesimal operators of the process and conditional mathematical expectations on the trajectories of processes, formulas are given that give probabilistic solutions of Cauchy problems for various parabolic equations with coefficients depending only on the spatial coordinates. In (Skorokhod, 1970 : 304) solved the problem of finding a joint distribution of functionals of the Wiener process W_t, $t \geq 0$, species, $\Phi(W_t)$, $\int\limits_0^t f(W_s)ds$ and $\int\limits_0^t g(W_s)dW_s$, where $\Phi(x)$ is a sufficiently smooth function, functions $f(x)$, $g(x)$ are piecewise continuous and bounded functions on each finite interval, and it is shown that the joint characteristic function of these quantities can be found as the solution of some parabolic equation. In (Akanbay, 2014 : 8), these results were generalized to a wider class of functions f and g.

In the paper, the results of the last two papers will be generalized, and the results (Akanbay, 2014 : 8) will be refined to the case when the functions f and g also depend additionally on the time, and the distributions of certain functionals of the Wiener process will be found as applications.

3 Methods of research

3.1 Derivation of the differential equation for conditional mathematical expectation $u(t,x) = M\left[\Phi(W_t)\exp\left\{I_2(t)\right\}\exp\left\{I_3(t)\right\}\right]$

Consider a function (conditional mathematical expectation)

$$u(t,x) = M_x\left[\Phi(W_t)e^{\int\limits_0^t g(t-s,W_s)dW_s} \cdot e^{\int\limits_0^t f(t-s,W_s)ds}\right] \qquad (3)$$

where W_t – is Wiener process, the sign M_x means the conditional mathematical expectation by all trajectories of a Wiener process that go from the point $x \in R$ in the initial moment $t = 0 : M_x(...) = M((...)/W_0 = x)$, and the integrals which are in the power of exponents are, respectively, the Ito stochastic integral in the Wiener process and the usual stochastic integral from the random functions (Gikhman, 1977), (Wentzel, 1996), where the integrals will be understood as the RMS limits of the corresponding integral sums.

Furthermore one can assume that the given functions $f(t,x)$, $g(t,x)$ and $\Phi(x)$ are continuous by t and x and limited with their derivatives up to second order functions.

Our goal here is to prove that the function $u(t,x)$ defined by (3) is a probability representation of the solution of the Cauchy problem for the parabolic equation

$$\frac{\partial u(t,x)}{\partial t} = \frac{1}{2}\frac{\partial^2 u(t,x)}{\partial x^2} + g(t,x)\frac{\partial u(t,x)}{\partial x} + \left[\frac{1}{2}g^2(t,x) + f(t,x)\right]u(t,x), \qquad (4)$$

where $t \geq 0$, $x \in R = (-\infty, +\infty)$, $u(0,x) = \Phi(x)$.

It is noted that the formula (3) can be rewritten in another form, in the form of unconditional mathematical expectation

$$u(t,x) = M\left[\Phi(W_t + x)e^{\int\limits_0^t g(t-s,W_s+x)dW_s+\int\limits_0^t f(t-s,W_s)ds}\right]. \tag{3'}$$

The formula (3′) is derived from the following equations:

$$M_x\left[\Phi(W_t)e^{\int\limits_0^t g(t-s,W_s)dW_s+\int\limits_0^t f(t-s,W_s)ds}\right] =$$

$$= M\left[\Phi((W_t - W_0) + W_0)\cdot e^{\int\limits_0^t g(t-s,(W_s-W_0)+W_0)dW_s+\int\limits_0^t f(t-s,(W_s-W_0)+W_0)ds} / W_0 = x\right] =$$

$$= M\left[\Phi(\widetilde{W}_t + x)\cdot e^{\int\limits_0^t g(t-s,\widetilde{W}_s+x)d\widetilde{W}_s+\int\limits_0^t f(t-s,\widetilde{W}_s+x)ds}\right],$$

$$\widetilde{W}_t = W_t(0) = W_t - W_0.$$

In the equation this fact was used: the process $\widetilde{W}_t = W_t - W_0$ is also (not dependent on W_0) Wiener process (in (3′) we have again designated this process $\widetilde{W}_t$ as W_t).

