IRSTI 27.31.21

Mathematical Analysis of the Euler-Bernoulli Beam Subject to Swelling Pressure

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Swelling pressures from materials confined by structures can cause structural deformations and instability. Due to the complexity of interactions between expansive solid and solid-liquid equilibrium, the forces exerting on retaining structures from swelling are highly nonlinear. In this paper, we consider the initial/boundary value problem of an Euler-Bernoulli elastic beam subject to the swelling pressure with one end clamped and another end free. We are interested in establishing and validating a mathematical model for dynamic deflections of an elastic Euler-Bernoulli beam with constant cross-sectional area subject to swelling pressure and some initial and boundary conditions. We built a sequence of functions by using the Galerkin approximation method and the eigenfunctions of the corresponding 4th order eigenvalue problem. It has been showed that the sequence of solutions to the ODE systems converges to the unique solution and that the weak solution is also a classical solution. This work is our initial attempt to study a semi-linear hyperbolic problem based on the Euler-elastic beam theory and some simplistic swelling pressure model in soil and rock mechanics.
Key words: Cantilever Euler-Beam, Expansive Swelling Pressure, Retaining Wall, Vibration Analysis, Existence and Uniqueness of Solution.

Математический анализ балки Эйлера-Бернулли с учетом давления набухания

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Давление набухания из материалов, ограниченных структурами, может вызвать структурные деформации и нестабильность. Из-за сложности взаимодействия между расширяющимися твердым и твердо-жидким равновесием силы, действующие на удерживающие структуры от набухания, сильно нелинейны. В настоящей работе рассматривается начальная / краевая задача для уравнения упругой балки Эйлера-Бернулли, с одним прикрепленным концом и другим свободным концом, с учетом давления набухания. Мы интересуемся вопросами установления и подтверждении математической модели для динамических прогибов упругой балки Эйлера-Бернулли с постоянной площадью поперечного сечения с учетом давления набухания и некоторых начальных и граничных условий. Построили последовательность функций, используя метод приближения Галеркина и собственные функции соответствующей спектральной задачи для дифференциального уравнения четвертого порядка. Было показано, что последовательность решений систем обыкновенных дифференциальных уравнений сходится к единственному решению и что слабое решение также является классическим решением. Эта работа представляет собой нашу первоначальную попытку изучения полулинейной гиперболической задачи, основанной на теории Эйлера упругой балки и некоторой модели упрощенного давления набухания в механике почв и горных пород.
Ключевые слова: Консольная Эйлерова балка, давление набухания, подпорная стенка, анализ колебаний, существование и единственность решения.

Бөгу қысымын ескерген жағдайдағы Эйлер-Бернулли білікшесінің математикалық талдауы

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Әр түрлі құрылымдармен шектелген материалдардың бөгу қысымы құрылымдық деформация мен тұрақсыздық тудыруы мүмкін. Кеңейтілген қатты дене және қатты-сұйық денелердің тепе-теңдігі арасындағы өзара әрекеттесудің күрделілігіне байланысты, құрылымдарды бөгуден ұстап қалушы әрекет ететін күштер күшті сызықты емес болып табылады. Бұл жұмыста бөгу қысымын ескере отырып, бір ұшы бекітілген және еінші ұшы еркін қозғалыстағы Эйлер-Бернулли серпімді білікше теңдеуі үшін бастапқы / шекаралық есепті қарастырамыз. Біз бөгу қысымы мен белгілі бір бастапқы және шекаралық шарттарды есепке ала отырып, көлденең қимасының ауданы тұрақты Эйлер-Бернулли серпімді білікшесінің динамикалық ауытқуларына арналған математикалық модельді құру және бекіту мәселелеріне мүддеміміз. Галеркиннің жуықтау әдісі мен төртінші ретті дифференциалдық теңдеуге сәйкес келетін спектрлік есептің меншікті функцияларын қолдану арқылы функциялар тізбегін құрамыз. Қарапайым дифференциалдық теңдеулер жүйесінің шешімдерінің жалғыз шешімге жинақталатыны және әлсіз шешімнің классикалық шешім болатыны көрсетілді. Бұл жұмыс Эйлердің серпімді білікше теориясына негізделген жартылай сызықты гиперболалық есепті және топырақ пен тау жыныстары механикасындағы жеңілдетілген бөгу қысымының кейбір моделін зерттеуге арналған алғашқы әрекетімізді білдіреді.
Түйін сөздер: Эйлер білікше консолі, бөгу қысымы, тіреу қабырғасы, тербеліс талдауы, шешімінің бар болуы мен жалғыздығы.

1 Introduction

Expansive solids such as elastomer, hydrogel, some rocks, and expansive clay present significant problems when lateral expansive solid pressures acting on retaining walls due to swelling, see, e.g., (Illeperuma, 2013), (Lou, 2012), and (Mansour, 2011) for descriptions of some of the materials and their swelling properties. Determination of the deflections and the stability of retaining walls (Illeperuma, 2013) or pipes (Rjeily, 2012) due to swelling pressures are important for design and integrity of such walls or pipes.

