

1-бөлім

Раздел 1

Section 1

Математика

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Mathematics

MSC 34K11, 39A11, 34C10, 39A12

Qualitative Behavior of the Solutions to Delay and Difference Equations

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It is noteworthy to observe that a first-order linear ordinary differential equation *without delay does not possess oscillatory* solutions. Therefore the investigation of oscillatory solutions is of interest for equations with delays or for the discrete analogue difference equations. Furthermore, the mathematical modelling of several real-world problems leads to differential equations that depend on the past history rather than only the current state. In this article conditions are presented such that all solutions of delay and difference equations *are oscillatory* while *all solutions* of the corresponding ordinary differential equations without delay are, for example, *decreasing and tend to zero*. Equations with constant and variable arguments are investigated. Several examples of delay and difference equations with applications to many sectors of life are presented.

Key words: oscillation, delay differential equations, difference equations.

1 Introduction

The oscillation theory of Ordinary Differential Equations (ODEs) was originated by Sturm [26] in 1836. Since then hundreds of papers have been published studying the oscillation theory of ODEs.

The oscillation theory of Delay Differential Equations (DDEs) was mainly developed after the 2nd world war. It was during the war that the admirals and officers in Navy (Fleet) observed that the ships were vibrating and asked the engineers and the scientists to solve the problem. Investigating the problem of vibrations (oscillations) the scientists found out that the equation which was to be taken into consideration was not an ODE (a usual equation without delays) but it was a differential equation with delays.

In the decade of 1970 a great number of papers were written extending known results from ODEs to DDEs. Of particular importance, however, has been the study of oscillations which are caused by the delay and which do not appear in the corresponding ODE. In recent years there has been a great deal of interest in the study of oscillatory behavior of the solutions to DDEs and also the discrete analogue Delay Difference Equations (DΔEs). See, for example, [1-31] and the references cited therein.

The problem of establishing sufficient conditions for the oscillation of all solutions to the differential equation

$$x'(t) + p(t)x(\tau(t)) = 0, \quad t \geq t_0, \quad (1)$$

where the functions $p, \tau \in C([t_0, \infty), \mathbb{R}^+)$ (here $\mathbb{R}^+ = [0, \infty)$), $\tau(t)$ is non-decreasing, $\tau(t) < t$ for $t \geq t_0$, and $\lim_{t \rightarrow \infty} \tau(t) = \infty$, has been the subject of many investigations. See, for example, [4-6, 8-12, 14-17, 19, 21, 22, 24, 28, 29, 31] and the references cited therein.

By a solution of Eq. (1) we understand a continuously differentiable function defined on $[\tau(T_0), \infty)$ for some $T_0 \geq t_0$ and such that (1) is satisfied for $t \geq T_0$. Such a solution is called *oscillatory* if it has arbitrarily large zeros, and otherwise it is called *nonoscillatory*.

The oscillation theory of the (discrete analogue) delay difference equation

$$\Delta x(n) + p(n)x(\tau(n)) = 0, \quad n = 0, 1, 2, \dots, \quad (1)'$$

where $p(n)$ is a sequence of nonnegative real numbers and $\tau(n)$ is a sequence of integers such that $\tau(n) < n - 1$ for $n \geq 0$ and $\lim_{n \rightarrow \infty} \tau(n) = \infty$, has also attracted growing attention in the recent few years. The reader is referred to [1-3, 7, 13, 18, 20, 23, 25, 27, 30] and the references cited therein.

By a solution of Eq. (1)' we mean a sequence $x(n)$ which satisfies (1)' for $n \geq 0$. A solution $x(n)$ of (1)' is said to be *oscillatory* if the terms of the solution are not eventually positive or eventually negative. Otherwise the solution is called *nonoscillatory*.

