

IRSTI 27.33.19

The Cauchy problem for singularly perturbed higher-order integro-differential equations

Mirzakulova A.E., al-Farabi Kazakh National University, Almaty, Kazakhstan,
+7-707-859-13-90, E-mail: aziza.mirzakulova@mail.ru
Dauylbaev M.K., al-Farabi Kazakh national university, Almaty, Kazakhstan,
+7-777-232-37-69, E-mail: dmk57@mail.ru
Akhmet M.U., Middle-east technical university, Ankara, Turkey,
+90 (312) 210-2970, E-mail: marat@metu.edu.tr
Dzhetpisbaeva A.K., al-Farabi Kazakh national university, Almaty, Kazakhstan

The article is devoted to research the Cauchy problem for singularly perturbed higher-order linear integro-differential equation with a small parameter at the highest derivatives, provided that the roots of additional characteristic equation have negative signs. The aim of this paper is to bring asymptotic estimation of the solution of a singularly perturbed Cauchy problem and the asymptotic convergence of the solution of a singularly perturbed initial value problem to the solution of an unperturbed initial value problem. In this paper the fundamental system of solutions, initial functions of a singularly perturbed homogeneous differential equation are constructed and their asymptotic estimates are obtained. By using the initial functions, we obtain an explicit analytical formula of the solution. The theorem about asymptotic estimate of a solution of the initial value problem is proved. The unperturbed Cauchy problem is constructed. We find the solution of the unperturbed Cauchy problem. An estimate difference of the solution of a singularly perturbed and unperturbed initial value problems. The asymptotic convergence of solution of a singularly perturbed initial value problem to the solution of the unperturbed initial value problem is proved.
Key words: singular perturbation, small parameter, the initial functions, asymptotics, passage to the limit.

Задачи Коши для сингулярно возмущенных интегро-дифференциальных уравнений высшего порядка

Мирзакулова А.Е., Казахский национальный университет имени аль-Фараби, г. Алматы, Республика Казахстан, +7-707-859-13-90, E-mail: aziza.mirzakulova@mail.ru
Дауылбаев М.К., Казахский национальный университет имени аль-Фараби, г. Алматы, Республика Казахстан, +7-777-232-37-69, E-mail: dmk57@mail.ru
Ахмет М.У., Средне-восточный технический университет, г. Анкара, Турция, +90 (312) 210-2970, E-mail: marat@metu.edu.tr
Джетписбаева А.К., Казахский национальный университет имени аль-Фараби, г. Алматы

В работе рассматривается сингулярно возмущенная задача Коши для линейного интегро-дифференциального уравнения высшего порядка с малым параметром при старших производных при условии, что корни дополнительного характеристического уравнения имеют отрицательные знаки. Работа посвящена получению асимптотических оценок решения сингулярно возмущенной задачи Коши и асимптотическая сходимость решения сингулярно возмущенной начальной задачи к решению вырожденной начальной задачи. В статье построена фундаментальная система решений, начальные функции сингулярно возмущенного однородного дифференциального уравнения, получены их асимптотические оценки. С помощью начальных функции получена явная аналитическая формула решений заданной начальной задачи. С помощью аналитической формулы доказана теорема об асимптотической оценке решения рассматриваемой начальной задачи. Построена невозмущенная задача Коши. Найдено решение невозмущенной задачи Коши. Получена оценка разности между решениями сингулярно возмущенной и невозмущенной начальных задач. Доказана асимптотическая сходимость решения заданной сингулярно возмущенной начальной задачи к решению невозмущенной начальной задачи.

Ключевые слова: сингулярное возмущение, малый параметр, начальные функции, асимптотика, предельный переход

Сингулярлы ауытқыған жоғарғы ретті интегралды-дифференциалдық теңдеулер үшін Коши есебі

Мирзакулова А.Е., әл-Фараби атындағы Қазақ ұлттық университеті, Алматы қ.,
 Қазақстан Республикасы, +7-707-859-13-90, Электрондық пошта: aziza.mirzakulova@mail.ru
 Дауылбаев М.Қ., әл-Фараби атындағы Қазақ ұлттық университеті, Алматы қ.,
 Қазақстан Республикасы, +7-777-232-37-69, Электрондық пошта: dmk57@mail.ru
 Ахмет М.У., Орталық-шығыс техникалық университеті, Анкара қ., Түркия,
 +90 (312) 210-2970, Электрондық пошта: marat@metu.edu.tr
 Джетписбаева А.К., әл-Фараби атындағы Қазақ ұлттық университеті, Алматы қ.,
 Қазақстан Республикасы, +7-707-826-52-17, Электрондық пошта: kakimovna_a@mail.ru