2 Review of literature

In 1994, Mesri et al., see (Mesri, 1994), developed a simplistic equation for swelling pressure as a function of mobilized volume strain which can be used to show that the pressure p acting on a wall due to swelling can be modeled by $p = \beta e^{-\alpha v(x,t)}$, where β is the swelling pressure against the unyielding wall, α a constant depending upon the solid-liquid equilibrium (see, e.g., (Schädlich, 2012), (Illeperuma, 2013)) and $v(x,t)$ the deflection of the wall modeled as a cantilever beam at location x along the beam from the clamped end and at time t. Similar and equivalent forms of the swelling pressure formula were also presented earlier in (Grob, 1972), (Gysel, 1977), and (Gysel, 1978) and are used in (Rjeily, 2012) and it is also called Grob's semi-logarithmic swelling law in literature.

In this paper, we consider the initial/boundary value problem of an Euler-Bernoulli elastic beam subject to the swelling pressure with one end clamped and another end free. We are interested in establishing and validating a mathematical model for dynamic deflections of an elastic Euler-Bernoulli beam with constant cross-sectional area subject to swelling pressure and some initial and boundary conditions.

3 Materials and methods

In Section 1, we state the dynamic beam problem and define the solution spaces. In Section 2, we prove that there exits an unique weak solution. In Section 3, we construct a sequence converging to the unique solution and show that the solution is also a classical solution.

3.1 The Semi-linear Hyperbolic Problem

Based on Euler's elastic beam theory and adapting the formula of the swelling force stated in the introduction, we consider the following semi-linear hyperbolic problem

$$\rho A \frac{\partial^2 v}{\partial t^2} + EI \frac{\partial^4 v}{\partial x^4} = F(v,t), \tag{1}$$

subject to the initial conditions

$$\begin{cases} v(x,0) = v_0(x), \\ \frac{\partial v}{\partial t}(x,0) = v_1(x), 0 < x < L \end{cases} \tag{2}$$

and the the boundary conditions

$$\begin{cases} v(0,t) = \frac{\partial v}{\partial x}(0,t) = 0, \\ \frac{\partial^2 v}{\partial x^2}(L,t) = \frac{\partial^3 v}{\partial x^3}(L,t) = 0, 0 < t < T, \end{cases} \tag{3}$$

where $v(x,t)$ is the vertical deflection of the beam at x and at time t, T and L are positive constants, $F(v,t) = \beta e^{-\alpha v} + g(x,t)$, $v_0(x)$ is the initial deflection, $v_1(x)$ the initial velocity. This semi-linear hyperbolic problem defined by (1), (2), and (3) is used to model an elastic cantilever beam subject to the swelling force $p = \beta e^{-\alpha v}$ and a dynamic driving force $g(x,t)$. The boundary conditions (3) correspond to the standard clamped-free ends conditions in structural mechanics as mentioned in the introduction. We are interested in studying the well-posedness of the problem defined by (1), (2), and (3). We will use $\dot{v}$ for the time derivative $\frac{\partial v}{\partial t}$ and prime v' for the spatial derivative $\frac{\partial v}{\partial x}$ respectively and the similarly for the higher order derivatives. We will use the standard Sobolev norm spaces $H_0^2(0,L) = \{v, v', v'' \in L^2(0,L)) | v$ satisfies the boundary condition (3) $\}$, $L^2(0,T; H_0^2(0,L))$, $L^2(0,T; L^2(0,L))$, and $L^2(0,T; H^{-1}(0,L))$ in the following sections. Here $H^{-1}(0,L)$ stands for the topological dual space of $H_0^2(0,L)$. We also use $X = \{v \in L^\infty(0,T; H_0^2(0,L) | v'' \in L^\infty(0,T; L^2(0,L)\}$with the norm

$$||v|| = \text{esssup}_{0 \le t \le T}(||v(t)||_{H_0^2(0,L)} + ||v(t)||_{L^2(0,L)})$$

as solution space.

3.2 Existence and Uniqueness of Solution

We use the following definition of a week solution:

Definition 1 *We say that a function* $u \in L^2(0,T;H_0^2(0,L))$*, for which* $\dot{u} \in L^2(0,T;L^2(0,L)), \ddot{u} \in L^2(0,T;H^{-1}(0,L))$*, is a weak solution of the semi-linear hyperbolic problem defined by (1), (2), and (3), if*

$$\int_0^L (\rho A\ddot{u}v + EIu''v'' - \beta e^{-\alpha u}v - gv)dx = 0$$

for each $v \in H_0^2(0,L)$ *and a.e. in* $[0,T]$*; and* $u(x,0) = v_0(x), \dot{u} = v_1(x), \forall x \in (0,L)$.

First, we show a priori boundedness of a solution by using the method of conservation of energy.