The performing of the initial condition $u(0,x) = \lim\limits_{t\downarrow 0} u(t,x) = \Phi(x)$ follows directly from the formula (1):

$$u(0,x) = M_x\Phi(W_0) = M(\Phi(W_0)/W_0 = x) = M(\Phi(x)/W_0 = x) = \Phi(x),$$

The existence of continuous derivatives $u'_x(t,x)$, $u''_{xx}(t,x)$ follows from the possibility of differentiating of the right side of (3′) by x twice under the sign of expectation. Introducing the function

$$h(t,x) = \int\limits_0^t g(t-s,W_s+x)dW_s + \int\limits_0^t f(t-s,W_s+x)ds, \quad v(t,x) = e^{h(t,x)},$$

we get

$$u(t,x) = M\left[\Phi(W_t + x)v(t,x)\right],$$

$$u'_x(t,x) = M\left[\Phi'_x(W_t + x)v(t,x) + \Phi(W_t + x)v'_x(t,x)\right] =$$

$$= M\left[\Phi'_x(W_t + x)v(t,x) + \Phi(W_t + x)v(t,x)h'_x(t,x)\right],$$

$$u''_{xx}(t,x) = M\left[\Phi''_{xx}(W_t + x)v(t,x) + 2\Phi'_x(W_t + x)v(t,x)h'_x(t,x) + \right.$$

$$+\Phi'(W_t+x)v(t,x)(h'_x(t,x))^2+\Phi(W_t+x)v(t,x)h''_{xx}(t,x)\Bigg]=$$

$$=M\Bigg\{v(t,x)\Bigg[\Bigg(\Phi''_{xx}(W_t+x)+2\Phi'_x(W_t+x)\times$$

$$\times\left(\int\limits_0^t g'_x(t-s,W_s+x)dW_s+\int\limits_0^t f'_x(t-s,W_s+x)ds\right)^2+$$

$$+\Phi(W_t+x)\left(\int\limits_0^t g'_x(t-s,W_s+x)dW_s+\int\limits_0^t f'_x(t-s,W_s+x)ds\right)^2+$$

$$+\Phi(W_t+x)\left(\int\limits_0^t g''_{xx}(t-s,W_s+x)dW_s+\int\limits_0^t f''_{xx}(t-s,W_s+x)ds\right)\Bigg]\Bigg\}.$$

Further, by defining a new Wiener process: $W_s(\Delta t)=W_{s+\Delta t}-W_{\Delta t}$, we can write

$$u(t+\Delta t,x)=M\Bigg[\Phi(W_s(\Delta t)+W_{\Delta t}+x)v(t+\Delta t,x)\Bigg],$$

where

$$v(t+\Delta t,x)=\exp\Bigg\{h(t+\Delta t,x)\Bigg\}=$$
$$=\exp\Bigg\{\int\limits_0^{\Delta t} g(t+\Delta t-s,W_s)dW_s+\int\limits_0^{\Delta t} f(t+\Delta t-s,W_s)ds+ \tag{5}$$
$$+\int\limits_0^t g(t-s,W_s(\Delta t)+W_{\Delta t})dW_s(\Delta t)+\int\limits_0^t f(t-s,W_s(\Delta t)+W_{\Delta t})ds\Bigg\}$$

As it is known from (Gikhman, 1977), (Wentzel, 1996), the process $W_s(\Delta t)$ is not dependent on $W_{\Delta t}$, and sigma algebra $\mathscr{F}_{\le\Delta t}=\sigma\Bigg\{W_s:0\le s\le\Delta t\Bigg\}$ ($\mathscr{F}_{\le\Delta t}$ – is the smallest σ – algebra containing all events of the form $\{W_s\in B,0\le s\le\Delta t,B\in\beta(R)\}$, $\beta(R)$ – is Borel σ – algebra on R) Wiener process.

Next, using the measurability of exponent of $\mathscr{F}_{\le\Delta t}$, which is in the right-hand side of (5) and contains the integrals from 0 to Δt independence of integrals from 0 to t of exponents of $\mathscr{F}_{\le\Delta t}$ and properties of conditional expectations with respect to the sigma-algebra, one can write:

$$u(t+\Delta t,x)=M\Bigg[M(\Phi(x+W_s(\Delta t)+W_{\Delta t})v(t+\Delta t,x)/\mathscr{F}_{\le\Delta t})\Bigg]=$$

$$= M \exp\left\{ \int_0^{\Delta t} g(t+\Delta t - s, x + W_s) dW_s + \int_0^{\Delta t} f(t+\Delta t - s, x+W_s) ds \right\} \cdot$$

$$\cdot M\Phi(x+W_s(\Delta t)+W_{\Delta t}) \exp\left\{ \int_0^{t} g(t+\Delta t - s, W_s(\Delta t)) dW_s(\Delta t) + \int_0^{t} f(t+\Delta t - s, W_s(\Delta t)) ds \right\}.$$