Жұмыста үлкен туындыларының алдында кіші параметрі бар жоғарғы ретті сингулярлы ауытқыған сызықты интегралды-дифференциалдық теңдеу үшін "қосымша сипаттауыш теңдеу" деп аталатын теңдеудің түбірлерінің таңбасы теріс болған жағдайында Коши есеп қарастырылған. Жұмыс сингулярлы ауытқыған Коши есеп шешімінің асимптотикалық бағалауын алуға және сингулярлы ауытқыған бастапқы есебінің шешімі сәйкес ауытқымаған бастапқы есебінің шешіміне асимптотикалық жинақталуын зерттеуге бағытталған. Мақалада берілген теңдеуге сәйкес біртекті сингулярлы ауытқыған дифференциалдық теңдеудің іргелі шешімдер жүйесі, бастапқы функциялары құрылып, олардың асимптотикалық бағалаулары алынған. Бұл функциялардың көмегімен қарастырылып отырған Коши есеп шешімінің аналитикалық формуласы алынған. Аналитикалық формуланың көмегімен қойылған бастапқы есеп шешімінің асимптотикалық бағалауы туралы теорема дәлелденген. Осы сингулярлы ауытқыған Коши есебіне сәйкес ауытқымаған Коши есебі құрылды. Ауытқымаған Коши есебінің шешімі алынды. Сингулярлы ауытқыған бастапқы есеп шешімі мен ауытқымаған бастапқы есеп шешімінің арасындағы айырым бағаланды. Берілген сингулярлы ауытқыған бастапқы есеп шешімінің ауытқымаған бастапқы есеп шешіміне ұмтылатыны дәлелденді.

Түйін сөздер: сингулярлы ауытқы, кіші параметр, бастапқы функциялар, асимптотика, шектік көшу.

1 Introduction and review of literature

Singularly perturbed equations act as mathematical models in many applied problems related to diffusion, heat and mass transfer, chemical kinetics and combustion, heat propagation in thin bodies, semiconductor theory, gyroscope motion, quantum mechanics, biology and biophysics, and many other branches of science and technology.

Various asymptotic methods exist to approximate solutions of certain singularly perturbed problems: the method of matching of outer and inner expansions (Cole, 1968), (Hinch, 1981), (Kevorkian, 1981), (Nayfeh, 2008), (Van Dyke, 1964), (Lagerstrom, 1988), (Eckhaus, 1973); the boundary layer function method (or composite asymptotic expansion) (O'Malley, 1974); the method of Lomov or regularization method (Lomov, 1992); the method WKB or Liouville-Green method (Olver, 1974); the method of multiple scales (Verhulst, 2005); the averaging method (Sanders, 1985); methods for relaxation oscillations (Grasman, 1987) and others. Development of different asymptotic methods can be found, e.g., in O'Malley (O'Malley, 1991) and Vasil'eva (Vasil'eva, 1994: 440-452). Each of these methods has a certain area of applicability; it successfully works in solving certain problems and becomes inapplicable in solving other problems. The initial value problem with initial jumps for a nonlinear ordinary differential equation of the second order with a small parameter was studied by M.I. Vishik and L.A. Lyusternik (Vishik, 1960: 1242-1245) and K. A. Kassymov (Kassymov, 1962: 187-188). They show that the solution of the original initial value problem tends to the solution of the

degenerate equation with changed initial conditions, when the small parameter approaches zero. Such problems became known as the Cauchy problems with initial jumps. Solution of a singularly perturbed problems have the phenomenon of an initial jump at some point (for example, at the initial point) in given segment if the value of the solution at this point is not the same as the solution of the unperturbed problem and the fast variable of solution at this point is unbounded as the small parameter tends to zero. Singularly perturbed problems with the initial jump possess specific characteristics, which is not typical for a singularly perturbed problems, do not have the phenomenon of the initial jump.

2 Material and methods

Consider the following singularly perturbed the Cauchy problem:

$$L_\varepsilon y \equiv \sum_{r=1}^m \varepsilon^r A_{n+r}(t) y^{(n+r)}(t, \varepsilon) + \sum_{k=0}^n A_k(t) y^{(k)}(t, \varepsilon) = F(t) + \int_0^1 \sum_{j=0}^{l+1} H_j(t, x) y^{(j)}(x, \varepsilon) dx, \quad (1)$$

$$y^{(i)}(0, \varepsilon) = \alpha_i, \quad i = \overline{0, n+m-1}, \quad (2)$$

where $\varepsilon > 0$ is a small parameter, α_i , $i = \overline{0, n+m-1}$ are known constants, $A_{n+m}(t) = 1$, $l = \text{fix}\{0, 1, \dots, n-2\}$.

