IRSTI 27.31.21+27.39.29

On positive solutions of Liouville-Gelfand problem

Kolosova S. V., Kharkiv National University of Radio Electronics,
Kharkiv, Ukraine, E-mail: lanakol@ukr.net
Lukhanin V. S., Kharkiv National University of Radio Electronics,
Kharkiv, Ukraine, E-mail: lukhanin.volodymyr@gmail.com
Sidorov M. V., Kharkiv National University of Radio Electronics,
Kharkiv, Ukraine, E-mail: maxim.sidorov@nure.ua

Modern science is highly interested in processes in nonlinear media. Mathematical models of these processes are often described by boundary-value problems for nonlinear elliptic equations. And the construction of two-sided approximations to the desired function is a perspective direction of solving such problems. The purpose of this work is to consider the existence and uniqueness of a regular positive solution to the Liouville-Gelfand problem and justify the possibility of constructing two-sided approximations to a solution. The two-sided approximations monotonically approximate the desired solution from above and below, and therefore have such an important advantage over other approximate methods that they provide an opportunity to obtain a convenient a posteriori estimate of the error of the calculations. The study of the Liouville-Gelfand problem is carried out by methods of the operator equations theory in partially ordered spaces. The mathematical model of the problem under consideration is the Dirichlet problem for a nonlinear elliptic equation with a positive parameter. The established properties of the corresponding nonlinear operator equation have given us an opportunity to obtain a condition for an input parameter, which guarantees the existence and uniqueness of the regular positive solution, as well as the possibility of constructing two-sided approximations, regardless of the domain geometry in which the problem is considered. The corresponding Liouville-Gelfand problem of the operator equation contains the Green's function for the Laplace operator of the first boundary value problem, and therefore the condition that the input parameter satisfies also contains it. Since the Green's function is known for a small number of relatively simple domains, Green's quasifunction method is used to solve the problem in domains of complex geometry. We note that the Green's quasifunction can be constructed practically for a domain of any geometry. The proposed approach allows us: a) to obtain a formula, which the parameter in the problem statement must satisfy, regardless of the domain geometry; b) for the first time, construct two-sided approximations to a solution to the Liouville-Gelfand problem; c) for the first time to obtain an a priori estimate of the solution depending on the selected value of the parameter in the problem statement. The proposed method has advantages over other approximate methods in relative simplicity of the algorithm implementation. The proposed method can be used for solving applied problems with mathematical models that are described by boundary value problems for nonlinear elliptic equations. In cases when the Green's function is unknown or has a complex form, the application of the Green's quasifunction method is proposed.
Key words: Green's function, Green's quasifunction, two-sided approximations, invariant cone segment, monotone operator.

О положительных решениях задачи Лиувилля-Гельфанда

Колосова С. В., Харьковский национальный университет радиоэлектроники,
Харьков, Украина, E-mail: lanakol@ukr.net
Луханин В. С., Харьковский национальный университет радиоэлектроники,
Харьков, Украина, E-mail: lukhanin.volodymyr@gmail.com
Сидоров М. В., Харьковский национальный университет радиоэлектроники,
Харьков, Украина, E-mail: maxim.sidorov@nure.ua

В современной науке наблюдается большой интерес к процессам, происходящим в нелинейных средах. Математическими моделями таких процессов зачастую являются краевые задачи для нелинейных эллиптических уравнений. Перспективными направлениями для решения таких задач есть построение двусторонних приближений к искомой функции. Целью данной работы является рассмотрение вопросов существования и единственности регулярного положительного решения у задачи Лиувилля-Гельфанда, а также обоснование возможности построения двусторонних приближений к решению. Двусторонние приближения монотонно сверху и снизу аппроксимируют искомое решение, и поэтому обладают тем важным преимуществом по сравнению с другими приближенными методами, что они дают возможность получить удобную апостериорную оценку погрешности вычислений. Исследование задачи Лиувилля-Гельфанда проводится методами теории операторных уравнений в полуупорядоченных пространствах. Математической моделью рассматриваемой задачи является задача Дирихле для нелинейного эллиптического уравнения с положительным параметром. Установленные свойства соответствующего нелинейного операторного уравнения дали возможность получить условие для входящего в постановку задачи параметра, которое гарантирует существование и единственность регулярного положительного решения, а также возможность построения двусторонних приближений независимо от геометрии области, в которой рассматривается задача. Соответствующее задачи Лиувилля-Гельфанда операторное уравнение содержит функцию Грина оператора Лапласа первой краевой задачи, а поэтому и условие, которому удовлетворяет параметр, также ее содержит. Так как функция Грина известна для небольшого числа достаточно простых областей, для решения задачи в областях сложной геометрии в работе применяется метод квазифункций Грина. Заметим, что квазифункцию Грина можно построить практически для области любой геометрии. Использованный в работе подход позволил: а) получить формулу, которой должен удовлетворять входящий в постановку задачи параметр, независимо от геометрии области; б) впервые для задачи Лиувилля-Гельфанда построить двусторонние приближения к решению; в) впервые получить априорную оценку решения в зависимости от выбранного значения параметра, который входит в постановку задачи. Предложенный метод решения имеет преимущества в сравнении с другими приближенными методами относительной простотой реализации алгоритма. Предлагаемый метод может быть использован при решении прикладных задач, математическими моделями которых являются краевые задачи для нелинейных эллиптических уравнений. В ситуациях, когда функция Грина неизвестна или имеет сложный вид, предложено применение метода квазифункций Грина.
Ключевые слова: функция Грина, квазифункция Грина, двусторонние приближения, инвариантный конусный отрезок, монотонный оператор.

