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Singularly perturbed linear oscillator with piecewise-constant argument

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The Cauchy problem for singularly perturbed linear differential equation the second order with piecewise-constant argument is considered in the article. The definition of singularly perturbed linear harmonic oscillator with piecewise-constant argument is given in the paper. The system of fundamental solutions of homogeneous singularly perturbed differential equation with piecewise-constant argument are constructed according to the nonhomogeneous singularly perturbed differential equation with piecewise-constant argument. With the help of the system of fundamental solutions, the initial functions are constructed and their asymptotic representation are obtained. By using the reduction method, the analytical formula of the solution of singularly perturbed the initial value problem with piecewise-constant argument is obtained. In addition, the unperturbed Cauchy problem is constructed according to the singularly perturbed Cauchy problem. The solution of the unperturbed Cauchy problem is obtained. When the small parameter tends to the zero, the solution of singularly perturbed the Cauchy problem with piecewise-constant argument approaches the solution of the unperturbed Cauchy problem with piecewise-constant argument. The theorem on the passage to the limit is proved.
Key words: piecewise-constant argument of generalized type, small parameter, singular perturbation.

Сингулярно возмущенный линейный осциллятор с кусочно-постоянным аргументом

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В статье рассматривается задача Коши для сингулярно возмущенного линейного дифференциального уравнения второго порядка с кусочно-постоянным аргументом. В статье дано определение сингулярно возмущенного линейного гармонического осциллятора с кусочно-постоянным аргументом. Построена система фундаментальных решений однородного сингулярно возмущенного дифференциального уравнения с кусочно-постоянным аргументом. С помощью системы фундаментальных решений, начальные функции построены и их асимптотических представлений получены. Используя метод редукции, получена аналитическая формула решения сингулярно возмущенной начальной задачи с кусочно-постоянным аргументом. Кроме того, невозмущенная задача Коши построена в соответствии с сингулярно возмущенной задачи Коши. Получено решение невозмущенной задачи Коши. Когда малый параметр стремится к нулю, доказана сходимость решения сингулярно возмущенной задачи Коши с кусочно-постоянным аргументом к решению невозмущенной задачи Коши с кусочно-постоянным аргументом. Доказана теорема о предельном переходе.
Ключевые слова: кусочно-постоянный аргумент обобщенного типа, малый параметр, сингулярное возмущение.

Құрақ-тұрақты аргументті сингулярлы ауытқыған сызықты осциллятор

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Мақалада құрақ-тұрақты аргументті екінші ретті сингулярлы ауытқыған сызықты дифференциалдық теңдеу үшін Коши есебі қарастырылған. Жұмыста құрақ-тұрақты аргументті сингулярлы ауытқыған сызықты гармоникалық осциллятордың анықтамасы келтірілді. Құрақ-тұрақты аргументті сингулярлы ауытқыған біртекті емес дифференциалдық теңдеуге сәйкес біртекті құрақ-тұрақты аргументті сингулярлы ауытқыған дифференциалдық теңдеудің іргелі шешімдер жүйесі құрылды. Іргелі шешімдер жүйесінің көмегімен бастапқы функциялары құрылып, олардың асимптотикалық сипаттары алынған. Редукция тәсілін қолданып, қойылған құрақ-тұрақты аргументті сингулярлы ауытқыған бастапқы есебінің шешімінің аналитикалық формуласы алынды. Сонымен қоса, сингулярлы ауытқыған Коши есебіне сәйкес ауытқымаған Коши есебі құрылды. Ауытқымаған Коши есебінің шешімі алынды. Кіші параметр нөлге ұмтылғанда, құрақ-тұрақты аргументті сингулярлы ауытқыған Коши есебінің шешімі құрақ-тұрақты аргументті ауытқымаған Коши есебінің шешіміне ұмтылатыны көрсетілді. Шектік көшу теоремасы дәлелденді.
Түйін сөздер: жалпыланған түрдегі құрақ-тұрақты аргумент, кіші параметр, сингулярлы ауытқу.