We will need the following assumptions:

(C1) $A_i(t) \in C^{n+m-1}([0, 1])$, $i = \overline{0, n+m}$, $F(t) \in C([0, 1])$ and $H_j(t, x)$, $j = \overline{0, l+1}$ are sufficiently smooth functions in the domain $D = \{0 \leq t \leq 1, 0 \leq x \leq 1\}$.

(C2) $A_n(t) \neq 0$, $0 \leq t \leq 1$.

(C3) The roots $\mu_1 \neq \mu_2 \neq \dots \neq \mu_m$ of "additional characteristic equation" $\mu^m + A_{n+m-1}(t)\mu^{m-1} + \dots + A_{n+1}(t)\mu + A_n(t) = 0$ satisfy the following inequalities $\text{Re}\mu_1 < 0$, $\text{Re}\mu_2 < 0$, \dots , $\text{Re}\mu_m < 0$.

Similarly the Cauchy problem (1),(2) for ordinary differential equation was considered in (Nurgabyl, 2012: 4-8). In the particular case, similarly boundary value problem with initial jumps for this case $m = 2$, $l = 2$ (Dauylbaev, 2016: 145-152), (Dauylbaev, 2017: 214-225).

2.1 Construction of the fundamental systems of solutions

We consider the following homogeneous singularly perturbed differential equation associated with (1):

$$L_\varepsilon y \equiv \sum_{r=1}^m \varepsilon^r A_{n+r}(t) y^{(n+r)}(t, \varepsilon) + \sum_{k=0}^n A_k(t) y^{(k)}(t, \varepsilon) = 0. \quad (3)$$

The fundamental systems of solutions of the equation (3) has the following asymptotic representation as $\varepsilon \rightarrow 0$:

$$y_i^{(q)}(t, \varepsilon) = y_{i0}^{(q)}(t) + O(\varepsilon), \quad i = \overline{1, n}, \quad q = \overline{0, n+m-1},$$

$$y_{n+r}^{(q)}(t, \varepsilon) = \frac{1}{\varepsilon^q} e^{\frac{1}{\varepsilon} \int_0^t \mu_r(x) dx} (\mu_r^q(t) y_{n+r,0}^{(q)}(t) + O(\varepsilon)), \quad r = \overline{1, m}, \quad q = \overline{0, n+m-1} \quad (4)$$

where $y_{i0}(t)$, $i = \overline{1, n}$ are solutions of the problem:

$$L_0 y_{i0}(t) = 0, y_{j0}^{(i-1)}(0) = \delta_{ij}, i = \overline{1, n}, j = \overline{1, n}, \tag{5}$$

δ_{ij} is a Kronecker symbol, $y_{n+r,0}(t)$, $r = \overline{1, m}$ are solutions of the following problem:

$$p_r(t)y'_{n+r,0}(t) + q_r(t)y_{n+r,0}(t) = 0, y_{n+r,0}(0) = 1, r = \overline{1, m}$$

where

$$p_r(t) = \sum_{i=0}^m A_{n+i}(t)(n+i)\mu_{r0}^{n+i-1}(t), r = \overline{1, m},$$

$$q_r(t) = \mu'_{r0}(t) \sum_{i=0}^m A_{n+i}(t)C_{n+i}^2 \mu_{r0}^{n+i-2}(t) + A_{n-1}(t)\mu_{r0}^{n-1}(t), r = \overline{1, m},$$

$$C_{n+i}^2 = \frac{(n+i)!}{2!(n+i-2)}, i = \overline{0, m}.$$

In view of (4), for the Wronskian $W(t, \varepsilon)$ the following asymptotic representation holds as $\varepsilon \rightarrow 0$:

$$W(t, \varepsilon) = \frac{1}{\varepsilon^\lambda} \overline{W}(t)\pi(t)\omega(t)\exp\left(\frac{1}{\varepsilon} \int_0^t \overline{\mu}(x)dx\right) (1 + O(\varepsilon)) \neq 0, \tag{6}$$

where $\overline{W}(t)$ is the Wronskian,

$$\overline{W}(t) = \begin{vmatrix} y_{10}(t) & \dots & y_{n0}(t) \\ \vdots & \ddots & \vdots \\ y_{10}^{(n-1)}(t) & \dots & y_{n0}^{(n-1)}(t) \end{vmatrix}, \lambda = \frac{2n+m-1}{2},$$

$$\overline{\mu}(x) = \mu_1(x) + \dots + \mu_m(x) = \sum_{k=1}^m \mu_k(x), \pi(t) = \prod_{k=1}^m y_{n+k}(t)\mu_k^n(t),$$

the determinant $\omega(t)$ is the m -th order Vandermond determinant,

$$\omega(t) = \begin{vmatrix} 1 & \dots & 1 \\ \mu_1(t) & \dots & \mu_m(t) \\ \vdots & \ddots & \vdots \\ \mu_1^{m-1}(t) & \dots & \mu_m^{m-1}(t) \end{vmatrix} \neq 0.$$