1 Introduction

Modern science is highly interested in processes that take place in nonlinear environments. Mathematical models of these processes typically are represented by nonlinear boundary value problems of mathematical physics of the following form

$$-\Delta u = f(x,u) \quad \forall x \in \Omega \subset R^N,$$

$$u > 0, \quad u|_{\partial\Omega} = 0.$$

It is important to identify among the analytical methods ones that provide specific ways of constructing the sought solution. These methods include iterative ones which are simpler than the others and can be implemented on a computer. Among the iterative methods we highlight a class of two-sided processes that approximate the sought solutions monotonically from above and below. They have such an important advantage in comparison with other approximate

methods that they place the sought solution in a "plug"at each step of the iterative process which makes it possible to obtain a posteriori error estimate of the calculations.

The aim of this paper is to prove the existence and uniqueness of the regular positive solution of the Liouville-Gelfand problem and the possibility of constructing two-sided approximations to it.

In this work we consider the boundary-value problem for the nonlinear elliptic equation in the bounded domain $\Omega \subset R^N$

$$\begin{aligned} -\Delta u = \lambda e^u \quad \forall x \in \Omega, \quad u > 0, \\ u|_{\partial\Omega} = 0 \quad (\lambda > 0). \end{aligned} \tag{1}$$

The equation of (1) is the stationary equation of the thermal-ignition theory at constant thermal conductivity, $u(x)$ is the temperature at the point x, the parameter λ represents all the quantities that are essential for problems of the thermal-ignition theory [1].

2 Literature review

The formulation of this problem belongs to Frank-Kamenetskii [1] and Zeldovich [2]. The same problem arises in the study of prescribed curvature problems [3, 4].

If the domain Ω is the unit ball in R^N, then by the classical result of Gidas, Ni and Nirenberg [5], all positive solutions of (1) are radially symmetric, reducing (1) to the boundary value problem

$$\begin{aligned} u'' + (N-1)/r u' + \lambda e^u = 0, \quad r \in (0,1), \\ u'(0) = u(1) = 0, \quad u(r) = u(|x|). \end{aligned} \tag{2}$$

For $N = 1$ this equation was first solved by Liouville in 1853 [6], using reduction of order methods. In 1914, Bratu [7] found an explicit solution of (2) when $N = 2$. For $N = 3$ numerical progress was made in 1934 by both Frank-Kamenetskii [1] in his study of combustion theory and Chandrasekhar [8] in his study of isothermal gas stars. In 1963, Gelfand published a comprehensive paper [9] hat included a review of (2) for $N = 1, 2, 3$. Approximately ten years later Joseph and Lundgren [10] determined the multiplicity of solutions for all N.

The problem (1) also attracted the attention of many other authors [11, 12, 13, 14]. However, they often considered (1) in fairly simple domains and found the exact solutions in cases where this was possible. In this paper we investigate a nonlinear operator equation that is equivalent to (1). The investigation is based on methods in nonlinear operator equations theory in half-ordered spaces [15, 16, 17]. This approach allows us to obtain theoretical results for almost any domain and justify the method two-sided approximations. Moreover, we impose a condition on the numerical parameter of the problem λ and on the introduced parameter β which is an a priori estimate of the sought solution.

3 Material and methods

The problem (1) is a particular case of a more general problem

$$\begin{aligned} -\Delta u = f(\lambda, x, u) \quad \forall x \in \Omega \subset R^N, \\ u > 0, \quad u|_{\partial\Omega} = 0. \end{aligned} \tag{3}$$

We assume that $f(\lambda, x, u) \geq 0$ in $\bar{\Omega}$, $\lambda > 0$. It is known [15, 16, 17] that (3) is equivalent to the operator equation in the class of continuous functions in Ω

$$u(x) = \int_{\Omega} G(x, s) f(\lambda, s, u(s)) ds, \tag{4}$$

where $G(x, s)$ is a Green's function of the operator Δ of the Dirichlet problem in the domain Ω, $x = (x_1, \ldots, x_N)$, $s = (s_1, \ldots, s_N)$.

Let $A(\lambda, u)$ be an operator with the domain $D(A) = K$

$$A(\lambda, u) = \int_{\Omega} G(x, s) f(\lambda, s, u(s)) ds,$$

where K is a cone of nonnegative functions in the space $C(\bar{\Omega})$.

We will investigate questions related to the positive solutions of (1) and hence the equivalent operator equation (4) using methods in nonlinear operator equations theory in half-ordered spaces. Let us give some definitions and main conclusions of this theory [15, 16, 17].

Definition 1 *Let E be a real Banach space. A convex closed set $K \subset E$ is called a cone if $au \in K$ $(a \geq 0)$ and $-u \notin K$ follows from $u \in K$, $u \neq 0$.*

Using the cone K in E we introduce a half-order as follows:

$$u < v, \text{ if } v - u \in K, \quad u, v \in E.$$

Definition 2 *The cone K is called normal if there exists an $N(K)$ such that $\|u\| \leq N(K)\|v\|$ for $0 < u < v$.*

It is known [15] that the cone of non-negative functions is normal in the space $C(\Omega)$.