1 Introduction and review of literature

Singularly perturbed equations are often used as mathematical models describing processes in physics, chemical kinetics and mathematical biology and they often arise during investigation of applied problems of technology and engineering (Damiano, 1996: 333-372), (Gondal, 1988: 1080-1085), (Hek, 2010: 347-386), (Kokotovic, 1984: 501-550), (Owen, 2001: 655-684). The first who emphasis application significance of singular problems and the necessity of their appearance as mathematical models was Prandtl. He attracted attention to them, when he developed the theory of the boundary layer in hydrodynamics in 1904. Several scientists such as S. Haykin, L. I. Gutenmakher, I. S. Gradstein, K. Friedrichs, W. Wazow, Levinson and others were interested in singularly perturbed equations. Systematic investigation of singularly perturbed equation by many mathematicians began after the proof by A. N. Tikhonov fundamental and well-known limit theorems for nonlinear systems of ordinary differential equations in (Tikhonov, 1952: 575-586). Note that other mathematical school of singularly perturbed equations in Kazakhstan and abroad investigate only boundary value problems, which does not have an initial jump. In works (Dauylbaev, 2017: 214-225), (Dauylbaev, 2016: 147-154), (Dauylbaev, 2015: 747-761), the initial and boundary value problems were considered that are equivalent to the Cauchy problem with the initial jump for differential and integro-differential equations in the stable case.

Systematic study of theoretical and practical problems involving piecewise constant arguments was initiated in the early 80's. Since then, differential equations with piecewise constant arguments have attracted great attention from the researches in mathematics, biology, engineering and other fields. A mathematical model including piecewise constant argument was first considered by Busenberg and Cooke (Busenberg, 1982: 179-187) in 1982. They constructed a first-order linear equation to investigate vertically transmitted diseases. Following this work, using the method of reduction to discrete equations, many authors have analyzed various types of differential equations with piecewise constant argument.

In (Akhmet, 2005: 11-20), (Akhmet,2011), (Akhmet, 2007: 367-383), (Akhmet, 2007: 646-663), (Akhmet, 2015: 2483-2495), (Akhmet, 2012: 337-352) M. Akhmet first proposed to investigate differential equations with piecewise constant argument of generalized type. Currently, these equations are widely used and the foundations of the theory are constructed. There are many works that are written in the framework of the theory created by M. Akhmet. Theoretically, this concerns theorems on the existence of bounded solutions, periodic, almost periodic solutions, exponential dichotomies, integral surfaces, differential equations of mathematical physics, and many others.

The movement of the weights on the spring, the pendulum, the charge in the electrical circuit and also the evolution of many systems in physics, chemistry, biology and other sciences in time under certain assumptions can be described by the same differential equations that in the theory of oscillations serves as the basic model. This model is called a linear harmonic oscillator.

The equation for the free oscillations of singularly perturbed linear harmonic oscillator with piecewise-constant argument has the form

$$\varepsilon y''(t) + ly'(t) + ky(\beta(t)) = 0,$$

where $y(t)$ is a variable describing the state of the system (weight shift, capacitor charge, etc.), l is a parameter characterizing the energy loss (friction in the mechanical system, resistance in the circuit), k is a natural frequency of oscillation, t is a time.

2 Material and methods

2.1 Consider the following Cauchy problem for singularly perturbed differential equation with piecewise-constant argument

$$\varepsilon y''(t) + ly'(t) + ky(\beta(t)) = F(t) \tag{1}$$

$$y(0,\varepsilon) = y_0, \;\; y'(0,\varepsilon) = y_1, \tag{2}$$

where $\varepsilon > 0$ is a small parameter, l, k, y_0, y_1 are known constants. The piecewise-constant argument is determined with the function $\beta(t) = \theta_i$, if $t \in [\theta_i, \theta_{i+1}), \;\; i = \overline{1,p}$, $0 < \theta_1 < \theta_2 < \ldots < \theta_p < T$.

The following conditions are hold:

(C1)$F(t)$ is a continuously differentiable function in the segment $0 \le t \le T$.

(C2) $l > 0, \;\; 0 \le t \le T$.