The presentation of $(3')$ shows that the latter expectation is equal to $u(t, x+W_{\Delta t})$. Thus, we find that

$$u(t+\Delta t, x) = \\ = M\left[\exp\left\{ \int_0^{\Delta t} g(t+\Delta t - s, x+W_s) dW_s + \int_0^{\Delta t} f(t+\Delta t - s, x+W_s) ds \right\} u(t, x+W_{\Delta t}) \right] \quad (6)$$

Now, referring to the conditions on the functions $f(t,x)$, $g(t,x)$ and to the fact that at small Δt $(\Delta t \to 0)$,

$$W_{\Delta t} \sim \sqrt{\Delta t}, \quad W_{\Delta t}^2 \sim \Delta t, \quad \int_0^{\Delta t} g(t+\Delta t - s, x+W_s) dW_s \sim g(t,x) W_{\Delta t},$$

$$\int_0^{\Delta t}\int_0^{\Delta t} g(t+\Delta t - s_1, x+W_{s_1}) g(t+\Delta t - s_2, x+W_{s_2}) dW_{s_1} dW_{s_2} \sim g^2(t,x) \cdot W_{\Delta t}^2,$$

$$\int_0^{\Delta t} f(t+\Delta t - s, x+W_s) ds \sim f(t,x)\Delta t,$$

let's expand the right-hand side of (6) by the accurency up to $o(\Delta t)$. Then we can write

$$u(t+\Delta t, x) = M\left[(1 + g(t,x) W_{\Delta t} + f(t,x)\Delta t + \frac{1}{2} g^2(t,x) W_{\Delta t}^2 + o(\Delta t)) u(t, x+W_{\Delta t}) \right] \quad (6')$$

In its turn

$$u(t, x+W_{\Delta t}) = u(t,x) + \frac{\partial u(t,x)}{\partial x} \cdot W_{\Delta t} + \frac{1}{2} \frac{\partial^2 u(t,x)}{\partial x^2} \cdot W_{\Delta t}^2 + o(\Delta t).$$

Putting this found expression in the right side of the $(6')$ and similarly, leaving only the terms up to order Δt, we get

$$u(t+\Delta t, x) = M\left[u(t,x) + \frac{\partial u(t,x)}{\partial x} \cdot W_{\Delta t} + \frac{1}{2} \frac{\partial^2 u(t,x)}{\partial x^2} \cdot W_{\Delta t}^2 + g(t,x) u(t,x) \cdot W_{\Delta t} + \right.$$

$$\left. + g(t,x) \frac{\partial u(t,x)}{\partial x} \cdot W_{\Delta t}^2 + f(t,x) u(t,x) \cdot \Delta t + \frac{1}{2} g^2(t,x) u(t,x) \cdot W_{\Delta t}^2 + o(\Delta t) \right].$$

In the latter ratio we calculate the required expectations. Then taking into account that $MW_{\Delta t} = 0$, $MW_{\Delta t}^2 = \Delta t$, we obtain

$$u(t+\Delta t, x) = u(t,x) + \frac{1}{2}\frac{\partial^2 u(t,x)}{\partial x^2}\cdot \Delta t + g(t,x)\frac{\partial u(t,x)}{\partial x}\cdot \Delta t +$$
$$+ f(t,x)u(t,x)\cdot \Delta t + \frac{1}{2}g^2(t,x)u(t,x)\cdot \Delta t + o(\Delta t).$$

thus

$$\lim_{\Delta t \downarrow 0} \frac{u(t+\Delta t, x) - u(t,x)}{\Delta t} = \frac{1}{2}\frac{\partial^2 u(t,x)}{\partial x^2} + g(t,x)\frac{\partial u(t,x)}{\partial x} + \left[\frac{1}{2}g^2(t,x) + f(t,x)\right]u(t,x). \quad (7)$$

From this relation (7) implies the existence of a right derivative of $u(t,x)$ with respect to t. Since it is continuous, there exists a two-sided derivative which coincides with the right one. Eventually we obtain the theorem.