Definition 3 *An operator A is positive if $AK \subset K$.*

Definition 4 *An operator A is monotone on the set $T \subset E$ if $Au \leq Av$ follows from $u \leq v$ $(u, v \in T)$.*

Definition 5 *A positive operator in K is called concave if there exists a fixed non-zero element $u_0 \in K$ such that for any non-zero $u \in K$*

$$B_1(u) u_0 \leq Au \leq B_2(u) u_0$$

where $B_1 > 0$, $B_2 > 0$, and also $\forall t \in (0, 1)$

$$A(tu) \geq tAu. \tag{5}$$

Definition 6 *A concave operator A is called u_0-concave if (5) is replaced by a stronger condition: $\forall t \in (0, 1)$ there exists an $\eta(u, t) > 0$ such that*

$$A(tu) \geq (1 + \eta)\ t(Au).$$

Definition 7 *A collection of elements* $\langle v_0, w_0\rangle = \{u : v_0 \leq u \leq w_0\}$ *is called the conical interval.*

Definition 8 *A conical interval* $\langle v_0, w_0\rangle$ *is called invariant for a monotone operator* A *if* A *transforms* $\langle v_0, w_0\rangle$ *into itself, that is* $Av_0 \geq v_0$, $Aw_0 \leq w_0$.

The following theorems hold.

Theorem 1 *[15, Theorem 4.1]. It suffices for the existence, for the monotone operator* A, *of at least one fixed point that there exists an invariant conical interval and that the cone* K *is normal and the operator* A *is completely continuous.*

Theorem 2 *[15, Theorem 4.4]. Let* A *be a monotone operator on the invariant conical interval* $\langle v_0, w_0\rangle$ *and has the unique fixed point* u^* *in* $\langle v_0, w_0\rangle$. *Let* K *be a normal cone and the operator* A *be completely continuous. Then successive approximations*

$$v_n = Av_{n-1}, \quad w_n = Aw_{n-1}, \quad n = 1, 2, \ldots, \tag{6}$$

converge in the norm of the space $C\left(\bar{\Omega}\right)$ *to the exact solution* u^* *of* (3), *whatever the initial approximation* $\tilde{u} \in \langle v_0, w_0\rangle$ *is.*

Remark 1 *From the uniqueness of the fixed point it follows that the limits of* (6) *coincide.*

If $A(\lambda, u) = \lambda Au$, the following theorem holds.

Theorem 3 *[15, Theorem 6.3]. If the operator* A *is* u_0*-concave and monotone, then the equation* $u = \lambda Au$ *does not have two distinct non-zero solutions in the cone* K *for any value of the parameter* λ.

Let us investigate the properties of the operator that corresponds to (1)

$$A(\lambda, u) = \lambda \int\limits_{\Omega} G(x, s)\, e^{u(s)} ds, \quad D(A) = K. \tag{7}$$

It is obvious that the operator A is monotone, since $u_1 \leq u_2$ is followed by $Au_1 \leq Au_2$. In addition, the operator A is completely continuous in the cone K [16, 17].

Let us build the invariant conical interval $\langle v_0, w_0\rangle \subset K$. We put $u = v_0 = 0$ in (7) and build the element $v_1 = \lambda \int\limits_{\Omega} G(x, s)\, e^{v_0(s)} ds = \lambda \int\limits_{\Omega} G(x, s)\, ds \geq v_0 = 0$. Having v_1, we build the element $v_2(x) = \lambda \int\limits_{\Omega} G(x, s)\, e^{v_1(s)} ds \geq v_1$. Continuing this process, we obtain the relations $0 = v_0 \leq v_1 \leq v_2 \leq \cdots \leq v_n$. If we put $u = w_0 = \beta = const > 0$ in (7) we obtain the element $w_1 = \lambda \int\limits_{\Omega} G(x, s)\, e^{w_0(s)} ds = \lambda e^{\beta} \int\limits_{\Omega} G(x, s)\, ds$. The parameters λ and β are chosen in such a way that $w_1 \leq w_0 = \beta$ which leads to the condition $\lambda e^{\beta} \int\limits_{\Omega} G(x, s)\, ds \leq \beta \ \forall x \in \Omega$. It now follows that

$$\max_{x \in \bar{\Omega}} \int\limits_{\Omega} G(x, s)\, ds \leq \frac{1}{\lambda} \beta e^{-\beta}. \tag{8}$$

Building the elements w_i is similar to the process for v_i. We obtain the inequalities

$$0 = v_0 \leq v_1 \leq \cdots \leq v_n \leq \cdots \leq w_n \leq \cdots \leq w_1 \leq w_0 = \beta,$$

therefore the conical interval $\langle v_0, w_0\rangle = \langle 0, \beta\rangle$ is invariant for the operator $A(\lambda, u)$.

In order to prove the concavity of the operator A we use Definition 5. We compose

$$A(\lambda, tu) - tA(\lambda, u) = \lambda \int\limits_{\Omega} G(x,s)\left(e^{tu(s)} - te^{u(s)}\right) ds.$$

It suffices for this difference to be nonnegative that $e^{tu} \geq te^u \ \forall t \in (0,1)$, $u > 0$, whence $tu \geq \ln t + u$, or $u(t-1) \geq \ln t$, or since $t \in (0,1)$, $u \leq \frac{\ln t}{t-1}$. Let φ denote the function $\varphi(t) = \frac{\ln t}{t-1}$, $0 < t < 1$. Since $\varphi(+0) = \lim\limits_{t\to+0} \varphi(t) = +\infty$, $\varphi(1-0) = \lim\limits_{t\to1-0} \varphi(t) = \lim\limits_{t\to1-0} \frac{\ln t}{t-1} = 1$, it follows that the sought solution $u^*(\lambda, x)$ of (1) satisfies the condition $0 < u^*(\lambda, x) < 1$, which coincides with the results of Frank-Kamenetskii [1]. Let u_0 be $u_0(x) = \int\limits_{\Omega} G(x,s)\, ds$. Then since $u \in \langle v_0, w_0\rangle$ it follows that the inequalities (5) are satisfied.