2.2 Construction of the initial functions

Definition. The functions $K_i(t, s, \varepsilon)$, $i = \overline{1, n+m}$ are called *initial functions*, if they satisfy the following problem:

$$L_\varepsilon K_i(t, s, \varepsilon) = 0, i = \overline{1, n+m}, 0 \leq s < t \leq 1,$$

$$K_i^{(j)}(s, s, \varepsilon) = \delta_{i-1, j}, \quad j = \overline{0, n+m-1},$$

and that can be represented in the form:

$$K_i(t, s, \varepsilon) = \frac{W_i(t, s, \varepsilon)}{W(s, \varepsilon)}, \quad i = \overline{1, n+m}, \quad (7)$$

$W_i(t, s, \varepsilon)$ is the $n+m$ -th order determinant obtained from the Wronskian $W(s, \varepsilon)$ by replacing the i -th row with $y_1(t, \varepsilon), y_2(t, \varepsilon), \dots, y_{n+m}(t, \varepsilon)$.

In view of (6),(7), for the initial functions $K_i^{(q)}(t, s, \varepsilon)$, $i = \overline{1, n+m}$, $q = \overline{0, n+m-1}$ the following asymptotic representation hold as $\varepsilon \rightarrow 0$:

$$K_i^{(q)}(t, s, \varepsilon) = \frac{\overline{W}_i^{(q)}(t, s)}{\overline{W}(s)} + \varepsilon^{n-q} \sum_{k=1}^m e^{\frac{1}{\varepsilon} \int_s^t \mu_k(x) dx} \frac{y_{n+k,0}(t) \mu_k^q(t)}{y_{n+k,0}(s) \mu_k^n(s)} \cdot \frac{\omega_{1k}(s)}{\omega(s)} \cdot \frac{\overline{W}_i(s)}{\overline{W}(s)} +$$

$$+ O \left(\varepsilon + \varepsilon^{n+1-q} \sum_{k=1}^m e^{\frac{1}{\varepsilon} \int_s^t \mu_k(x) dx} \right), \quad i = \overline{1, n}, \quad q = \overline{0, n+m-1}; \quad (8)$$

$$K_{n+r}^{(q)}(t, s, \varepsilon) = \varepsilon^r \left((-1)^r \frac{\omega_{r+1}(s)}{\omega(s) A_n(s)} \cdot \frac{\overline{W}_n^{(q)}(t, s)}{\overline{W}(s)} + \varepsilon^{n-1-q} \sum_{k=1}^m e^{\frac{1}{\varepsilon} \int_s^t \mu_k(x) dx} \cdot \right.$$

$$\left. \frac{y_{n+k,0}(t) \mu_k^q(t)}{y_{n+k,0}(s) \mu_k^n(s)} \cdot \frac{\omega_{rk}(s)}{\omega(s)} + O \left(\varepsilon + \varepsilon^{n-q} \sum_{k=1}^m e^{\frac{1}{\varepsilon} \int_s^t \mu_k(x) dx} \right) \right), \quad r = \overline{1, m}, \quad q = \overline{0, n+m-1}, \quad (9)$$

where $\omega_{ij}(s)$ is the $m-1$ -th order determinant obtained from the determinant $\omega(s)$ by deleting i -th row and j -th column, $\omega_i(s)$ is the m -th order determinant obtained from the following determinant

$$\begin{vmatrix} 1 & \dots & 1 \\ \mu_1(s) & \dots & \mu_m(s) \\ \vdots & \ddots & \vdots \\ \mu_1^m(s) & \dots & \mu_m^m(s) \end{vmatrix}$$

by deleting i -th row, $\overline{W}_n^{(q)}(t, s)$ is the determinant obtained from the Wronskian $\overline{W}(s)$ by replacing the n -th row with $y_{10}^{(q)}(t), \dots, y_{n0}^{(q)}(t)$, $\overline{W}_i(t)$ is the $n+1$ -th order determinant obtained from the following determinant

$$\begin{vmatrix} y_{10}(t) & \dots & y_{n0}(t) \\ y'_{10}(t) & \dots & y'_{n0}(t) \\ \vdots & \ddots & \vdots \\ y_{10}^{(n)}(t) & \dots & y_{n0}^{(n)}(t) \end{vmatrix}$$

by deleting the i -th row.