2 Oscillation Criteria for Delay Equations

In this section we study the delay equation

$$x'(t) + p(t)x(\tau(t)) = 0, \quad t \geq t_0. \quad (1)$$

The first systematic study for the oscillation of all solutions to Eq.(1) was made by Myshkis. In 1950 [22] he proved that all solutions of Eq.(1) oscillate if

$$\limsup_{t \rightarrow \infty} [t - \tau(t)] < \infty \quad \text{and} \quad \liminf_{t \rightarrow \infty} [t - \tau(t)] \liminf_{t \rightarrow \infty} p(t) > \frac{1}{e}. \quad (C_1)$$

In 1972, Ladas, Lakshmikantham and Papadakis [19] and in 1982 Koplataдзе and Canturija [15] concluded the same result if

$$A := \limsup_{t \rightarrow \infty} \int_{\tau(t)}^t p(s) ds > 1, \quad (C_2)$$

or

$$\alpha := \liminf_{t \rightarrow \infty} \int_{\tau(t)}^t p(s) ds > \frac{1}{e}; \quad (C_3)$$

respectively, while ([15]) if

$$\limsup_{t \rightarrow \infty} \int_{\tau(t)}^t p(s) ds < \frac{1}{e}, \quad (N_1)$$

then Eq.(1) has a nonoscillatory solution.

It is obvious that there is a gap between the conditions (C_2) and (C_3) when the limit $\lim_{t \rightarrow \infty} \int_{\tau(t)}^t p(s) ds$ does not exist. How to fill this gap is an interesting problem which has been investigated by several authors.

In 1988, Erbe and Zhang [6] proved that all the solutions of Eq.(1) are oscillatory, if $0 < \alpha \leq \frac{1}{e}$ and

$$A > 1 - \frac{\alpha^2}{4}. \quad (C_4)$$

In 1991, Jian [12] derived the condition

$$A > 1 - \frac{\alpha^2}{2(1-\alpha)}, \quad (C_5)$$

while in 1992, Yu and Wang [28] and Yu, Wang, Zhang and Qian [29] obtained the condition

$$A > 1 - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}. \quad (C_6)$$

In 1990, Elbert and Stavroulakis [4] and in 1991, Kwong [17], using different techniques, improved (C_4) , in the case where $0 < \alpha \leq \frac{1}{e}$, to the conditions

$$A > 1 - \left(1 - \frac{1}{\sqrt{\lambda_1}}\right)^2 \quad (C_7)$$

and

$$A > \frac{\ln \lambda_1 + 1}{\lambda_1}, \quad (C_8)$$

respectively, where λ_1 is the smaller root of the equation $\lambda = e^{\alpha\lambda}$.

In 1994, Koplatadze and Kvinikadze [16] improved (C_6) , while in 1998, Philos and Sficas [23] and in 1999, Zhou and Yu [31] and Jaroš and Stavroulakis [11] derived the conditions

$$A > 1 - \frac{\alpha^2}{2(1-\alpha)} - \frac{\alpha^2}{2}\lambda_1, \quad (C_9)$$

$$A > 1 - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2} - \left(1 - \frac{1}{\sqrt{\lambda_1}}\right)^2, \quad (C_{10})$$

and

$$A > \frac{\ln \lambda_1 + 1}{\lambda_1} - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}, \quad (C_{11})$$

respectively.

Consider Eq.(1) and assume that $\tau(t)$ is continuously differentiable and that there exists $\theta > 0$ such that $p(\tau(t))\tau'(t) \geq \theta p(t)$ eventually for all t . Under this additional condition, in 2000, Kon, Sficas and Stavroulakis [14] and in 2003, Sficas and Stavroulakis [24] established the conditions

$$A > 2\alpha + \frac{2}{\lambda_1} - 1, \quad (C_{12})$$

$$A > \frac{\ln \lambda_1 - 1 + \sqrt{5 - 2\lambda_1 + 2\alpha\lambda_1}}{\lambda_1}. \quad (C_{13})$$

In the case where $\alpha = \frac{1}{e}$, then $\lambda_1 = e$, and (C_{13}) leads to $A > \sqrt{7-2e}/e \approx 0.459987065$.

It is to be noted that as $\alpha \rightarrow 0$, then all the previous conditions $(C_4) - (C_{12})$ reduce to the condition (C_2) , i.e. $A > 1$. However, the condition (C_{13}) leads to $A > \sqrt{3} - 1 \approx 0.732$, which is an essential improvement. Moreover (C_{13}) improves all the above conditions when $0 < \alpha \leq \frac{1}{e}$ as well. Note that the value of the lower bound on A can not be less than $\frac{1}{e} \approx 0.367879441$. Thus the aim is to establish a condition

which leads to a value *as close as possible to* $\frac{1}{e}$. For illustrative purpose, we give the values of the lower bound on A under these conditions when $\alpha = \frac{1}{e}$. (C_4) :0.966166179, (C_5) :0.892951367, (C_6) :0.863457014, (C_7) :0.845181878, (C_8) :0.735758882, (C_9) :0.709011646, (C_{10}) :0.708638892, (C_{11}) :0.599215896, (C_{12}) :0.471517764, (C_{13}) :0.459987065.