In order to prove the u_0-concavity of the operator A, where $u_0(x) = \int\limits_{\Omega} G(x,s)\, ds$, we compose the difference

$$A(\lambda, tu) - (1+\eta)\, tA(\lambda, u) = \lambda \int\limits_{\Omega} G(x,s)\left(e^{tu(s)} - (1+\eta)\, te^{u(s)}\right) ds.$$

It suffices for this difference to be nonnegative that $e^{tu} - (1+\eta)\, te^u \geq 0 \ \forall t \in (0,1)$, $u > 0$, whence it follows that $0 < \eta(u,t) \leq \frac{e^{tu} - te^u}{te^u}$, which proves the u_0-concavity of the operator A. Thus, we have just proved the following theorem.

Theorem 4 *The problem* (1) *has the unique nonnegative regular solution* $u^* \in C\left(\bar{\Omega}\right)$ *in the cone segment* $\langle v_0, w_0\rangle$, $v_0 = 0$, $w_0 = \beta$ *which can be constructed with two-sided approximations according to the scheme*

$$\begin{aligned} v_n(x) &= \lambda \int\limits_{\Omega} G(x,s)\, e^{v_{n-1}(s)} ds, \quad n = 1, 2, \ldots, \\ w_n(x) &= \lambda \int\limits_{\Omega} G(x,s)\, e^{w_{n-1}(s)} ds, \quad n = 1, 2, \ldots, \end{aligned} \tag{9}$$

which converge uniformly to the sought solution if λ *and* β *satisfy* (8).

Remark 2 *It follows from Theorem 3 that* (1) *does not have two distinct nonnegative regular solutions for any value of the parameter* λ *in the cone* K.

Now we prove the following theorem which has a direct relation to (1) using the technique of proving a similar theorem in [18].

Theorem 5 *Let operator* $A(\lambda, u)$ *be monotone and concave for each* $\lambda > 0$ *and monotonically increasing for each* $u \in K$ *with respect to* λ *and satisfy the condition*

$$A(t\lambda, u) \leq \frac{1}{t} A(\lambda, u), \quad t \in (0, 1]. \tag{10}$$

Let u_1 *and* u_2 *be positive solutions of the equation* $u = A(\lambda, u)$ *which correspond to two distinct values* λ_1 *and* λ_2, $\lambda_1 < \lambda_2$. *Then* $u_1 < u_2$.

Proof. Suppose that it follows from $\lambda_1 < \lambda_2$ that $u_1 > u_2$. Let τ_0 be the maximum constant such that $\tau_0 u_1 \leq u_2$ and $tu_1 \geq u_2$ if $t > \tau_0$, $t \in (0, 1]$. Obviously, $\tau_0 \in (0, 1]$. According to the statement of the theorem we have $u_2 = A(\lambda_2, u_2) \geq [\text{since } u_2 \geq \tau_0 u_1] \geq A(\lambda_2, \tau_0 u_1) \geq *$. It follows from the operator A concavity in the variable u that the inequality (5) can be rewritten as: $A(\lambda_2, \tau_0 u_1) \geq \tau_0 A(\lambda_2, u_1)$ since $\tau_0 \in (0, 1)$, and therefore we have

$$* \geq \tau_0 A(\lambda_2, u_1) = \tau_0 A\left(\frac{\lambda_2}{\lambda_1}\lambda_1, u_1\right) \geq [(10)] \geq \tau_0 \frac{\lambda_2}{\lambda_1} A(\lambda_1, u_1) = \tau_0 \frac{\lambda_2}{\lambda_1} u_1.$$

Thus, we have obtained that $u_2 \geq \tau_0 \frac{\lambda_2}{\lambda_1} u_1$. Further, it follows from the maximality of the constant τ_0 that $\frac{\lambda_2}{\lambda_1} \leq 1$ or $\lambda_2 \leq \lambda_1$ which contradicts the assumption $\lambda_1 < \lambda_2$. This completes the proof of the theorem.

Now we show that all conditions of Theorem 5 are satisfied with respect to (1). The monotonicity and concavity of the operator $A(\lambda, u)$ of the form (7) are shown at the beginning of this section. Assume that $\lambda_1 < \lambda_2$, it follows that $A(\lambda_1, u) - A(\lambda_2, u) = (\lambda_1 - \lambda_2) \int\limits_\Omega G(x, s) e^{u(s)} ds < 0$, that is, the operator A is increasing in the variable λ $\forall u \in K$. Next, we compose the difference $A(t\lambda, u) - \frac{1}{t} A(\lambda, u) = \frac{\lambda(t^2-1)}{t} \int\limits_\Omega G(x, s) e^{u(s)} ds \leq 0$ $\forall t \in (0, 1]$, which proves (10). Thus, two different values λ_1 and λ_2, $\lambda_1 < \lambda_2$, correspond to two positive solutions u_1 and u_2, having $u_1 < u_2$.

4 Results and discussion

Computational experiments for (1) are conducted in four domains for different values of the parameter λ and the corresponding values of the parameter β.

For the domain $\Omega_1 = \{(x_1, x_2) \,|\, 1 - x_1^2 - x_2^2 > 0\}$ the maximum value of λ^* which satisfies (8) is $\lambda^* = 1.47151$, the corresponding value of β is $\beta = 0.99999$.