2.3 The analytical formula of solution

Let us denote by the right-hand side of the equation (1):

$$z(t, \varepsilon) = F(t) + \int_0^1 \sum_{j=0}^{l+1} H_j(t, x) y^{(j)}(x, \varepsilon) dx. \quad (10)$$

We seek to find the solution of the differential $L_\varepsilon y \equiv z(t, \varepsilon)$ equation in the form:

$$y(t, \varepsilon) = \sum_{i=1}^{n+m} C_i K_i(t, 0, \varepsilon) + \frac{1}{\varepsilon^m} \int_0^t K_{n+m}(t, s, \varepsilon) z(s, \varepsilon) ds, \quad (11)$$

where $K_i(t, s, \varepsilon)$, $i = \overline{1, n+m}$ are the initial functions, $z(t, \varepsilon)$ is a unknown function.

Substituting (11) into (10), we obtain the following expression:

$$\begin{aligned} z(t, \varepsilon) = & F(t) + \int_0^1 \sum_{j=0}^{l+1} H_j(t, x) \sum_{i=1}^{n+m} C_i K_i^{(j)}(x, 0, \varepsilon) dx + \\ & + \int_0^1 \sum_{j=0}^{l+1} H_j(t, x) \frac{1}{\varepsilon^m} \int_0^x K_{n+m}^{(j)}(x, s, \varepsilon) z(s, \varepsilon) ds dx. \end{aligned}$$

By replacing the order of sum and integral, we obtain

$$\begin{aligned} z(t, \varepsilon) = & F(t) + \sum_{i=1}^{n+m} C_i \int_0^1 \sum_{j=0}^{l+1} H_j(t, x) K_i^{(j)}(x, 0, \varepsilon) dx + \\ & + \int_0^1 z(s, \varepsilon) ds \frac{1}{\varepsilon^m} \int_s^1 \sum_{j=0}^{l+1} H_j(t, x) K_{n+m}^{(j)}(x, s, \varepsilon) dx. \end{aligned}$$

By introducing of additional symbols, we obtain the following Fredholm integral equation of the second kind:

$$z(t, \varepsilon) = f(t, \varepsilon) + \int_0^1 H(t, s, \varepsilon) z(s, \varepsilon) ds, \quad (12)$$

where

$$\begin{aligned} f(t, \varepsilon) = & F(t) + \sum_{i=1}^{n+m} C_i \int_0^1 \sum_{j=0}^{l+1} H_j(t, x) K_i^{(j)}(x, 0, \varepsilon) dx, \\ H(t, s, \varepsilon) = & \frac{1}{\varepsilon^m} \int_s^1 \sum_{j=0}^{l+1} H_j(t, x) K_{n+m}^{(j)}(x, s, \varepsilon) dx. \end{aligned}$$

(C4) 1 is not an eigenvalue of the kernel $H(t, s, \varepsilon)$.

In view of condition (C4) integral equation (12) has an unique solution, that can be represented in the form:

$$z(t, \varepsilon) = f(t, \varepsilon) + \int_0^1 R(t, s, \varepsilon) f(s, \varepsilon) ds, \quad (13)$$

where $R(t, s, \varepsilon)$ is a resolvent of the kernel $H(t, s, \varepsilon)$.

Substituting (13) into (11), we obtain the analytical formula of solution:

$$y(t, \varepsilon) = \sum_{i=1}^{n+m} C_i \left(K_i(t, 0, \varepsilon) + \frac{1}{\varepsilon^m} \int_0^t K_{n+m}(t, s, \varepsilon) \bar{\varphi}_i(s, \varepsilon) ds \right) + \frac{1}{\varepsilon^m} \int_0^t K_{n+m}(t, s, \varepsilon) \bar{F}(s, \varepsilon) ds, \quad (14)$$

where C_i , $i = \overline{1, n+m}$ are unknown constants, $K_i(t, s, \varepsilon)$, $i = \overline{1, n+m}$ are the initial functions,

$$\bar{\varphi}_i(s, \varepsilon) = \int_0^1 \sum_{j=0}^{l+1} \bar{H}_j(s, x, \varepsilon) K_i^{(j)}(x, s, \varepsilon) dx, \quad i = \overline{1, n+m},$$

$$\bar{H}_j(s, x, \varepsilon) = H_j(s, x) + \int_0^1 R(s, p, \varepsilon) H_j(p, x) dp, \quad (15)$$

$$\bar{F}(s, \varepsilon) = F(s) + \int_0^1 R(s, p, \varepsilon) F(p) dp.$$

By using initial conditions (2) in (14), we find the constants $C_i = \alpha_{i-1}$, $i = \overline{1, n+m-1}$.