We see that the condition (C_{13}) essentially improves all the known results in the literature.

Example 2.1 ([24]) Consider the delay differential equation

$$x'(t) + px(t - q \sin^2 \sqrt{t} - \frac{1}{pe}) = 0, \quad p > 0, \quad q > 0 \text{ and } pq = 0.46 - \frac{1}{e}.$$

Then $\alpha = \liminf_{t \rightarrow \infty} \int_{\tau(t)}^t p ds = \liminf_{t \rightarrow \infty} p(q \sin^2 \sqrt{t} + \frac{1}{pe}) = \frac{1}{e}$ and

$$A = \limsup_{t \rightarrow \infty} \int_{\tau(t)}^t p ds = \limsup_{t \rightarrow \infty} p(q \sin^2 \sqrt{t} + \frac{1}{pe}) = pq + \frac{1}{e} = 0.46.$$

Thus, according to (C_{13}) , all solutions of this equation oscillate. Observe that none of the conditions (C_4) - (C_{12}) apply to this equation.

3 Oscillation Criteria for Difference Equations

Consider the first order linear delay difference equation

$$\Delta x(n) + p(n)x(\tau(n)) = 0, \quad n = 0, 1, 2, \dots, \quad (1)'$$

where $p : N \rightarrow R_+$, $\tau : N \rightarrow N$, $\tau(n)$ is nondecreasing $\tau(n) \leq n - 1$ and $\lim_{n \rightarrow +\infty} \tau(n) = +\infty$, and the particular case of the equation with constant delay

$$\Delta u(n) + p(n)u(n - k) = 0, \quad k \in N \quad (1)''$$

which has been the subject of many recent investigations.

In 1981, Domshlak [3] studied this problem in the case where $k = 1$. In 1989, Erbe and Zhang [7] proved that all solutions of $(1)''$ oscillate if

$$\beta := \liminf_{n \rightarrow \infty} p(n) > 0 \quad \text{and} \quad \limsup_{n \rightarrow \infty} p(n) > 1 - \beta \quad (D_1)$$

or

$$\liminf_{n \rightarrow \infty} p(n) > \frac{k^k}{(k+1)^{k+1}} \quad (D_2)$$

or

$$A := \limsup_{n \rightarrow \infty} \sum_{i=n-k}^n p(i) > 1. \quad (C_2)''$$

while Ladas, Philos and Sficas [20] improved the above condition (D_2) as follows

$$\liminf_{n \rightarrow \infty} \frac{1}{k} \sum_{i=n-k}^{n-1} p(i) > \frac{k^k}{(k+1)^{k+1}}. \quad (C_3)''$$

Concerning the constant $\frac{k^k}{(k+1)^{k+1}}$ in (D_2) and $(C_3)''$ it should be emphasized that, as it is shown in [7], if

$$\sup p(n) < \frac{k^k}{(k+1)^{k+1}}, \quad (N_1)$$

then $(1)''$ has a nonoscillatory solution. Moreover, when $p(n)$ is a constant, say $p(n) = p$, then conditions (D_2) and $(C_3)''$ reduce to

$$p > \frac{k^k}{(k+1)^{k+1}},$$

oscillation of all solutions to Eq.(1)''.

In 1990, Ladas [18] conjectured that Eq.(1)'' has a nonoscillatory solution if

$$\frac{1}{k} \sum_{i=n-k}^{n-1} p(i) \leq \frac{k^k}{(k+1)^{k+1}}$$

holds eventually. However this conjecture is not correct and a counter-example was given in 1994 by Yu, Zhang and Wang [30]. Moreover, in 1999 Tang and Yu [27], using a different technique, showed that Eq.(1)'' has a nonoscillatory solution if the so-called "corrected Ladas conjecture"

$$\sum_{i=n-k}^n p(i) \leq \left(\frac{k}{k+1}\right)^{k+1} \quad \text{for all large } n, \quad (N_2)$$

is satisfied.