For the domain $\Omega_2 = \{(x_1, x_2) \,|\, x_2 (1 - x_1^2 - x_2^2) > 0\}$ the maximum value of λ^* which satisfies (8) is $\lambda^* = 3.79257$ with $\beta = 0.99999$. Table 1 lists the values of $w_{11}(x)$ (in the numerator) and $v_{11}(x)$ (in the denominator) at the points of Ω_2 with the polar coordinates (ρ_i, φ_j), where $\rho_i = 0.2i$, $\varphi_j = \frac{\pi j}{10}$, $i = \overline{0, 5}$, $j = \overline{0, 5}$ (the values in the other quarter are symmetric). Figure 1 and 2 show the surface and the level lines of the approximate solution $w_{11}(x)$ respectively and Figure 3 shows the graphs of $w_n(0, x_2)$ (solid line) and $v_n(0, x_2)$ (dashed line) for $n = \overline{0, 5}$.

For the domain $\Omega_3 = \{(x_1, x_2) \,|\, (1 - x_1^2)(1 - x_2^2) > 0\}$ the maximum value of λ^* which satisfies (8) is $\lambda^* = 1.24704$ with $\beta = 0.99999$. Table 2 lists the values of $w_{11}(x)$ (in the numerator) and $v_{11}(x)$ (in the denominator) at the points of Ω_3 with coordinates $(-1 + 0.2i, -1 + 0.2j)$, where $i = \overline{0, 5}$, $j = \overline{0, 5}$ (the values in the other quarters are symmetrical).

The dependency of the norm $\|u_n\|$ in the space $C(\Omega_i)$, $i = \overline{1, 3}$ from λ is shown in Figure 4 in the form of graphs for Ω_1 (solid line), Ω_2 (dashed line) and Ω_3 (dotted line), where $u_n = \frac{v_n + w_n}{2}$.

Hence it follows that if λ tends to zero then the desired solution $u(x)$ tends to zero too.

Since the Green's function is known for several fairly simple domains we apply the Green's quasifunction method for the solution of (3) in the regions Ω_2 and Ω_3 and compare the results

Table 1: The values of $w_{11}(x)$ and $v_{11}(x)$ at the points of Ω_2

ρ	φ					
	0	$\frac{\pi}{10}$	$\frac{\pi}{5}$	$\frac{3\pi}{10}$	$\frac{2\pi}{5}$	$\frac{\pi}{2}$
0	0	0	0	0	0	0
	0	0	0	0	0	0
0.2	0	0.12494	0.22831	0.30383	0.34954	0.36482
	0	0.12493	0.22828	0.30378	0.34949	0.36477
0.4	0	0.21457	0.36929	0.47301	0.53271	0.55213
	0	0.21455	0.36924	0.47294	0.53262	0.55204
0.6	0	0.23401	0.37849	0.46710	0.51620	0.53179
	0	0.23399	0.37845	0.46703	0.51612	0.53170
0.8	0	0.16504	0.24750	0.29450	0.31992	0.32787
	0	0.16503	0.24748	0.29446	0.31987	0.32782
1	0	0	0	0	0	0
	0	0	0	0	0	0

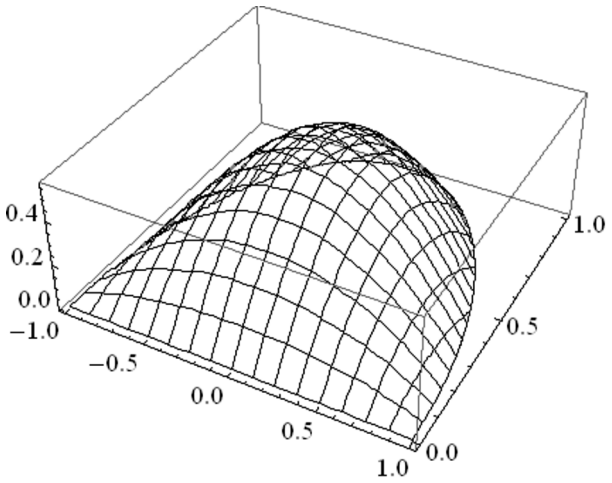

Figure 1: The surface of $w_{11}(x)$

with those obtained according to the scheme (9).

The essence of the Green's quasifunction method in Rvachev's interpretation [19] (for linear partial differential equations) with our adjustments for nonlinear partial differential equations [20, 21, 22] consists in the transition from the boundary value problem (1) to the equivalent nonlinear integral equation

$$u(x) = \int\limits_{\Omega} G_q(x,s)\,\lambda e^{u(s)} ds + \int\limits_{\Omega} u(s)\,K(x,s)\,ds, \tag{11}$$

where

$$G_q(x,s) = \frac{1}{2\pi}\left(\ln\frac{1}{r} - \zeta(x,s)\right), \quad \zeta(x,s) = -\frac{1}{2}\ln\left(r^2 + 4\omega(x)\,\omega(s)\right),$$

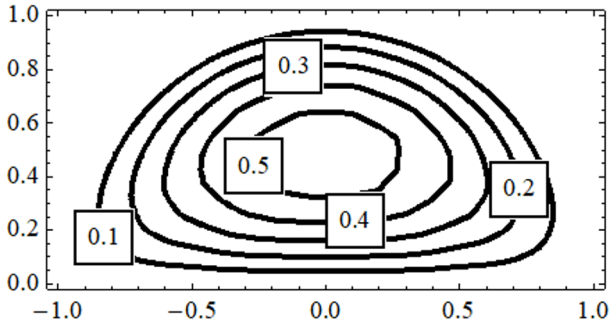

Figure 2: The level lines of $w_{11}(x)$

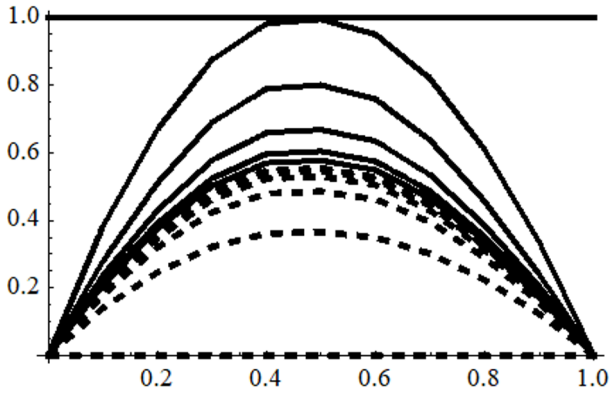

Figure 3: The graphs of $w_n(0, x_2)$ (solid line) and $v_n(0, x_2)$ (dashed line) for $n = \overline{0,5}$