Theorem 1. Let assumptions (C1)-(C4) hold. Then the Cauchy problem (1),(2) on the interval $0 \leq t \leq 1$ has an unique solution and expressed by the formula:

$$y(t, \varepsilon) = \sum_{i=1}^{n+m} \alpha_{i-1} \left(K_i(t, 0, \varepsilon) + \frac{1}{\varepsilon^m} \int_0^t K_{n+m}(t, s, \varepsilon) \bar{\varphi}_i(s, \varepsilon) ds \right) + \frac{1}{\varepsilon^m} \int_0^t K_{n+m}(t, s, \varepsilon) \bar{F}(s, \varepsilon) ds, \quad (16)$$

where $K_i(t, s, \varepsilon)$, $i = \overline{1, n+m}$ are initial functions, functions $\bar{\varphi}_i(s, \varepsilon)$, $\bar{F}(s, \varepsilon)$, $\bar{H}_j(s, x, \varepsilon)$ defined by the formula (15).

2.4 Asymptotic estimations of solution

Theorem 2. Let assumptions (C1)-(C4) hold. Then for the solution of the Cauchy problem (1),(2) and its derivatives the following asymptotic estimation hold as $\varepsilon \rightarrow 0$:

$$|y^{(q)}(t, \varepsilon)| \leq C \left(\sum_{i=0}^{n-1} |\alpha_i| + \sum_{r=1}^m \varepsilon^r \cdot |\alpha_{n-1+r}| + \max_{0 \leq t \leq 1} |F(t)| \right) + C\varepsilon^{n-q} e^{-\gamma \frac{t}{\varepsilon}} \left(\sum_{i=0}^n |\alpha_i| + \sum_{r=1}^{m-1} \varepsilon^r \cdot |\alpha_{n+r}| + \max_{0 \leq t \leq 1} |F(t)| \right) \cdot \left| \sum_{k=1}^m \frac{\bar{\mu}_k^q(t) \omega_{mk}(t)}{\bar{\mu}_k^{n+1}(0)} \right|, \quad (17)$$

where $q = \overline{0, n + m - 1}$, $C > 0, \gamma > 0$ is a constant independent of ε ,

$$\sum_{k=1}^m \frac{\bar{\mu}_k^q(t) \omega_{mk}(t)}{\bar{\mu}_k^{n+1}(0)} \Big|_{t=0} \equiv 0, \quad j = \overline{n + 1, n + m - 1}.$$

Proof. In view of (7)-(9) and conditions (C1)-(C3), for the initial functions $K_i(t, s, \varepsilon)$, $i = \overline{1, n + m}$ the following asymptotic estimation hold:

$$|K_i^{(q)}(t, s, \varepsilon)| \leq C \left(1 + \varepsilon^{n-q} \exp \left(-\frac{t-s}{\varepsilon} \right) \right), \quad i = \overline{1, n}, \quad q = \overline{0, n + m - 1}, \quad (18)$$

$$|K_{n+r}^{(q)}(t, s, \varepsilon)| \leq C\varepsilon^r \left(1 + \varepsilon^{n-1-q} \exp \left(-\frac{t-s}{\varepsilon} \right) \right), \quad r = \overline{1, m}, \quad q = \overline{0, n + m - 1}. \quad (19)$$

By applying the asymptotic estimations of the initial functions (18),(19) in (15), we obtain the following asymptotic estimations for the function $\bar{\varphi}_i(s, \varepsilon)$, $i = \overline{1, n + m}$:

$$|\bar{\varphi}_i(s, \varepsilon)| \leq C, \quad i = \overline{1, n}, \quad |\bar{\varphi}_{n+r}(s, \varepsilon)| \leq C\varepsilon^r, \quad r = \overline{1, m}. \quad (20)$$

By applying (18)-(20) in (16), we obtain asymptotic estimations of the solution (17). Theorem 2 is proved.

2.5 The unperturbed initial problem

We consider the following unperturbed initial value problem:

$$L_0 \bar{y} \equiv \sum_{k=0}^n A_k(t) \bar{y}^{(k)}(t) = F(t) + \int_0^1 \sum_{j=0}^{l+1} H_j(t, x) \bar{y}^{(j)}(x) dx, \quad (21)$$

$$\bar{y}^{(i)}(0) = \alpha_i, \quad i = \overline{0, n - 1}. \quad (22)$$

Let us denote by

$$\bar{z}(t) = F(t) + \int_0^1 \sum_{j=0}^{l+1} H_j(t, x) \bar{y}^{(j)}(x) dx. \quad (23)$$