Theorem 1 *Let $f(t,x)$, $g(t,x)$ and $\Phi(x)$ are continuous with respect to x and t and limited, together with their derivatives up to the second order (inclusive) functions.*

Then defined by formula (3) (or the formula (3′)) function $u(t,x)$ in $t > 0$, $x \in (-\infty, +\infty)$ satisfies the differential equation (4).

Comment 1 *One should note that in (Skorokhod, 1970) a similar theorem was proved for the case when the functions $f(t,x)$ and $g(t,x)$ do not depend on time, and the first theorem is proved in the case of $g(x) = 0$, after that the final result is obtained by replacing the original function. We also note that in (Akanbai, 2014 : 8) another, but not completely rigorous, proof of this theorem is given.*

3.2 The equation for the joint characteristic function

The proven theorem allows us to find the joint characteristic function of the functionals $I_1(t) = W_t$, $I_2(t) = \int_0^t g(t-s, W_s)dW_s$, $I_3(t) = \int_0^t f(t-s, W_s)ds$, as a solution of some parabolic partial differential equation.

In fact, let $\varphi(t, x, z_1, z_2, z_3)$ is a (conditional) joint characteristic function of the functionals $I_1(t)$, $I_2(t)$, $I_3(t)$ $(z_1, z_2, z_3 \in R^1)$:

$$\varphi(t, x, z_1, z_2, z_3) = M_x(e^{iz_1 I_1(t) + iz_2 I_2(t) + iz_3 I_3(t)}). \quad (8)$$

If in theorem there are $\overline{\Phi}(x) = e^{iz_1 x}$ instead $\Phi(x)$, the function $\overline{g}(t,x) = iz_2 g(t,x)$ instead of $g(t,x)$, the function $\overline{f}(t,x) = iz_3 f(t,x)$ instead of $f(t,x)$, then according to the proven theorem, the conditional characteristic function $\varphi(t,x,z)$ (see (8)) where $z = (z_1, z_2, z_3)$, can be found as a solution of the equation:

$$\frac{\partial \varphi(t,x,z)}{\partial t} = \frac{1}{2}\frac{\partial^2 \varphi(t,x,z)}{\partial x^2} + iz_2 g(t,x)\frac{\partial \varphi(t,x,z)}{\partial x} + (iz_3 f(t,x) - \frac{z_2^2}{2}g^2(t,x))\varphi(t,x,z) \quad , (9)$$
$$\varphi(0,x,z) = e^{ixz_1}.$$

It is clear, that the function of $\varphi(t,0,z) = \varphi(t,0,z_1,z_2,z_3)$ is what we need (unconditional) joint characteristic function of $I_1(t)$, $I_2(t)$, $I_3(t)$. In the general case it is not possible to find a solution of equation (9). Therefore, we consider the particular case of equation (9). We will assume that the functions g and f are independent of t: $g(t,x) = g(x)$, $f(t,x) = f(x)$. Then, using the Laplace transformation with respect to t, the solution of the particular case of equation (9) can be reduced to the solution of an ordinary differential equation. To do this, we multiply both sides of the new equation by $e^{-\lambda t}$ $(\lambda > 0)$, then integrate with respect to t from zero to infinity.

If we denote the Laplace transformation $\varphi(t,x,z)$ with respect to t by $\widetilde{\varphi}(\lambda,x,z)$ we obtain the equation for this function

$$\frac{1}{2}\frac{\partial^2\widetilde{\varphi}(\lambda,x,z)}{\partial x^2} + iz_2 g(x)\frac{\partial\widetilde{\varphi}(\lambda,x,z)}{\partial x} + (iz_3 f(x) - \frac{z_2^2}{2}g^2(x) - \lambda)\widetilde{\varphi}(\lambda,x,z) = -e^{ixz_1}. \quad (10)$$

The solution of the equation (10) should be restricted to the entire real axis, and since $|\varphi(t,x,z)| \le 1$, it shall be $|\widetilde{\varphi}(t,x,z)| \le \dfrac{1}{\lambda}$.

In this case, the equation (10) is performed at all points of continuity of $f(x)$ and $g(x)$, and at points of discontinuity of these functions its derivative is continuous $\dfrac{\partial\widetilde{\varphi}(t,x,z)}{\partial x}$.