Theorem 1 *Suppose that* $v_0 \in H_0^2(0,L)$ *and* $v_1 \in L^2(0,L)$*and both satisfy the boundary condition (3), and suppose that* u *is a solution of the problem in the sense of Definition 1 with these initial conditions, and with* $g \in L^\infty(0,T;L^2(0,L))$*. Then* $u \in X$ *and there exists a positive constant* $M > 0$ *such that* $||u|| \leq M$.

Proof. Suppose that u is a solution of the problem. Let

$$E(t) = \int_0^L (\frac{\rho A}{2}|\dot{u}|^2 + \frac{EI}{2}|u''|^2 + \frac{\beta}{\alpha}e^{-\alpha u} - gu)dx,$$

we have

$$\dot{E}(t) = \int_0^L (\rho A\dot{u}\ddot{u} + EIu''\dot{u}'' + \beta e^{-\alpha u}\dot{u} - g\dot{u})dx.$$

By using integrations by parts on the second term and applying the boundary conditions, we have $\int_0^L u''\dot{u}''dx = \int_0^L u^{(IV)}\dot{u}dx$. Therefore

$$\dot{E}(t) = \int_0^L (\rho A\ddot{u} + EIu^{(IV)} - \beta e^{-\alpha u} - g)\dot{u}dx = 0,$$

which implies that

$$E(t) = E(0) = \int_0^L (\frac{\rho A}{2}|v_1|^2 + \frac{EI}{2}|v_0''|^2 + \frac{\beta}{\alpha}e^{-\alpha v_0} - g_0 v_0)dx = C,$$

where C equals a constant, where g_0 denotes $g(x,0)$. Therefore, we have

$$\int_0^L (\frac{\rho A}{2}|\dot{u}|^2 + \frac{EI}{2}|u''|^2 + \frac{\beta}{\alpha}e^{-\alpha u})dx = \int_0^L gudx + C,$$

which implies that $||u''(t)||^2_{L^2(0,L)} \leq C_1||g(t)||_{L^2}||u(t)||_{L^2(0,L)} + C$. By Sobolev inequalities, we have

$$||u(t)||^2_{L^2(0,L)} \leq C_2||u''(t)||^2_{L^2} \leq C||g(t)||_{L^2}||u(t)||_{L^2(0,L)} + C_3,$$

where C_1, C_2, C_3 are all positive constants. This last inequality implies that there exists $M > 0$ independent on u, such that $||u||_{L^\infty(0,T;L^2(0,L))} + ||u''||_{L^\infty(0,T;L^2(0,L))} \leq M$.

Theorem 2 *Suppose that $v_0 \in H_0^2(0,L)$ and $v_1 \in L^2(0,L)$ both satisfy the boundary condition (3), and $g \in L^\infty(0,T;L^2(0,L))$. Then problem defined by (1), (2), and (3) can have only one weak solution in $u \in L^2(0,T;H_0^2(0,L))$ in the sense of Definition 1.*

Proof. To prove existence and uniqueness of solutions, we shall assume that $v_0 \in H^2(0,L)$ and $v_1 \in L^2(0,L)$. Therefore $E(t)$ is a constant for the given initial conditions $\dot{u}(x,0) = v_0(x), u(x,0) = v_1(x), 0 < x < L$.

We shall adapt the standard energy method to prove uniqueness of the solution to our problem. Suppose that there are two solutions which are denoted by v_1 and v_2 respectively. Let $w = v_1 - v_2$, then we have

$$\rho A\ddot{w} + EIw^{(IV)} = F(v_1,t) - F(v_2,t).$$

Multiply both sides of this equation by $\dot{w}$, perform integration by parts and applying the initial/boundary conditions, we have

$$\frac{\rho A}{2}\int_0^L |\dot{w}|^2 dx + \frac{EI}{2}\int_0^L |w''|^2 dx = \int_0^t \int_0^L [\beta(e^{-\alpha v_1} - e^{-\alpha v_2}) + g]\dot{w}dxdt.$$

By Hölder's inequality, the Sobolev inequality, and by Theorem 1, we have

$$\frac{\rho A}{2}\int_0^L |\dot{w}|^2 dx + \frac{EI}{2}\int_0^L |w''|^2 dx \le C\int_0^t (||w||_{L^2(0,L)} + ||g||_{L^2(0,L)})||\dot{w}||_{L^2(0,L)} dt.$$

Whence, an application of Gronwall's lemma yields $w = 0$ and $v_1 = v_2$. By using Brouwer's Fixed Point Theorem in the Banach space X defined above, we can prove the existence of solution by the method similar to the what is presented in B§12.2.1 of (Evans, 2010) and obtain the following:

Theorem 3 *Suppose that $v_0 \in H_0^2(0,L)$ and $v_1 \in L^2(0,L)$ both satisfy the boundary condition (3), and $g \in L^\infty(0,T;L^2(0,L))$. Then, problem defined by (1), (2), and (3) has an unique solution in $u \in L^2(0,T;H_0^2(0,L))$ in the sense of Definition 1.*

For simplicity, we do not present the proof here.