We seek to find the solution of the problem (21),(22):

$$\bar{y}(t) = \sum_{i=1}^n C_i \bar{K}_i(t, 0) + \frac{1}{A_n(t)} \int_0^t \bar{K}_n(t, s) \bar{z}(s) ds, \quad (24)$$

where C_i , $i = \overline{1, n}$ are unknown constants, $\bar{K}_i(t, s)$, $i = \overline{1, n}$ are the initial functions, $\bar{z}(t)$ is a unknown function. By substituting function (24) in (23), we obtain the following Fredholm integral equation of the second kind:

$$\bar{z}(t) = F(t) + \sum_{i=1}^n C_i \int_0^1 \sum_{j=0}^{l+1} H_j(t, x) \bar{K}_i^{(j)}(x, 0) dx + \int_0^1 \bar{H}(t, s) \bar{z}(s) ds, \quad (25)$$

where

$$\bar{H}(t, s) = \int_s^1 \sum_{j=0}^{l+1} H_j(t, x) \left(\frac{\bar{K}_n(x, s)}{A_n(x)} \right)^{(j)} dx.$$

(C5) 1 is not an eigenvalue of the kernel $\bar{H}(t, s)$. Then the integral equation (25) has an unique solution and that can be represented in the form:

$$\bar{z}(t) = \bar{F}^0(t) + \sum_{i=1}^n C_i \bar{\varphi}_i^0(t), \quad (26)$$

where $\bar{R}(t, s)$ is a resolvent of the kernel $\bar{H}(t, s)$,

$$\bar{\varphi}_i^0(t) = \int_0^1 \sum_{j=0}^{l+1} \bar{H}_j^0(t, x) \bar{K}_i^{(j)}(x, 0) dx, \quad i = \overline{1, n},$$

$$\bar{H}_j^0(t, x) = H_j(t, x) + \int_0^1 \bar{R}(t, s) H_j(s, x) ds, \quad j = \overline{0, l+1}, \quad (27)$$

$$\bar{F}^0(t) = F(t) + \int_0^1 \bar{F}(t, s) F(s) ds.$$

Substituting (26) into (24) and by using initial conditions (22) into obtained solution, we obtain the solution of the initial value problem (21),(22):

$$\bar{y}(t) = \sum_{i=1}^n \alpha_{i-1} \left(\bar{K}_i(t, 0) + \frac{1}{A_n(t)} \int_0^t \bar{K}_n(t, s) \bar{\varphi}_i^0(s) ds \right) + \frac{1}{A_n(t)} \int_0^t \bar{K}_n(t, s) \bar{F}^0(s) ds, \quad (28)$$

where $\bar{K}_i(t, s)$, $i = \overline{1, n}$ are the initial functions, functions $\bar{\varphi}_i^0(s)$, $\bar{F}^0(s)$ are defined by the formula (27).

Let us denote by

$$u(t, \varepsilon) = y(t, \varepsilon) - \bar{y}(t) \Rightarrow y(t, \varepsilon) = u(t, \varepsilon) + \bar{y}(t). \tag{29}$$

Substituting (29) into the initial value problem (1),(2), in view of the problem (21),(22), we obtain the following problem for $u(t, \varepsilon)$:

$$L_\varepsilon u \equiv \sum_{r=1}^m \varepsilon^r A_{n+r}(t) u^{(n+r)}(t, \varepsilon) + \sum_{k=0}^n A_k(t) u^{(k)}(t, \varepsilon) = - \sum_{r=1}^m \varepsilon^r A_{n+r}(t) \bar{y}^{(n+r)}(t) + \int_0^1 \sum_{j=0}^{l+1} H_j(t, x) u^{(j)}(x, \varepsilon) dx, \tag{30}$$

$$u^{(i)}(0, \varepsilon) = 0, \quad i = \overline{0, n-1}, \quad u^{(i)}(0, \varepsilon) = \alpha_i - \bar{y}^{(i)}(0), \quad i = \overline{n, n+m-1}. \tag{31}$$

The problem (30),(31) is of the same type as the problem (1),(2), by applying estimates (17), we obtain asymptotic estimations for the function $u(t, \varepsilon)$ as $\varepsilon \rightarrow 0$:

$$|u^{(q)}(t, \varepsilon)| \leq C \left(\sum_{r=1}^m \varepsilon^r \cdot |\alpha_{n-1+r} - \bar{y}^{(n-1+r)}(0)| + \varepsilon \right) + C \varepsilon^{n-q} e^{-\gamma \frac{t}{\varepsilon}} \left(\sum_{r=1}^{m-1} \varepsilon^r \cdot |\alpha_{n+r} - \bar{y}^{(n+r)}(0)| + \varepsilon \right) \cdot \left| \sum_{k=1}^m \frac{\bar{\mu}_k^q(t) \omega_{mk}(t)}{\bar{\mu}_k^{n+1}(0)} \right|, \tag{32}$$

$$q = \overline{0, n+m-1}.$$

From the estimations (32), the following limiting equalities hold:

$$\lim_{\varepsilon \rightarrow 0} u^{(q)}(t, \varepsilon) = 0, \quad 0 \leq t \leq 1, \quad q = \overline{0, n},$$

$$\lim_{\varepsilon \rightarrow 0} u^{(q)}(t, \varepsilon) = 0, \quad 0 < t \leq 1, \quad q = \overline{n+1, n+m-1}.$$