Comment 2 *With the passage to the limit, one can make himself sure that the equation* (10) *holds for piecewise constant and bounded on every finite interval function of $f(x)$ and $g(x)$.*

3.3 Finding the distribution of the functional $I(t) = \int\limits_0^t signW_s dW_s$

In our equation (4) $g(x) = signx$, $f(x) = 0$, and thus the equation (4) is written as follows:

$$\frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} + signx \cdot \frac{\partial u(t,x)}{\partial x} + \frac{1}{2}u(t,x),$$

$$u(0,x) = 1.$$

Further,

$$\varphi(t,x,z) = M_x\left(e^{iz\int\limits_0^t signW_s dW_s}\right), \quad \widetilde{\varphi}(\lambda,x,z) = \int\limits_0^\infty e^{-\lambda t}\varphi(t,x,z)dt$$

and the equation (10) has the form

$$\frac{1}{2}\frac{\partial^2\widetilde{\varphi}(\lambda,x,z)}{\partial x^2} + izsignx \cdot \frac{\partial\widetilde{\varphi}(\lambda,x,z)}{\partial x} - (\frac{z^2}{2} + \lambda)\widetilde{\varphi}(\lambda,x,z) = -1. \quad (11)$$

The equation (11) can be written separately for the cases of $x > 0$ and $x < 0$.

In the case $x > 0$ the equation (11) is transformed into the equation with constant (i.e. independent of x) coefficients

$$\frac{1}{2}\frac{\partial^2\widetilde{\varphi}(\lambda,x,z)}{\partial x^2} + iz \cdot \frac{\partial\widetilde{\varphi}(\lambda,x,z)}{\partial x} - (\frac{z^2}{2} + \lambda)\widetilde{\varphi}(\lambda,x,z) = -1, \quad (12)$$

and in the case of $x<0$ in the equation with constant coefficients

$$\frac{1}{2}\frac{\partial^2\widetilde{\varphi}(\lambda,x,z)}{\partial x^2}-iz\cdot\frac{\partial\widetilde{\varphi}(\lambda,x,z)}{\partial x}-(\frac{z^2}{2}+\lambda)\widetilde{\varphi}(\lambda,x,z)=-1. \tag{13}$$

The roots of the characteristic equation of the homogeneous part of the equation (12) are $s_1=iz+\sqrt{2\lambda}$, $s_2=-iz-\sqrt{2\lambda}$, and the particular solution (12) is a function $\widetilde{\varphi}_1(\lambda,x,z)=\dfrac{1}{\dfrac{z^2}{2}+\lambda}=\dfrac{2}{2\lambda+z^2}$. In a similar way we solve (13). As a result, we find that the general solutions of the equations (12) and (13) are the functions (respectively)

$$\widetilde{\varphi}(\lambda,x,z)=C_1e^{(-iz+\sqrt{2\lambda})x}+C_2e^{(-iz-\sqrt{2\lambda})x}+\frac{2}{2\lambda+z^2},\quad x>0;$$

$$\widetilde{\varphi}(\lambda,x,z)=C_3e^{(iz+\sqrt{2\lambda})x}+C_4e^{(iz-\sqrt{2\lambda})x}+\frac{2}{2\lambda+z^2},\quad x<0,$$

where C_1,C_2,C_3,C_4 – are constant.

Since the function $\widetilde{\varphi}(\lambda,x,z)$ is bounded, we get that $C_1=0$, $C_4=0$, from the continuity of $\widetilde{\varphi}(\lambda,x,z)$ with respect to x it follows that $C_2=C_3$.

In this way

$$\widetilde{\varphi}(\lambda,x,z)=C_2e^{(-iz-\sqrt{2\lambda})x}+\frac{2}{2\lambda+z^2},\quad x>0;$$

$$\widetilde{\varphi}(\lambda,x,z)=C_2e^{(iz+\sqrt{2\lambda})x}+\frac{2}{2\lambda+z^2},\quad x<0.$$

Further, using continuity of $\dfrac{\partial\widetilde{\varphi}(\lambda,x,z)}{\partial x}$ by x we get that $C_2=0$.