If $t \in [0, \theta_1)$, then the Cauchy problem (1),(2) has a form:

$$\varepsilon y''(t) + ly'(t) = F(t) - ky_0, \tag{3}$$

$$y(0,\varepsilon) = y_0, \;\; y'(0,\varepsilon) = y_1. \tag{4}$$

The system of fundamental solutions of the homogeneous equation according to the equation (3) is determined the following type:

$$y_1(t,\varepsilon) = 1 + O(\varepsilon), \;\; y_2(t,\varepsilon) = e^{-\frac{l}{\varepsilon}t}(1 + O(\varepsilon)). \tag{5}$$

Definition. Let the functions $K_i(t,s,\varepsilon), \;\; i = 1,2, \;\; 0 \le s < t \le \theta_1$ be a solution for the following problem

$$\varepsilon K_i''(t,s,\varepsilon) + lK_i'(t,s,\varepsilon) = 0, \;\; i = 1,2, \tag{6}$$

$$K_i^{(j)}(s,s,\varepsilon) = \delta_{i-1,j}, \;\; j = 0,1, \tag{7}$$

the functions $K_i(t,s,\varepsilon), \;\; i = 1,2$ are called *the initial functions* and can be represented as

$$K_i(t,s,\varepsilon) = \frac{W_i(t,s,\varepsilon)}{W(s,\varepsilon)} \;\; i = 1,2, \tag{8}$$

where $\delta_{i-1,j}$ is the Kronecker symbol, $W(s,\varepsilon)$ is the Wronskian of the fundamental set of solutions $y_1(s,\varepsilon), y_2(s,\varepsilon)$, $W_i(t,s,\varepsilon)$ are the second order determinant obtained from the Wronskian $W(s,\varepsilon)$ by replacing the $i-$th row with the system of fundamental solutions (5).

By the formula (5),(8), we obtain the asymptotic representation of the initial $K_i(t,s,\varepsilon), \;\; i = 1,2$ functions as $\varepsilon \to 0:$

$$K_1(t,s,\varepsilon) = 1 + O(\varepsilon), \quad K_2(t,s,\varepsilon) = \frac{\varepsilon}{l}\left(1 - e^{-\frac{l}{\varepsilon}(t-s)} + O(\varepsilon)\right). \tag{9}$$

The solution of the Cauchy problem (3),(4) is as follows:

$$y(t,\varepsilon) = y_0 + \frac{\varepsilon}{l}\left(\frac{ky_0}{l} + y_1\right)\left(1 - e^{-\frac{l}{\varepsilon}t}\right) - \frac{ky_0}{l}t + \frac{1}{l}\int\limits_0^t \left(1 - e^{-\frac{l}{\varepsilon}(t-s)}\right)F(s)ds. \tag{10}$$

We get the unperturbed problem according to the Cauchy problem (3),(4) as $\varepsilon = 0:$

$$l\overline{y}'(t) + ky_0 = F(t), \quad \overline{y}(0) = y_0. \tag{11}$$

The solution of unperturbed initial problem is as follows:

$$\overline{y}(t) = y_0 - \frac{ky_0}{l}t + \frac{1}{l}\int_0^t F(s)ds. \tag{12}$$

Thus, if $t \in [\theta_i, \theta_{i+1}),\ \ i = \overline{1,p}$, then the Cauchy problem (1),(2) has a form:

$$\varepsilon y''(t) + ly'(t) = F(t) - ky(\theta_i), \tag{13}$$

$$y(\theta_i, \varepsilon) = y(\theta_i),\ \ y'(\theta_i, \varepsilon) = y'(\theta_i). \tag{14}$$

The system of fundamental solutions of the homogeneous equation according to the equation (13) is determined the following type:

$$y_1(t,\varepsilon) = 1 + O(\varepsilon),\ \ y_2(t,\varepsilon) = e^{-\frac{l}{\varepsilon}(t-\theta_i)}(1 + O(\varepsilon)). \tag{15}$$

By the formula (8),(15), we obtain the asymptotic representation of the initial $K_i(t,s,\varepsilon),\ \ i = 1,2$ functions as $\varepsilon \to 0:$

$$K_1(t,s,\varepsilon) = 1 + O(\varepsilon), \quad K_2(t,s,\varepsilon) = \frac{\varepsilon}{l}\left(1 - e^{-\frac{l}{\varepsilon}(t-s)} + O(\varepsilon)\right). \tag{16}$$

Let us try to define the solution of the Cauchy problem (1),(2) in the interval $t \in [\theta_i, \theta_{i+1}),\ \ \ i = \overline{1,p}$, we make the change of variable $s = t - \theta_i,\ \ t = \theta_i \Longrightarrow s = 0$, we obtain

$$\varepsilon\frac{d^2y}{ds^2} + l\frac{dy}{ds} = F(t) - ky(\theta_i),\ \ y(0,\varepsilon) = y(\theta_i),\ \ y'(0,\varepsilon) = y'(\theta_i). \tag{17}$$