Table 2: The values of $w_{11}(x)$ and $v_{11}(x)$ at the points of Ω_2

x_1	x_2					
	-1	-0.8	-0.6	-0.4	-0.2	0
-1	0	0	0	0	0	0
	0	0	0	0	0	0
-0.8	0	0.08458	0.13846	0.17550	0.19564	0.20311
	0	0.08457	0.13845	0.17549	0.19562	0.20310
-0.6	0	0.13846	0.23664	0.30471	0.34332	0.35683
	0	0.13845	0.23662	0.30468	0.34328	0.35679
-0.4	0	0.17550	0.30471	0.39602	0.44870	0.46700
	0	0.17549	0.30468	0.39598	0.44865	0.46695
-0.2	0	0.19564	0.34332	0.44870	0.51022	0.53146
	0	0.19562	0.34328	0.44865	0.51016	0.53140
0	0	0.20311	0.35683	0.46700	0.53146	0.55376
	0	0.20310	0.35679	0.46695	0.53140	0.55370

$$K(x,s) = -\frac{1}{2\pi}\Delta_s \zeta(x,s),$$

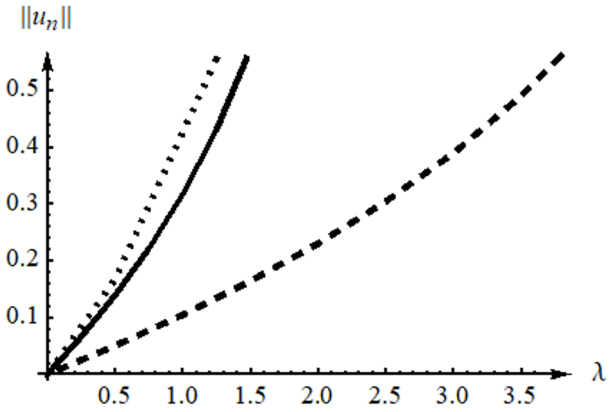

Figure 4: The dependency of the norm $\|u_n\|$ from λ for Ω_1 (solid line), Ω_2 (dashed line) and Ω_3 (dotted line)

for $\Omega \subset R^2$ and

$$G_q(x,s) = \frac{1}{4\pi}\left(\frac{1}{r} - \zeta(x,s)\right), \quad \zeta(x,s) = \left(r^2 + 4\omega(x)\,\omega(s)\right)^{-\frac{1}{2}},$$

$$K(x,s) = -\frac{1}{4\pi}\Delta_s \zeta(x,s),$$

for $\Omega \subset R^3$. Also in both cases

$$r = |x-s|; \quad \Delta_s = \sum_{i=1}^{N} \frac{\partial^2}{\partial s_i^2}, \quad s \in \Omega \subset R^N; \quad \omega(x) = \begin{cases} > 0 & \forall x \in \Omega, \\ 0 & \forall x \in \partial\Omega. \end{cases}$$

We use the method of successive approximations in Svirsky's interpretation [23] to construct an approximate solution of (11) which leads us to a sequence of linear integral equations

$$u_{n+1}(x) - \int_\Omega u_{n+1}(s)\,K(x,s)\,ds = \int_\Omega G_q(x,s)\,\lambda e^{u_n(s)} ds, \quad n = 1,2,\ldots,$$

where we put $u_1(x) = 0$.

Each of these equations can be solved by the Bubnov-Galerkin method [23]. We obtain the following sequence of approximate solutions

$$u_n(x) = \sum_{i=1}^{k} c_{n,i}\phi_i(x), \quad n = 1,2,\ldots,$$

where $\{\phi_i(x)\}_{i=1}^k$ is a coordinate sequence, $c_{n,i}$ $(i = \overline{1,k},\ n = 2,3,\ldots)$ is a solution of a system of linear algebraic equations

$$\sum_{i=1}^{k} c_{2,i}\left[\int_\Omega \phi_i(x)\,\phi_j(x)\,dx - \int_\Omega\int_\Omega K(x,s)\,\phi_i(s)\,\phi_j(x)\,dsdx\right] =$$

$$= \int\limits_{\Omega}\int\limits_{\Omega} G_q(x,s)\,\lambda e^{u_1(s)}\phi_j(x)\,dsdx, \quad j=\overline{1,k},$$

$$\sum_{i=1}^{k} c_{n,i}\left[\int\limits_{\Omega}\phi_i(x)\,\phi_j(x)\,dx - \int\limits_{\Omega}\int\limits_{\Omega} K(x,s)\,\phi_i(s)\,\phi_j(x)\,dsdx\right] =$$

$$= \int\limits_{\Omega}\int\limits_{\Omega} G_q(x,s)\,\lambda e^{u_{n-1}(s)}\phi_j(x)\,dsdx, \quad j=\overline{1,k}, \quad n=3,4,\ldots.$$