Thus, the solution of the equation (9) is the function

$$\widetilde{\varphi}(\lambda,x,z)=\widetilde{\varphi}(\lambda,0,z)=\frac{2}{2\lambda+z^2}=\frac{1}{\lambda(1+\frac{z^2}{2\lambda})}.$$

Now the goal is to represent $\widetilde{\varphi}(\lambda,0,z)$ as a Laplace transformation of some function $\varphi(t,0,z)$. i.e. as a representation of $\widetilde{\varphi}(\lambda,0,z)$ as

$$\widetilde{\varphi}(\lambda,0,z)=\int\limits_0^\infty e^{-\lambda t}\varphi(t,0,z)dt. \tag{14}$$

For this purpose we expand the function $\widetilde{\varphi}(\lambda,0,z)$ in the domain $\left|\dfrac{z^2}{2\lambda}\right|<1$ in the series:

$$\widetilde{\varphi}(\lambda,0,z)=\frac{1}{\lambda(1+\frac{z^2}{2\lambda})}=\frac{1}{\lambda}\sum_{n=0}^{\infty}\frac{(-1)^n\cdot z^{2n}}{2^n\lambda^n}=$$

$$=\int\limits_0^\infty\sum_{n=0}^{\infty}\frac{(-1)^n\cdot z^{2n}}{2^n}\cdot\frac{e^{-\lambda t}\cdot t^n}{n!}dt=\int\limits_0^\infty e^{-\lambda t}\varphi(t,0,z)dt.$$

We used the well-known relation

$$\int_0^\infty \frac{e^{-\lambda tn} \cdot t^n}{n!} dt = \frac{1}{\lambda^{n+1}}.$$

Further we can write:

$$\widetilde{\varphi}(\lambda, 0, z) = \int_0^\infty e^{-\lambda t} \left(\sum_{n=0}^\infty \frac{(-1)^n \cdot (tz^2)^n}{2^n \cdot n!} \right) dt = \int_0^\infty e^{-\lambda t} e^{-\frac{tz^2}{2}} dt.$$

Thus, the characteristic function of a functional $I(t) = \int_0^t signW_s dW_s$ is a function $\varphi(t, 0, z) = e^{-\frac{tz^2}{2}}$, i.e. $I(t)$ is a normal random variable with zero mathematical expectation and a variance equal to t: $I(t) \sim \mathscr{N}(0, t)$. ((We note that another erroneous result was obtained in (Akanbai, 2014 : 8)). However, this result could be obtained directly from the relation (see formula (14)) $\frac{2}{2\lambda + z^2} = \int_0^\infty \exp\left\{ -\lambda t - \frac{tz^2}{2} \right\} dt$.

3.4 Finding the distribution of the functional $W_t + \int_0^t signW_s dW_s$

Next we consider the functional

$$I_1(t) = \int_0^t g_{0,1}(W_s) dW_s,$$

where $g_{0,1}(x) = \frac{1 + signx}{2} = 1,\ x > 0;\ g_{0,1}(x) = 0,\ x < 0$. Then

$$I_1(t) = \frac{1}{2}(W_t + \int_0^t signW_s dW_s),$$

as a sum of normal random variables, $I_1(t)$ – is a normal random variable. Let us find the distribution parameters $I_1(t)$.

Obviously $MI_1(t) = 0$, variance

$$DI_1(t) = \frac{1}{4}(t + 2cov(W_t, \int_0^t signW_s dW_s) + t) = \frac{1}{2}(t + cov(W_t, \int_0^t signW_s dW_s)).$$

for simplify the notation, we denote below $f(x) = signx$. Then

$$\begin{gathered} cov(W_t, \int_0^t f(W_s) dW_s) = M(W_t \int_0^t f(W_s) dW_s) = \\ \lim_{n\to\infty} \sum_{i=1}^n M\Big[f(W_{s_{i-1}})(W_{s_i} - W_{s_{i-1}}) W_t \Big], \end{gathered} \tag{15}$$

where the points of division $0 = s_0 < s_1 < s_2 < ... < s_{n-1} < s_n = t$, $\max_i (s_i - s_{i-1}) \to 0$.

Denoting, as before, through $\mathscr{F}_s = \sigma\{W_u : u \le s\}$ the smallest σ – algebra generated by a Wiener process up to the moment s and using the properties of the conditional mathematical expectation, for the i – th summand of formula (15) we can write:

$$M\left[f(W_{s_{i-1}})(W_{s_i} - W_{s_{i-1}})W_t\right] = MM\left[f(W_{s_{i-1}})(W_{s_i} - W_{s_{i-1}})W_t/\mathscr{F}_{s_i}\right] =$$

$$= M\left[f(W_{s_{i-1}})(W_{s_i} - W_{s_{i-1}})M(W_t/\mathscr{F}_{s_i})\right] = M\left[f(W_{s_{i-1}})(W_{s_i} - W_{s_{i-1}})W_{s_i}\right],$$

because the random variable $f(W_{s_{i-1}})(W_{s_i} - W_{s_{i-1}})$ is measurable with respect to the σ – algebra $\mathscr{F}_{s_i}$, and the Wiener process forms a martingale with respect to the family of the σ – algebra $\mathscr{F}_s : M(W_t/\mathscr{F}_{s_i}) = W_s$, $0 \le s \le t$.