Theorem 3. Let assumptions (C1)-(C5) hold. Then for the solution $y(t, \varepsilon)$ of the Cauchy problem the following limiting equalities hold:

$$\lim_{\varepsilon \rightarrow 0} y^{(q)}(t, \varepsilon) = \bar{y}^{(q)}(t), \quad 0 \leq t \leq 1, \quad q = \overline{0, n},$$

$$\lim_{\varepsilon \rightarrow 0} y^{(q)}(t, \varepsilon) = \bar{y}^{(q)}(t), \quad 0 < t \leq 1, \quad q = \overline{n+1, n+m-1}$$

where function $\bar{y}(t)$ is the solution of the unperturbed problem (21),(22) and defined by the formula (28).

3 Conclusion

The article is devoted to research the Cauchy problem for singularly perturbed $n + m$ order linear integro-differential equation with a small parameter at the m -th derivatives. In the work the fundamental system of solutions, initial functions of a singularly perturbed homogeneous differential equation are constructed and their asymptotic representation are obtained. By using the initial functions, we obtain an explicit analytical formula of the solution. The asymptotic convergence of solution of a singularly perturbed initial value problem to the solution of the unperturbed initial value problem is proved.

References

- [1] *Cole J.D.* Perturbation methods in applied mathematics. - New York: Blaisdell Publishing Company, 1968.
- [2] *Hinch E. J.* Perturbation methods. - Cambridge: Cambridge University Press, 1981.
- [3] *Kevorkian J., Cole J.D.* Perturbation methods in applied mathematics. - New York: Springer-Verlag, 1981.
- [4] *Nayfeh A.H.* Perturbation methods. - New York: John Wiley & Sons, 2008.
- [5] *Van Dyke M.* Perturbation methods in fluid dynamics. -New York: Academic Press, 1964.
- [6] *Lagerstrom* Matched asymptotic expansions. - New York: Springer-Verlag, 1988.
- [7] *Eckhaus W.* Matched asymptotic expansions and singular perturbations. -London, 1973.
- [8] *O'Malley R.E.* Introduction to singular perturbation. -New York: Academic Press, 1974.
- [9] *Lomov S.A.* Introduction to the general theory of singular perturbations. -Providence, Rhode Island: AMS, 1992.
- [10] *Olver F.* Introduction to asymptotics and special functions. - New York: Academic Press, 1974.
- [11] *Verhulst F.* Methods and applications of singular perturbations: Boundary layers and multiple timescale dynamics. - New York: Springer, 2005.
- [12] *Sanders J.A., Verhulst F.* Averaging methods in nonlinear dynamical systems. - New York: Springer-Verlag, 1985.
- [13] *Grasman J.* Asymptotic methods for relaxation oscillations and applications. - New York: Springer-Verlag, 1987.
- [14] *O'Malley R.E.* Singular perturbations methods for ordinary differential equations. -New York: Springer-Verlag, 1991.
- [15] *Vasil'eva A.* On the development of singular perturbation theory at Moscow State University and elsewhere // SIAM Review. -1994.- №36. -P. 440-452.
- [16] *Vishik I., Lyusternik L. A.* On the initial jump for non-linear differential equations containing a small parameter // Doklady Akademii Nauk SSSR. -1960. - Vol. 132, № 6. -P. 1242-1245.
- [17] *Kassymov K. A.* On the asymptotics of the solution of the Cauchy problem with large initial conditions for nonlinear ordinary differential equations containing a small parameter // Uspehi Mat. Nauk. -1962.-Vol. 17, № 5. -P. 187-188.
- [18] *Нургабыл Д.Н.* Асимптотические оценки решения начальной задачи для линейных дифференциальных уравнений с малым параметром при старших производных // Вестник ЖГУ им.И.Жансугурова. -2012. - № 2. - С.4-8.
- [19] *Dauylbaev M.K., Mirzakulova A.E.* Asymptotic behavior of solutions of singular integro-differential equations // Discontinuity, Nonlinearity and Complexity. -2016. -Vol.5, №2. - P.147-154.
- [20] *Dauylbaev M.K., Mirzakulova A.E.* Boundary-value problems with initial jumps for singularly perturbed integro-differential equations // Journal of Mathematical Sciences. -2017. -Vol.222, №2. - P.214-225.