The problem (17) is similar to (3),(4), the solution of the problem (17) is as follows $s \in [0, \theta_1)$:

$$y(s,\varepsilon) = y(\theta_i) + \frac{\varepsilon}{l}\left(\frac{ky(\theta_i)}{l} + y'(\theta_i)\right)\left(1 - e^{-\frac{l}{\varepsilon}s}\right) - \frac{ky(\theta_i)}{l}s + \frac{1}{l}\int_0^s \left(1 - e^{-\frac{l}{\varepsilon}(t-p)}\right)F(p)dp. \tag{18}$$

As a result, the solution of the problem (13),(14) is as follows:

$$y(t,\varepsilon) = y(\theta_i) + \frac{\varepsilon}{l}\left(\frac{ky(\theta_i)}{l} + y'(\theta_i)\right)\left(1 - e^{-\frac{l}{\varepsilon}(t-\theta_i)}\right) -$$

$$-\frac{ky(\theta_i)}{l}(t - \theta_i) + \frac{1}{l}\int_{\theta_i}^t \left(1 - e^{-\frac{l}{\varepsilon}(t-s)}\right)F(s)ds. \tag{19}$$

For determining vector $\begin{pmatrix} y(\theta_i) \\ y'(\theta_i) \end{pmatrix}$, $i=\overline{1,p}$, we obtain system of difference equation:

$$y(\theta_{i+1},\varepsilon)=\left(1-\frac{k}{l}(\theta_{i+1}-\theta_i)+\frac{\varepsilon k}{l^2}\left(1-e^{-\frac{l}{\varepsilon}(\theta_{i+1}-\theta_i)}\right)\right)y(\theta_i)+$$

$$+\frac{\varepsilon}{l}\left(1-e^{-\frac{l}{\varepsilon}(\theta_{i+1}-\theta_i)}\right)y'(\theta_i)+\frac{1}{l}\int\limits_{\theta_i}^{\theta_{i+1}}\left(1-e^{-\frac{l}{\varepsilon}(\theta_{i+1}-s)}\right)F(s)ds, \tag{20}$$

$$y'(\theta_{i+1},\varepsilon)=-\frac{k}{l}\left(1-e^{-\frac{l}{\varepsilon}(\theta_{i+1}-\theta_i)}\right)y(\theta_i)+e^{-\frac{l}{\varepsilon}(\theta_{i+1}-\theta_i)}y'(\theta_i)+\frac{1}{\varepsilon}\int\limits_{\theta_i}^{\theta_{i+1}}e^{-\frac{l}{\varepsilon}(\theta_{i+1}-s)}F(s)ds.$$

We get the unperturbed problem according to the Cauchy problem (13),(14) as $\varepsilon=0:$

$$l\overline{y}'(t)+ky(\theta_i)=F(t), \quad \overline{y}|_{t=\theta_i}=\overline{y}(\theta_i). \tag{21}$$

The solution of the unperturbed initial problem (21) is as follows:

$$\overline{y}(t)=\overline{y}(\theta_i)-\frac{k\overline{y}(\theta_i)}{l}t+\frac{1}{l}\int\limits_{\theta_i}^{t}F(s)ds. \tag{22}$$

Theorem. If the conditions (C1),(C2) are true, then the Cauchy problem (1),(2) has a unique solution on the interval $t\in[\theta_i,\theta_{i+1}),\quad i=\overline{0,p}$ and expressed by the formula (19), such that the following limiting equalities hold:

$$\lim_{\varepsilon\to 0}y(t,\varepsilon)=\overline{y}(t), \quad \theta_i\le t<\theta_{i+1},$$

$$\lim_{\varepsilon\to 0}y'(t,\varepsilon)=\overline{y}'(t), \quad \theta_i\le t<\theta_{i+1}.$$

Proof. Consider first the interval $t\in[0,\theta_1)$. With the help of the solution (10) of singularly perturbed initial problem (3),(4) and the solution (12) of the unperturbed problem (11), when $\varepsilon\to 0$, as a result

$$\lim_{\varepsilon\to 0}y(t,\varepsilon)=y_0-\frac{ky_0}{l}t+\frac{1}{l}\int\limits_0^t F(s)ds\equiv\overline{y}(t), \quad 0\le t<\theta_1,$$

$$\lim_{\varepsilon\to 0}y'(t,\varepsilon)=-\frac{ky_0}{l}\equiv\overline{y}'(t), \quad 0\le t<\theta_1.$$

Continue in this way to prove the limiting equalities of the Theorem in the interval $t\in[\theta_i,\theta_{i+1}),\quad i=\overline{1,p}$. Theorem is proved.