3.3 Construction of a classical Solution

In the above, we have shown the well-posedness of our problem for a general force g. In this sections, we show that an explicit construction of a sequence of functions can be made for smooth g and this sequence converges to a classical solution of our problem. We denote by $C_0^4(\bar{G}, \mathbf{R})$ the set of fourfold continuously differentiable functions $v(x,t)$ defined in $\bar{G} = \{(x,t): 0 \le x \le L, 0 \le t \le T\}$ satisfying the initial and boundary conditions (2), and (3). We rewrite equation (1) in the following operator form

$$Bv \equiv \rho A\ddot{v} + EIv^{(IV)} = F(v,t). \tag{4}$$

We use $C_0^4[0,L]$ for the set of fourfold continuously differentiable functions of x in $[0,L]$ satisfying the boundary condition (3).

Definition 2 *A function $v(x,t) \in C(\bar{G}, \mathbf{R})$ is said to be a solution of the problem (1), (2), and (3), if there exists a sequence $\{v_m\}_{m=1}^{\infty}$ and a function $v(x,t)$ in $C_0^4(0,L)$ such that $\|v_m - v\|_{C_0^4[0,L]} \to 0$ and $\|Bv_m - F(v)\|_{C[0,L]} \to 0$ as $m \to \infty$ for all fixed $t \in [0,T]$ and moreover that the limit function $v(x,t)$ satisfies conditions (3).*

We first consider the free vibration case $g = 0$, which means $F(v,t) = \beta e^{-\alpha v}$. For simplicity, we shall use v' for the spatial derivative $\frac{\partial v}{\partial x}$, $\dot{v}$ for the time-derivative $\frac{\partial v}{\partial t}$, and similar notations for the higher order derivatives. We use $G = \{(x,t) : \ 0 < x < L, \ 0 < t < T\}$ as the space-time domain. The main result of this section is

Theorem 4 *For the given functions $F(v,t) = \beta e^{-\alpha v}$, v_0, v_1, suppose that*

$$v_0(x), v_1(x) \in C_0^4[0,L], \tag{5}$$

then there exists a unique solution $v(x,t) \in C(\bar{G}, \mathbf{R})$ of the nonlinear problem (1), (2), and (3) in the sense of Definition 2.

The proof of this theorem requires some classical results of the corresponding linear Euler beam problem which can be found in, e.g., (Collatz, 1963). For completeness, we state the results as auxiliary lemmas below.

3.4 Some classical results

We denote by $\tilde{B}$ the operator, corresponding to the boundary value problem

$$\varphi^{IV}(x) = f(x), 0 < x < L,$$
$$\varphi(0) = 0, \varphi^{'}(0) = 0, \varphi^{''}(L) = 0, \varphi^{'''}(L) = 0,$$

which maps a function $f \in L_2(0,L)$ to $\varphi = \tilde{B}(f)$ as the solution to the problem. It is well known that it is a closed operator and domain, denoted by $D(\tilde{B})$, of the operator $\tilde{B}$ is dense in the functional space $L_2(0,L)$.

Lemma 1 *Operator $\tilde{B}$ is self-adjoint in the space $L_2(0,L)$.*

Proof. Since

$$\left(\tilde{B}\varphi, \psi\right) = \int_0^L \varphi^{(IV)}(x)\psi(x)dx = \varphi^{'''}(x)\psi(x)|_0^L - \varphi^{''}(x)\psi^{'}(x)|_0^L + \varphi^{'}(x)\psi^{''}(x)|_0^L \tag{6}$$
$$-\varphi(x)\psi^{'''}(x)|_0^L \int_0^L \varphi(x)\psi^{(IV)}(x)dx = \int_0^L \varphi(x)\psi^{(IV)}(x)dx = \left(\varphi, \tilde{B}\psi\right)$$

for all $\varphi(x), \psi(x) \in D(\tilde{B})$ and since the range of the symmetric operator $\tilde{B}$ coincides with $L_2(0,L)$, the operator $\tilde{B}$ is self-adjoint. The eigenvalue problem for the operator $\tilde{B}$ is

$$\varphi^{(IV)}(x) = \lambda\varphi, 0 < x < L,$$
$$\varphi(0) = 0, \varphi^{'}(0) = 0, \varphi^{''}(L) = 0, \varphi^{(IV)}(L) = 0.$$

It is also well known that the operator $\tilde{B}$ has a discrete spectrum. The eigenvalues of the operator $\tilde{B}$ can be arranged in non-decreasing order $\lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots$. The proof of this Lemma is omitted here. We will state and outline the proof of the following lemma.

Lemma 2 *The asymptotic distribution of the eigenvalues of the operator $\tilde{B}$ satisfies the following expression*

$$\lim_{k\to\infty} \frac{\lambda_k}{(2k+1)^4} = \tilde{\delta},$$

where $\tilde{\delta}$ is a constant.

Proof. It is known (cf., e.g., (Collatz, 1963)) that the spectrum of the operator $\tilde{B}$ can be determined uniquely by the zeroes of entire function

$$\Delta(\lambda) = \cos rL \cosh rL = -1, \tag{7}$$

where $r^4 = \lambda$.