We use the Legendre polynomials which are orthogonal on the segment $[-1,1]$ to construct the coordinate sequence

$$P_i(z) = \frac{1}{2^i i!}\frac{d^i}{dz^i}\left(z^2-1\right)^i, \quad z\in R.$$

For the domain $\Omega_2 = \{(x_1,x_2)\,|x_2\,(1-x_1^2-x_2^2)>0\}$, $\lambda^* = 3.79257$ and $\beta = 0.99999$ Table 3 lists the values of $u_n(x)$ for $n=10$ at the points of Ω_2 with the polar coordinates (ρ_i,φ_j), where $\rho_i = 0.2i$, $\varphi_j = \frac{\pi j}{10}$, $i=\overline{0,5}$, $j=\overline{0,5}$ (the values in the other quarter are symmetric).

Table 3: The values of $u_n(x)$ for $n=10$ at the points of Ω_2

ρ	φ					
	0	$\frac{\pi}{10}$	$\frac{\pi}{5}$	$\frac{3\pi}{10}$	$\frac{2\pi}{5}$	$\frac{\pi}{2}$
0	0	0	0	0	0	0
0.2	0	0.12847	0.23407	0.31131	0.35798	0.37355
0.4	0	0.21253	0.37045	0.47570	0.53492	0.55393
0.6	0	0.22795	0.37949	0.47028	0.51744	0.53199
0.8	0	0.15910	0.25238	0.30189	0.32573	0.33292
1	0	0	0	0	0	0

For the domain $\Omega_3 = \{(x_1,x_2)\,|\,(1-x_1^2)\,(1-x_2^2)>0\}$, $\lambda^* = 1.24704$ and $\beta = 0.99999$ Table 4 lists the values of $u_n(x)$ for $n=10$ at the points of Ω_3 with coordinates $(-1+0.2i,-1+0.2j)$, where $i=\overline{0,5}$, $j=\overline{0,5}$ (the values in the other quarters are symmetric).

Now we apply the Green's quasifunction method to (3) for the domain $\Omega_4 = \{(x_1,x_2)\,|1-x_1^8-x_2^8>0\}$. We use the inequality $\lambda < \frac{\beta e^{-\beta}}{\max\limits_{x\in\Omega_3}\int\limits_{\Omega_3} G(x,s)ds}$ to select the values of the parameter λ, where Ω_3 is the smallest square containing Ω_4. Hence we have $\lambda^* = 1.24704$, $\beta = 0.99999$. Table 5 lists the values of $u_n(x)$ for $n=9$ at the points of Ω_4 with polar coordinates (ρ_i,φ_j), where $\rho_i = 0.2i$, $\varphi_j = \frac{\pi j}{10}$, $i=\overline{0,5}$, $j=\overline{0,5}$ (the values in the other quarters are symmetric).

In contrast to the authors who solved the Liouville-Gelfand problem in some rather simple domains and for the most part found solutions in cases where the equations of the problem could be reduced to an ordinary differential equation, in our work we propose a technique for finding a regular solution in almost any domain. However, it should be noted that we have not considered the solutions multiplicity, but proved the existence and uniqueness of a regular solution of (1).

Table 4: The values of $u_n(x)$ for $n = 10$ at the points of Ω_3

x_1	x_2					
	-1	-0.8	-0.6	-0.4	-0.2	0
-1	0	0	0	0	0	0
-1	0	0	0	0	0	0
-0.8	0	0.07651	0.13407	0.17415	0.19778	0.20559
-0.6	0	0.13406	0.23487	0.30502	0.34638	0.36004
-0.4	0	0.17411	0.30500	0.39605	0.44971	0.46744
-0.2	0	0.19772	0.34632	0.44969	0.51060	0.53071
0	0	0.20553	0.35998	0.46741	0.53071	0.55161

Table 5: The values of $u_n(x)$ for $n = 9$ at points of Ω_4

ρ	φ					
	0	$\frac{\pi}{10}$	$\frac{\pi}{5}$	$\frac{3\pi}{10}$	$\frac{2\pi}{5}$	$\frac{\pi}{2}$
0	0.50295	0.50295	0.50295	0.50295	0.50295	0.50295
0.2	0.48881	0.48883	0.48884	0.48885	0.48885	0.48885
0.4	0.44608	0.44621	0.44636	0.44635	0.44621	0.44611
0.6	0.36934	0.37146	0.37444	0.37440	0.37140	0.36930
0.8	0.23022	0.24560	0.26746	0.26739	0.24544	0.23008
1	—	0.04938	0.11977	0.11966	0.04927	—

5 Conclusion

In this paper we have proven the possibility of constructing of two-sided approximations to regular positive solutions of the Liouville-Gelfand problem. We have obtained the conditions that guarantee the convergence of the two-sided iterative process. Constructing the cone segment $\langle v_0, w_0\rangle$, we have obtained an a priori estimate of the sought solution u^*, since $v_0 \leq u^* \leq w_0$. The obtained two-sided approximations to the solution of the problem makes it possible to make a posteriori conclusions.

One of the advantages of the applied method in comparison with others is the relatively simple algorithm in terms of implementation.

We note that for the first time we have constructed two-sided approximations for the Liouville-Gelfand problem in certain domains for which Green's function of the problem is known. We propose to use Green's quasifunction method in case of complex domains where Green's function is unknown. We have improved the method to solve boundary value problems for nonlinear elliptic equations. The above-mentioned represents the scientific novelty of the results.