2.2 Example. Consider the following initial value problem:

$$\varepsilon y''(t) + y'(t) - y([t]) = 0, \tag{23}$$

$$y(0,\varepsilon) = 1, \;\; y'(0,\varepsilon) = 3, \tag{24}$$

where $[\cdot]$ denotes the greatest integer function.

If $t \in [0,1)$, then the Cauchy problem (23),(24) has a form:

$$\varepsilon y''(t) + y'(t) = 1, \;\; y(0,\varepsilon) = 1, \;\; y'(0,\varepsilon) = 3. \tag{25}$$

The solution of the initial value problem (25) is as follows:

$$y(t,\varepsilon) = 1 + 2\varepsilon\left(1 - e^{-\frac{t}{\varepsilon}}\right) + t, \;\; t \in (0,1).$$

Let us try to define the solution of the Cauchy problem (23),(24) in the interval $t \in [n, n+1)$, we make the change of variable $s = t - n, \;\; t = n \Longrightarrow s = 0$, we obtain the solution of the Cauchy problem (23),(24)

$$y(t,\varepsilon) = \left(1 - \varepsilon\left(1 - e^{-\frac{t-n}{\varepsilon}}\right) + t - n\right) y(n) + \varepsilon\left(1 - e^{-\frac{t-n}{\varepsilon}}\right) y'(n), \;\; t \in [n, n+1). \tag{26}$$

For determining vector $\widehat{y}(n) = \begin{pmatrix} y(n) \\ y'(n) \end{pmatrix}$, we obtain system of difference equation:

$$\widehat{y}(n+1) = A(\varepsilon)\widehat{y}(n), \tag{27}$$

where

$$A(\varepsilon) = \begin{pmatrix} 2 - \varepsilon\left(1 - e^{-\frac{1}{\varepsilon}}\right) & \varepsilon\left(1 - e^{-\frac{1}{\varepsilon}}\right) \\ \left(1 - e^{-\frac{1}{\varepsilon}}\right) & e^{-\frac{1}{\varepsilon}} \end{pmatrix}. \tag{28}$$

Vector $\widehat{y}(n)$ has a form

$$\widehat{y}(n) = A^n(\varepsilon)\widehat{y}(0), \tag{29}$$

where

$$A^n(\varepsilon) = T(\varepsilon)J_A(\varepsilon)T^{-1}(\varepsilon), \quad \widehat{y}(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \tag{30}$$

$J_A(\varepsilon) = \begin{pmatrix} \lambda_1(\varepsilon) & 0 \\ 0 & \lambda_2(\varepsilon) \end{pmatrix}$ is a matrix Jordan, $T(\varepsilon)$ is a transforming matrix.

We construct the characteristic equation respectively system (27):

$$\lambda^2 - \left(e^{-\frac{1}{\varepsilon}} + 2 - \varepsilon\left(1 - e^{-\frac{1}{\varepsilon}}\right)\right)\lambda + 2e^{-\frac{1}{\varepsilon}} - \varepsilon\left(1 - e^{-\frac{1}{\varepsilon}}\right) = 0. \tag{31}$$

Roots of the characteristic equation (31) is defined by the formula

$$\lambda_{1,2}(\varepsilon)=\frac{e^{-\frac{1}{\varepsilon}}+2-\varepsilon\left(1-e^{-\frac{1}{\varepsilon}}\right)\pm\sqrt{\left(e^{-\frac{1}{\varepsilon}}+2-\varepsilon\left(1-e^{-\frac{1}{\varepsilon}}\right)\right)^2-4\left(2e^{-\frac{1}{\varepsilon}}-\varepsilon\left(1-e^{-\frac{1}{\varepsilon}}\right)\right)}}{2}. \tag{32}$$

Eigenvectors according to the found eigenvalues is determined by linear algebraic system

$$\begin{cases} \left(2-\varepsilon\left(1-e^{-\frac{1}{\varepsilon}}\right)-\lambda_{1,2}\right)\xi_1+\varepsilon\left(1-e^{-\frac{1}{\varepsilon}}\right)\xi_2=0 \\ \left(1-e^{-\frac{1}{\varepsilon}}\right)\xi_1+\left(e^{-\frac{1}{\varepsilon}}-\lambda_{1,2}\right)\xi_2=0. \end{cases} \tag{33}$$