Applying the well-known Rouche's theorem, one can determine the asymptotic behavior of the zeroes of entire function. The two sets of zeros of equation (7) as the spectrum of the operator are found to be asymptotically expressed by

$$r_k = \pm\frac{(2k+1)\pi}{2L}\left(1+\overline{\overline{o}}(1)\right), r_k^{'} = \pm i\frac{(2k+1)\pi}{2L}\left(1+\overline{\overline{o}}(1)\right),\ k = 1, 2, ...$$

with the corresponding eigenfunctions also expressed asymptotically as

$$\tilde{y}_k(x) = \left(-\frac{e^{\frac{(2k+1)\pi}{2}}}{r_k} e^{\frac{(2k+1)\pi}{2L}x} + \frac{e^{-\frac{(2k+1)\pi}{2}}}{r_k} e^{-\frac{(2k+1)\pi}{2L}x}\right)(1+o(1)), k = 1, 2, ...$$

Here we denote by $\overline{\overline{o}}(1)$ and $o(1)$ the infinitesimal quantities as $k \to \infty$.

Let $m_k \le \tilde{y}_k(x) \le M_k$ for all $k = 1, 2, ...$, where $m_k = \min_{x\in[0,L]} \tilde{y}_k(x)$ and $M_k = \max_{x\in[0,L]} \tilde{y}_k(x)$.

We note that the eigenfunctions $\{\tilde{y}_k(x)\}_{k\ge1}$ of the operator $\tilde{B}$ is orthogonal system of functions in $L_2(0, L)$, since $\tilde{B} = \tilde{B}^*$ by Lemma 1. For further purposes it is convenient to normalize the system of functions $\{\tilde{y}_k(x)\}_{k\ge1}$, namely traverse to system

$$\left\{ y_k(x) = \frac{\tilde{y}_k(x)}{\|\tilde{y}_k\|_{L_2(0,L)}} \right\}_{k\ge1},$$

where $\|\cdot\|_{L_2(0,L)}$ is the norm of the space $L_2(0, L)$.

Let m be a fixed natural number. We consider the system of nonlinear differential equations with respect to $c_1(t)$, $c_2(t)$,..., $c_m(t)$

$$\begin{cases} \rho A\ddot{c}_1(t) + EI\lambda_1 c_1(t) = \beta \int_0^L e^{-\alpha \sum_{k=1}^m c_k(t) y_k(x)} y_1(x)dx, \\ \rho A\ddot{c}_2(t) + EI\lambda_2 c_2(t) = \beta \int_0^L e^{-\alpha \sum_{k=1}^m c_k(t) y_k(x)} y_2(x)dx, \\ \qquad \cdots \qquad \cdots \qquad \cdots \\ \rho A\ddot{c}_m(t) + EI\lambda_m c_m(t) = \beta \int_0^L e^{-\alpha \sum_{k=1}^m c_k(t) y_k(x)} y_m(x)dx, \end{cases} \tag{8}$$

with initial conditions

$$\begin{cases} c_1(0) = d_1, \ldots, c_m(0) = d_m, \\ \dot{c}_1(0) = h_1, \ldots, \dot{c}_m(0) = h_m \end{cases} \tag{9}$$

for a given set of numbers $d_1, \ldots, d_m$ and $h_1, \ldots, h_m$.

Lemma 3 *The system of nonlinear differential equations (8), (9) is equivalent to the following system of nonlinear integral equations*

$$c_k(t) = d_k \cos\sqrt{\widetilde{\lambda}_k}t + \frac{h_k}{\sqrt{\widetilde{\lambda}_k}} \sin\sqrt{\widetilde{\lambda}_k}t + \tag{10}$$

$$\frac{\widetilde{\beta}}{\sqrt{\widetilde{\lambda}_k}} \int_0^t \sin\sqrt{\widetilde{\lambda}_k}(t-\tau) \int_0^L e^{-\alpha\sum_{j=1}^m c_j(\tau)y_j(x)} y_k(x)dxd\tau.$$

where $\widetilde{\lambda}_k = \frac{EI\lambda_k}{\rho A}$, $\widetilde{\beta} = \frac{\beta}{\rho A}$, $k = 1, 2, \ldots, m$.

Proof. The proof of the Lemma 3 is based on the direct verification that the right-hand side of (10) satisfies relations (8), (9). The reverse is also true.

Lemma 3 implies immediately that

Corollary 1 *If there exist the solutions of (10), then they are infinitely differentiable with respect to* t.

Lemma 4 *There exists an unique solution of the system of integral equations (10) in the class of smooth functions with respect to* t.