The practical value lies in the fact that this approach can be used to find solutions to applied problems with mathematical models represented by boundary value problems for nonlinear elliptic equations.

References

[1] D. A. Frank-Kamenetskii *Diffusion and heat exchange in chemical kinetics* (Princeton: Princeton Univ. Press, 1955), 382.

[2] Ya. B. Zel'dovich "K teorii rasprostranenija plameni [Theory of flame propagation]", *Zhurnal fizicheskoj himii*. Vol. 22, No. 1 (1948) : 27-48.

[3] T. Aubin *Some nonlinear problems in Rie-mannian geometry* (Berlin: Springer-Verlag, 1998), 398. doi: 10.1007/978-3-662-13006-3.

[4] C. Bandle *Isoperimetric inequalities and applications* (London: Pitman, 1980), 228.

[5] B. Gidas, W. Ni and L. Nirenberg "Symmetry and related properties via the maximum principle" *Comm. Math. Phys.* Vol. 68, No. 3 (1979) : 209-243.

[6] J. Liouville "Sur l'equation aux derivees partielles $d^2 \log \lambda / dudv \pm \frac{\lambda}{2a^2} = 0$" *J. Math. Pures Appl.* Vol. 18 (1853) : 71-72.

[7] G. Bratu "Sur les equations integrales non lineaires" *Bulletin de la Societe Mathematique de France*. Vol. 42 (1914) : 113-142. doi: 10.24033/bsmf.943.

[8] S. Chandrasekhar *An introduction to the study of stellar structure* (New York: Dover Pub., Inc., 1957), 509.

[9] I. M. Gelfand "Some problems in the theory of quasilinear equations" *American Mathematical Society Translations: Series 2*. Vol. 29 (1963) : 295-381. doi: 10.1090/trans2/029.

[10] D. Joseph and T. Lundgren "Quasilinear Dirichlet problems driven by positive sources" *Arch. Rat. Mech. Anal.* Vol. 49, No. 4 (1973) : 241–269. doi: 10.1007/BF00250508.

[11] D. Ye and F. Zhou "A generalized two dimensional Emden-Fowler equation with exponential nonlinearity", *Calculus of Variations and Partial Differential Equations*. Vol. 13, No. 2 (2001) : 141-158. doi: 10.1007/s005260000069.

[12] Y. Bozhkov "Noether Symmetries and Critical Exponents" *Symmetry, Integrability and Geometry: Methods and Applications*. Vol. 1, No. 022 (2005) : 1-12. doi: 10.3842/SIGMA.2005.022.

[13] V. L. Rvachev, A. P. Slesarenko and N. A. Safonov "Matematicheskoe modelirovanie teplovogo samovosplamenenija dlja stacionarnyh uslovij metodom R-funkcij [Mathematical modeling of thermal autoignition for stationary conditions using R-functions method]" *Doklady AN Ukrainy. Serija A*. No. 12 (1992) : 24-27.

[14] N. S. Kurpel *Projection-iterative methods for solution of operator equations* (Providence: American Mathematical Society, 1976), 196.

[15] M. A. Krasnoselskii *Positive solutions of operator equations* (Groningen: P. Noordhoff, 1964), 381.

[16] V. I. Opoitsev "A generalization of the theory of monotone and concave operators" *Trans. Moscow, Math. Soc.* Vol. 2 (1979) : 243-279.

[17] V. I. Opoitsev and T. A. Khurodze *Nelinejnye operatory v prostranstvah s konusom [Nonlinear operators in spaces with a cone]* (Tbilisi: Izd-vo Tbilisskogo un-ta, 1984), 270.

[18] A. I. Kolosov "Ob odnom klasse uravnenij s vognutymi operatorami, zavisjashhimi ot parametra [A class of equations with concave operators that depend on a parameter]" *Matematicheskie zametki*. Vol. 49, No. 4 (1991) : 74-80.

[19] V. L Rvachev *Teorija R-funkcij i nekotorye ee prilozhenija [Theory of R-functions and some applications]* (Kiev: Nauk. dumka, 1982), 552.

[20] S. V. Kolosova, M. V. Sidorov "Primenenie iteracionnyh metodov k resheniju jellipticheskih kraevyh zadach s jeksponencial'noj nelinejnost'ju [Application of iterative methods to the solution of elliptic boundary value problems with exponential nonlinearity]" *Radiojelektronika i informatika*. No. 3 (62) (2013) : 28-31.

[21] S. V. Kolosova, V. S. Lukhanin, M. V. Sidorov "O nekotoryh podhodah k resheniju kraevyh zadach dlja nelinejnyh jellipticheskih uravnenij [On some approaches to the solution of boundary value problems for nonlinear elliptic equations]" *Trudy XVI Mezhdunarodnogo simpoziuma «Metody diskretnyh osobennostej v zadachah matematicheskoj fiziki» (MDOZMF-2013)*. (2013) : 205-208.

[22] S. V. Kolosova, V. S. Lukhanin "Pro dodatni rozv'jazki odniyeyi zadachi z geterotonnim operatorom ta pro pobudovu poslidovnih nablizhen' [On positive solutions of one problem with heterotone operator and the construction of successive approximations]" *Visnik Harkivs'kogo nacional'nogo universitetu imeni V.N. Karazina. Serija Matematichne modeljuvannja. Informacijni tehnologiyi. Avtomatizovani sistemi upravlinnja*. No. 31 (2016) : 59-72.

[23] I. V. Svirsky *Metody tipa Bubnova-Galerkina i posledovatel'nyh priblizhenij [Methods of the Bubnov-Galerkin type and a sequence of approximation]* (Moscow: Nauka, 1968), 199.