Further, since $W_{s_{i-1}}$ is measurable with respect to $\mathscr{F}_{s_{i-1}}$, and $W_{s_i} - W_{s_{i-1}}$ does not depend on $\mathscr{F}_{s_{i-1}}$, we obtain

$$M\Big[f(W_{s_{i-1}})(W_{s_i} - W_{s_{i-1}})W_{s_i}\Big] =$$

$$= M\Big[f(W_{s_{i-1}})(W_{s_i} - W_{s_{i-1}})^2 + f(W_{s_{i-1}})W_{s_{i-1}} \cdot (W_{s_i} - W_{s_{i-1}})\Big] =$$

$$= M\Big[M(f(W_{s_{i-1}})(W_{s_i} - W_{s_{i-1}})^2/\mathscr{F}_{s_i})\Big] + MM(\Big|W_{s_{i-1}}\Big|(W_{s_i} - W_{s_{i-1}})/\mathscr{F}_{s_i}) =$$

$$= Mf(W_{s_{i-1}})M(W_{s_i} - W_{s_{i-1}})^2 + M\Big|W_{s_{i-1}}\Big| \cdot M(W_{s_i} - W_{s_{i-1}}) =$$

$$= Mf(W_{s_{i-1}})(s_i - s_{i-1}) = Mf(W_{s_{i-1}})\Delta s_i = 0$$

Thus

$$cov(W_t, \int_0^t signW_s dW_s) = M\int_0^t (W_t signW_s) dW_s = 0.$$

So,

$$DI_1(t) = \frac{1}{2}t, \text{ т.е. } I_1(t) \sim \mathscr{N}(0, \frac{1}{2}t).$$

Then,

$$I_2(t) = W_t + \int_0^t signW_s dW_s = 2I_1(t) \sim \mathscr{N}(0, 2t).$$

4 Results and discussion

Using the methods of the theory of Markov processes, it is proved that the function $u(t,x)$ defined by (3) gives a probabilistic solution of equation (4).

Further, from (4) we obtain an equation for the joint characteristic function of the functionals under consideration and show that in one particular case the solution of this equation can be reduced to the solution of an ordinary differential equation. As an application of the results obtained, the distributions of the functionals $I(t) = \int_0^t signW_s dW_t$ and $I_1(t) = W_t + I(t)$ are found.

Apparently, in what follows it will be possible to introduce some concrete functions f and g so that the solution of equation (9), or even equations (10), can be found explicitly. In turn, this would allow us to find a distribution in a more complicated way of certain functionals from the Wiener process. For example, if instead of the Wiener process W_t, $t \geq 0$, consider the process $\widetilde{W}_t = W_t + ct$, where c is a nonzero constant ($\widetilde{W}_t$ is a Wiener process with drift), then the infinitesimal operator $\widetilde{W}_t$ would be the operator $A_c = \frac{1}{2}\frac{d^2}{dx^2} + c\frac{d}{dx}$, and equation (4) would be written in terms of this operator. Further, to solve the analogue of equation (4), we could apply the method of this paper.

5 Conclusion

In this paper, we found the joint characteristic function of the functionals of the Wiener process $I_1(t) = \Phi(W_t)$, $I_2(t) = \int\limits_0^t g(t-s, W_s)dW_s$, $I_3(t) = \int\limits_0^t f(t-s, W_s)ds$, where the integrals are understood as mean-square integrals ($I_2(t)$ – is a integral Ito, $I_3(t)$ – is the integral of a random function), and the functions Φ, g, f satisfy certain smoothness conditions. It is further shown that when the functions g and f do not depend on the time variable, then finding the Laplace transformation of the joint characteristic function reduces to solving an ordinary differential equation. Unfortunately, it is possible to solve this last equation explicitly only in certain special cases.

As an application of the results obtained, the distributions of some functionals of the Wiener process are found explicitly.

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