Proof. For each positive integer k between 1 and m, we construct the sequence of approximations to solution of system (10)

$$c_k^{(n)}(t) = c_k^{(0)}(t) + \frac{\widetilde{\beta}}{\sqrt{\widetilde{\lambda}_k}} \int_0^t \sin\sqrt{\widetilde{\lambda}_k}(t-\tau) \int_0^L y_k(x) e^{-\alpha\sum_{j=1}^m c_j^{(n-1)}(\tau)y_j(x)} dxd\tau$$

for $n \geq 1$, with $c_k^{(0)}(t) = d_k \cos\sqrt{\widetilde{\lambda}_k}t + \frac{h_k}{\sqrt{\widetilde{\lambda}_k}} \sin\sqrt{\widetilde{\lambda}_k}t$. For $n \geq 2$, we have the following difference equations

$$\left|c_k^{(n)}(t) - c_k^{(n-1)}(t)\right| =$$

$$\frac{\widetilde{\beta}}{\sqrt{\widetilde{\lambda}_k}} \int_0^t \left|\sin\sqrt{\widetilde{\lambda}_k}(t-\tau)\right| \times$$

$$\times \int_0^L |y_k(x)| \left|e^{-\alpha\sum_{j=1}^m c_j^{(n-1)}(\tau)y_j(x)} - e^{-\alpha\sum_{j=1}^m c_j^{(n-2)}(\tau)y_j(x)}\right| dxd\tau.$$

$$(k = 1, 2, \ldots, m)$$

By the mean value theorem, we obtain the inequality

$$\left|e^{-\alpha\sum_{j=1}^m c_j^{(n-1)}(\tau)y_j(x)} - e^{-\alpha\sum_{j=1}^m c_j^{(n-2)}(\tau)y_j(x)}\right| \leq$$

$$\leq m\alpha \max_{1\leq j\leq m} |y_j(x)|\, e^{-\alpha\sum_{j=1}^m (c_j^{(n-1)}(\tau)+\theta(x)c_j^{(n-2)}(\tau))y_j(x)} \cdot \left|c_j^{(n-1)}(\tau) - c_j^{(n-2)}(\tau)\right|,$$

$$(k = 1, 2, \ldots, m)$$

where values of $\theta(x)$ lie between 0 and 1. By virtue of boundedness of the quantities $\left|\sin\sqrt{\widetilde{\lambda}_k}(t-\tau)\right|$, $|y_j(x)|$, $\left|c_j^{(n-1)}(\tau)+\theta(x)c_j^{(n-2)}(\tau)\right|$, we have the inequalities

$$\left|c_k^{(n)}(t)-c_k^{(n-1)}(t)\right| \leq \frac{\widetilde{\beta}}{\sqrt{\widetilde{\lambda}_k}}tC\max_{1\leq j\leq m}\left|c_j^{(n-1)}(t)-c_j^{(n-2)}(t)\right|.,\quad (k=1,2,\ldots,m)$$

where C does not depend on k. We note, if inequalities

$$\frac{\widetilde{\beta}}{\sqrt{\widetilde{\lambda}_k}}tC \leq 1,(k=1,2,\ldots,m) \tag{11}$$

hold, it follows the convergence of the sequence $\left\{c_k^{(n)}(t)\right\}_{n\geq 1}$ for all k. The proof of the Lemma 4 is complete. Inequality (11) holds for large values of k, since Lemma 2 holds. Thus, for all k there exist limits $\lim_{n\to\infty}c_k^{(n)}(t)=c_k(t)$, which are solutions of the system (10). It is convenient to denote the above set with two subscripts by $\{c_{1m}(t),\cdots,c_{mm}(t)\}$.

Lemma 5 *The sequences* $\{c_{1m}(t),\cdots,c_{mm}(t)\}_{m\geq 1}$, $\{\dot{c}_{1m}(t),\cdots,\dot{c}_{mm}(t)\}_{m\geq 1}$, *and* $\{\ddot{c}_{1m}(t),\cdots,\ddot{c}_{mm}(t)\}_{m\geq 1}$ *are all Cauchy sequences for fixed* $t\in[0,T]$, *if* $d_k\to 0$ *and* $h_k\to 0$ *as* $k\to\infty$.

Proof. Together with the sequence $\{c_{km}(t)\}_{k=1}^m$, we consider the sequence of functions $\{c_{k,m+p}(t)\}_{k=1}^{m+p}$ for arbitrary $p\geq 1$, that satisfies the integral equations system

$$c_{k,m+p}(t) = d_k\cos\sqrt{\widetilde{\lambda}_k}t+\frac{h_k}{\sqrt{\widetilde{\lambda}_k}}\sin\sqrt{\widetilde{\lambda}_k}t+ \frac{\widetilde{\beta}}{\sqrt{\widetilde{\lambda}_k}}\int_0^t\sin\sqrt{\widetilde{\lambda}_k}(t-\tau)\int_0^L e^{-\alpha\sum_{j=1}^{m+p}c_{j,m+p}(\tau)y_j(x)}y_k(x)dxd\tau. \quad (k=m+1,\ldots,m+p) \tag{12}$$

We take, that the sequences $d_k\to 0$ and $h_k\to 0$ as $k\to\infty$. Eigenvalues $\widetilde{\lambda}_k\to\infty$ by Lemma 2. Therefore, (12) implies that $c_{k,m+p}\to 0$ as $m\to\infty$, $k\geq m$ and for all $t\in[0,T]$.