The solution of linear algebraic system (33) is as follows

$$\xi_1(\varepsilon)=\frac{e^{-\frac{1}{\varepsilon}}-2+\varepsilon\left(1-e^{-\frac{1}{\varepsilon}}\right)\mp\sqrt{\left(e^{-\frac{1}{\varepsilon}}+2-\varepsilon\left(1-e^{-\frac{1}{\varepsilon}}\right)\right)^2-4\left(2e^{-\frac{1}{\varepsilon}}-\varepsilon\left(1-e^{-\frac{1}{\varepsilon}}\right)\right)}}{2\left(e^{-\frac{1}{\varepsilon}}-1\right)}\xi_2(\varepsilon). \tag{34}$$

By the found eigenvectors according to eigenvalues are defined the elements of matrix $T(\varepsilon)$:

$$T_{11}(\varepsilon)=\frac{e^{-\frac{1}{\varepsilon}}-2+\varepsilon\left(1-e^{-\frac{1}{\varepsilon}}\right)-\sqrt{\left(e^{-\frac{1}{\varepsilon}}+2-\varepsilon\left(1-e^{-\frac{1}{\varepsilon}}\right)\right)^2-4\left(2e^{-\frac{1}{\varepsilon}}-\varepsilon\left(1-e^{-\frac{1}{\varepsilon}}\right)\right)}}{2\left(e^{-\frac{1}{\varepsilon}}-1\right)}, \tag{35}$$

$$T_{12}(\varepsilon)=\frac{e^{-\frac{1}{\varepsilon}}-2+\varepsilon\left(1-e^{-\frac{1}{\varepsilon}}\right)+\sqrt{\left(e^{-\frac{1}{\varepsilon}}+2-\varepsilon\left(1-e^{-\frac{1}{\varepsilon}}\right)\right)^2-4\left(2e^{-\frac{1}{\varepsilon}}-\varepsilon\left(1-e^{-\frac{1}{\varepsilon}}\right)\right)}}{2\left(e^{-\frac{1}{\varepsilon}}-1\right)},$$

$$T_{21}(\varepsilon)=1, \quad T_{22}(\varepsilon)=1.$$

The elements of the inverse matrix $T^{-1}(\varepsilon)$ are as follow:

$$T^{-1}_{11}(\varepsilon)=\frac{1-e^{-\frac{1}{\varepsilon}}}{\sqrt{\left(e^{-\frac{1}{\varepsilon}}+2-\varepsilon\left(1-e^{-\frac{1}{\varepsilon}}\right)\right)^2-4\left(2e^{-\frac{1}{\varepsilon}}-\varepsilon\left(1-e^{-\frac{1}{\varepsilon}}\right)\right)}}, \tag{36}$$

$$T^{-1}_{12}(\varepsilon)=\frac{e^{-\frac{1}{\varepsilon}}-2+\varepsilon\left(1-e^{-\frac{1}{\varepsilon}}\right)+\sqrt{\left(e^{-\frac{1}{\varepsilon}}+2-\varepsilon\left(1-e^{-\frac{1}{\varepsilon}}\right)\right)^2-4\left(2e^{-\frac{1}{\varepsilon}}-\varepsilon\left(1-e^{-\frac{1}{\varepsilon}}\right)\right)}}{\sqrt{\left(e^{-\frac{1}{\varepsilon}}+2-\varepsilon\left(1-e^{-\frac{1}{\varepsilon}}\right)\right)^2-4\left(2e^{-\frac{1}{\varepsilon}}-\varepsilon\left(1-e^{-\frac{1}{\varepsilon}}\right)\right)}},$$

$$T_{21}^{-1}(\varepsilon) = -\frac{1-e^{-\frac{1}{\varepsilon}}}{\sqrt{\left(e^{-\frac{1}{\varepsilon}}+2-\varepsilon\left(1-e^{-\frac{1}{\varepsilon}}\right)\right)^2-4\left(2e^{-\frac{1}{\varepsilon}}-\varepsilon\left(1-e^{-\frac{1}{\varepsilon}}\right)\right)}},$$