Let $p\geq 1$. Now we show that the differences $c_{km}(t)-c_{k,m+p}(t)$ for fixed t approach zero as $m\to\infty$. Systems (10) and (12) imply that the differences $c_{km}(t)-c_{k,m+p}(t)$ satisfy the following equations

$$c_{k,m+p}(t)-c_{km}(t) = \frac{\widetilde{\beta}}{\sqrt{\widetilde{\lambda}_k}}\int_0^t\sin\sqrt{\widetilde{\lambda}_k}(t-\tau)\int_0^L\left[e^{-\alpha\sum_{j=1}^{m+p}c_{j,m+p}(\tau)y_j(x)}-e^{-\alpha\sum_{j=1}^{m}c_{jm}(\tau)y_j(x)}\right]y_k(x)dxd\tau. \tag{13}$$

$$(k = 1, 2, \dots, m)$$

Further, we transform (13)

$$c_{k,m+p}(t) - c_{km}(t) = \tag{14}$$
$$\frac{\widetilde{\beta}}{\sqrt{\widetilde{\lambda_k}}} \int_0^t \sin\sqrt{\widetilde{\lambda_k}}(t-\tau)d\tau \int_0^L \left[e^{-\alpha\sum_{j=1}^{m+p}(c_{j,m+p}(\tau)-c_{jm}(\tau))y_j(x)} - 1\right] \times$$
$$e^{-\alpha\sum_{j=1}^{m} c_{jm}(\tau)y_j(x)} \left(1 + \bar{o}(1)\right) y_k(x)dx.(k = 1, 2, \dots, m)$$

If we introduce the notation $\delta_m(t) = c_{km}(t) - c_{k,m+p}(t)$, then (14) can be rewritten in the following form

$$\delta_m(t) = \tag{15}$$
$$\frac{\widetilde{p_{sl}}}{\sqrt{\widetilde{\lambda_k}}} \int_0^t \sin\sqrt{\widetilde{\lambda_k}}(t-\tau)d\tau \int_0^L \left[e^{-\alpha\sum_{j=1}^{m+p}\delta_m(\tau)y_j(x)} - 1\right] \times$$
$$e^{-\alpha\sum_{j=1}^{m} c_{jm}(\tau)y_j(x)} \left(1 + \bar{o}(1)\right) y_k(x)dx.(k = 1, 2, \dots, m)$$

Thus, we have obtained the nonlinear integral equation (15). Since, the right hand side of (15) is infinitesimal quantity, then it has only solution $\delta_m(t)$ that approaches zero as $m \to \infty$. So the proof of Lemma 5 is complete.

By using

$$\{c_{1m}(t), \cdots, c_{mm}(t)\},$$

we construct the sequence of functions in (x, t)

$$v_m(x,t) = \sum_{k=1}^{m} c_{km}(t)y_k(x). \tag{16}$$

Lemma 6 *The sequences of functions $\{v_m(x,t)\}_{m\geq 1}$, $\{\dot{v}_m(x,t)\}_{m\geq 1}$ and $\{\ddot{v}_m(x,t)\}_{m\geq 1}$ are Cauchy sequence in* max*-norm of $C(\bar{G}, R)$ with respect to x for all fixed $t \in [0, T]$.*

Proof. By (15), we have

$$|v_{m+p}(x,t) - v_m(x,t)| \leq$$
$$\sum_{k=1}^{m} |c_{k,m+p}(t) - c_{km}(t)||y_k(x)| +$$
$$\sum_{k=m+1}^{m+p} |c_{k,m+p}(t)||y_k(x)| \to 0$$

as $m \to \infty$, $k \geq m$ and for all fixed $t \in [0, T]$. Hence

$$\max_{x\in[0,L]} |v_{m+p}(x,t) - v_m(x,t)| \leq \tag{17}$$
$$\max_{x\in[0,L]} \sum_{k=1}^{m} |c_{k,m+p}(t) - c_{km}(t)||y_k(x)| +$$
$$\max_{x\in[0,L]} \sum_{k=m+1}^{m+p} |c_{k,m+p}(t)||y_k(x)| \to 0,$$

By arguing as in (17), we obtain

$$\max_{x\in[0,L]} \left|v'_{m+p}(x,t) - v'_m(x,t)\right| \leq \tag{18}$$
$$\max_{x\in[0,L]} \sum_{k=1}^{m} |c_{k,m+p}(t) - c_{km}(t)||y'_k(x)| +$$
$$\max_{x\in[0,L]} \sum_{k=m+1}^{m+p} |c_{k,m+p}(t)||y'_k(x)| \to 0,$$

$$\max_{x\in[0,L]}\left|v''_{m+p}(x,t)-v''_m(x,t)\right|\le \tag{19}$$
$$\max_{x\in[0,L]}\sum_{k=1}^{m}|c_{k,m+p}(t)-c_{km}(t)||y''_k(x)|+ \max_{x\in[0,L]}\sum_{k=m+1}^{m+p}|c_{k,m+p}(t)||y''_k(x)|\to 0,$$

$$\max_{x\in[0,L]}\left|v'''_{m+p}(x,t)-v'''_m(x,t)\right|\le \tag{20}$$
$$\max_{x\in[0,L]}\sum_{k=1}^{m}|c_{k,m+p}(t)-c_{km}(t)||y'''_k(x)|+ \max_{x\in[0,L]}\sum_{k=m+1}^{m+p}|c_{k,m+p}(t)||y'''_k(x)|\to 0$$

as $m\to\infty$, $k\ge m$ and for all fixed $t\in[0,T]$.

By combining (17)- (20), we have the uniformly convergence of the sequence $v_m(x,t)_{m=1}^{\infty}$ with respect to x in the norm of the space $C^{(4)}[0,L]$ for all fixed $t\in[0,T]$, i.e.

$$\|v_{m+p}(x,t)-v_m(x,t)\|_{C^{(4)}[0,L]}=\sum_{k=0}^{4}\max_{x\in[0,L]}\left|v^{(k)}_{m+p}(x,t)-v^{(k)}_m(x,t)\right|\to 0$$

as $m\to\infty$, $k\ge m$ and for all fixed $t\in[0,T]$. The proof of Lemma 6 is complete. Similarly, we can show that $\{\dot v_m(x,t)\}_{m\ge1}$ and $\{\ddot v_m(x,t)\}_{m\ge1}$ also satisfy estimates like (17)- (20) and therefore are also Cauchy-sequences in the norm of $C(\bar G,R)$ for each $t\in[0,T]$.

3.5 Proof of Theorem 4

We will search for a solution $v(x,t)$ of the problem (1), (2), and (3) as a limit in the norm $C(\bar G,R)$ of the sequence $\{v_m(x,t)\}_{m=1}^{\infty}$, i.e.

$$\|v_m(x,t)-v(x,t)\|_{C(\bar G,R)}\to 0 \;\; as \;\; m\to\infty.$$

Usually $v_m(x,t)$ can be found as the following linear combination (16) by some system of functions $\{y_k(x)\}$, and besides the coefficients $c_{km}(t)$ may vary together with numbers m.

We note that the set of functions $\{c_{1m}(t),\cdots,c_{mm}(t)\}$ is the solution of the Cauchy problem (8)- (9), where the set of numbers $d_1,\dots,d_m$ and $h_1,\dots,h_m$ are the first m Fourier coefficients of functions $v_0(x,t)$ and $v_1(x,t)$, respectively, by system $\{y_k(x)\}_{k\ge1}$.

Further, by Lemmas 3, 4, 5 and 6, the sequences of functions $\{v_m(x,t)\}_{m\ge1}$, $\{\dot v_m(x,t)\}_{m\ge1}$, and $\{\ddot v_m(x,t)\}_{m\ge1}$ are all Cauchy sequences in max-norm with respect to x in $C_0^4(\bar G,\mathbf{R})$ and for all fixed $t\in[0,T]$. Thus, the proof of Theorem 4 is complete.

4 Results and discussions

We prove existence and uniqueness of a weak solution to this semi-linear hyperbolic problem in certain function spaces by using the standard energy estimates and a fixed point argument (see, e.g., (Evans, 2010) for the notations and argument). We construct a sequence of functions by using systems of ODEs, the Galerkin approximation method, and the eigenfunctions of the corresponding 4th order eigenvalue problem. We demonstrate that the sequence of solutions to the ODE systems converges to the unique solution and that the weak solution is also a classical solution. Our results validates the well-posedness of the hyperbolic problem and and provide an explicit numerical procedure to compute a sequence of functions converging to the solution. In the following sections of this paper, the density of the beam is denoted by ρ, the Young's modulus of the beam is E, the constant cross-sectional area is A, and the area moment of inertia is I, all of which are assumed to be positive constants. To our knowledge, this problem and our results have not been available in literature and therefore our results are novel in this regard, although some numerical results are reported in (Rjeily, 2012) for the corresponding steady state problem which determines the deflection of the beam subject to such force and similar boundary conditions.

5 Summary

We have proved the well-posedness of a semi-linear hyperbolic problem which is a model for the the dynamic behavior of a cantilever Euler beam subject to a nonlinear swelling load with fixed-free ends. For smooth initial conditions and smooth force g, we also show existence of classical solutions by using explicit construction of a sequence of functions corresponding to a sequence of nonlinear DOEs convergent to the solution. Our proof also provides explicit algorithms for computing solutions to the sequence of approximating ODE problems. The novelty here is the consideration of the nonlinear effect of the swelling force acting on the lateral side of the Euler beam.

Acknowledgments

This work was supported by the projects grant of the Committee of Science of the Ministry of Education and Science of the Republic of Kazakhstan (project theme «Identification of the boundary conditions of differential operators», 2018-2020).

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