$$T_{22}^{-1}(\varepsilon) = -\frac{e^{-\frac{1}{\varepsilon}}-2+\varepsilon\left(1-e^{-\frac{1}{\varepsilon}}\right)-\sqrt{\left(e^{-\frac{1}{\varepsilon}}+2-\varepsilon\left(1-e^{-\frac{1}{\varepsilon}}\right)\right)^2-4\left(2e^{-\frac{1}{\varepsilon}}-\varepsilon\left(1-e^{-\frac{1}{\varepsilon}}\right)\right)}}{\sqrt{\left(e^{-\frac{1}{\varepsilon}}+2-\varepsilon\left(1-e^{-\frac{1}{\varepsilon}}\right)\right)^2-4\left(2e^{-\frac{1}{\varepsilon}}-\varepsilon\left(1-e^{-\frac{1}{\varepsilon}}\right)\right)}}.$$

$$J_A(\varepsilon) = \begin{pmatrix} \lambda_1^n(\varepsilon) & 0 \\ 0 & \lambda_2^n(\varepsilon) \end{pmatrix}. \tag{37}$$

Considering the formulas (30),(32),(35)-(37), we obtain the limiting equalities for the elements of the matrix $A^n(\varepsilon)$:

$$\lim_{\varepsilon\to 0} a_{11}^n(\varepsilon) = 2^n, \quad \lim_{\varepsilon\to 0} a_{12}^n(\varepsilon) = 0, \quad \lim_{\varepsilon\to 0} a_{21}^n(\varepsilon) = 2^{n-1}, \quad \lim_{\varepsilon\to 0} a_{22}^n(\varepsilon) = 0. \tag{38}$$

The solution of the initial value problem (23),(24) on the interval $[n, n+1)$ is as follows:

$$y(t,\varepsilon) = A(t,\varepsilon)\cdot A^n(\varepsilon)\cdot \widehat{y}(0), \tag{39}$$

where

$$A(t,\varepsilon) = \begin{pmatrix} 1-\varepsilon\left(1-e^{-\frac{t-n}{\varepsilon}}\right)+t-n & \varepsilon\left(1-e^{-\frac{t-n}{\varepsilon}}\right) \\ \left(1-e^{-\frac{t-n}{\varepsilon}}\right) & e^{-\frac{t-n}{\varepsilon}} \end{pmatrix}, \tag{40}$$

and $A^n(\varepsilon)$, $\widehat{y}(0)$ are expressed by the formula (30).

Consider unperturbed initial value problem:

$$\begin{cases} \overline{y}'(t) = \overline{y}([t]), \\ \overline{y}(0) = 1. \end{cases} \tag{41}$$

If $t \in [0,1)$, then the problem (41) has a form

$$\begin{cases} \overline{y}'(t) = 1, \\ \overline{y}(0) = 1. \end{cases} \tag{42}$$

The solution of unperturbed problem (42) is as follows

$$\overline{y}(t) = t+1. \tag{43}$$

As shown in the above, the solution of the unperturbed problem (41) in the interval $t \in [n, n+1)$ is defined by the formula

$$\overline{y}(t) = (t-n+1)\overline{y}(n). \tag{44}$$

The unknown value $\overline{y}(n)$ is determined by the following difference equation:

$$\overline{y}(n+1) = 2\overline{y}(n). \tag{45}$$

By solving the equation (45), we obtain $\overline{y}(n) = 2^n$.

The results can be seen to perform the following limiting equalities:

$$\lim_{\varepsilon\to 0} y(t,\varepsilon) = t+1 \equiv \overline{y}(t), \quad 0 < t < 1,$$

$$\lim_{\varepsilon\to 0} y'(t,\varepsilon) = 1 \equiv \overline{y}'(t), \quad 0 < t < 1,$$

$$\lim_{\varepsilon\to 0} y(t,\varepsilon) = 2^n(t-n+1) \equiv \overline{y}(t), \quad n < t < n+1,$$

$$\lim_{\varepsilon\to 0} y'(t,\varepsilon) = 2^n \equiv \overline{y}'(t), \quad n < t < n+1.$$

3 Conclusion

The Cauchy problem for a singularly perturbed linear differential equation with piecewise-constant is considered in the article. A solution of a singularly perturbed Cauchy problem is obtained using the reduction method. A solution of the unperturbed problem is obtained. The theorem on the passage to the limit is proved.

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