In 2017 Karpuz [13] studied this problem and derived the following conditions. If

$$\liminf_{n \rightarrow \infty} \inf_{\lambda \geq 1} \left[\frac{1}{\lambda} \prod_{i=n-k}^n [1 + \lambda p(i)] \right] > 1,$$

then every solution of Eq.(1)'' oscillates, while if there exists $\lambda_0 \geq 1$ such that

$$\frac{1}{\lambda_0} \prod_{i=n-k}^n [1 + \lambda_0 p(i)] \leq 1 \quad \text{for all large } n,$$

nonoscillatory solution. From the above conditions, using the Arithmetic-Geometric mean, it follows that if

$$\sum_{i=n-k}^n p(i) \leq \left(\frac{k}{k+1}\right)^k \quad \text{for all large } n, \quad (N_3)$$

then Eq.(1)'' has a nonoscillatory solution. That is, Karpuz [13] replaced condition (N_2) by (N_3) , which is a weaker condition.

As in Section 2, it is interesting to establish sufficient conditions for the oscillation of all solutions to Eq.(1)'' when both $(C_2)''$ and $(C_3)''$ are not satisfied.

In 2004 Stavroulakis [25] established the following: Assume that $0 < \alpha \leq \left(\frac{k}{k+1}\right)^{k+1}$. Then either one of the conditions

$$\limsup_{n \rightarrow \infty} \sum_{i=n-k}^{n-1} p(i) > 1 - \frac{\alpha^2}{4} \quad (C_4)''$$

or

$$\limsup_{n \rightarrow \infty} \sum_{i=n-k}^{n-1} p(i) > 1 - \alpha^k \quad (D_3)$$

implies that all solutions of (1)' oscillate.

In 2008, Chatzarakis, Koplatadze and Stavroulakis [1,2] investigated for the first time the oscillatory behaviour of equation (1)' in the case of a variable delay argument $\tau(n)$ and derived the following. If

$$\limsup_{n \rightarrow \infty} \sum_{j=\tau(n)}^n p(j) > 1 \quad (C_2)'$$

or $\limsup_{n \rightarrow \infty} \sum_{i=\tau(n)}^{n-1} p(i) < +\infty$ and

$$\alpha := \liminf_{n \rightarrow \infty} \sum_{i=\tau(n)}^{n-1} p(i) > \frac{1}{e} \quad (C_3)'$$

then all solutions of equation (1)' oscillate.

4 Applications

1. Nicholson's blowflies

The delay differential equation

$$\dot{N}(t) = -\delta N(t) + PN(t - \tau)e^{-aN(t-\tau)}, \quad t \geq 0 \quad (4.1)$$

was used by Gurney et al. [9, p.51] to describe the dynamics of Nicholson's blowflies. Here P is the maximum per capita daily egg production rate, $1/a$ is the size at which the population reproduces at its maximum rate, δ is per capita daily adult death rate, τ is the generation time and $N(t)$ is the size of the population at time t .

2. Delay logistic equation

The delay differential equation

$$\dot{N}(t) = rN(t)[1 - N(t - \tau)/K], \quad (4.2)$$

where $r, \tau, K \in (0, \infty)$ is known as *delay logistic equation* and has been investigated by numerous authors [9, p.85]. This equation is a prototype in modelling the dynamics of single-species population systems whose biomass or density is denoted by a differentiable function N . The constant r is called the *growth rate* and the constant K is called the *carrying capacity* of the habitat.

3. The Lasota-Ważewska model for the survival of red blood cells

The delay differential equation

$$\dot{N}(t) = -\mu N(t) + pe^{-\gamma N(t-\tau)}, \quad t \geq 0 \quad (4.3)$$

has been used by Wazewska-Cryzewska and Lasota [9, p.89] as a model for the survival of red blood cells in an animal. Here $N(t)$ denotes the number of red blood cells at time t , μ is the probability of death of a red blood cell, p and γ are positive constants related to the production of red blood cells per unit time, and τ is the time required to produce a red blood cell.

4. Discrete delay logistic equation

The delay difference equation

$$N_{t+1} = \frac{\alpha N_t}{1 + \beta N_{t-k}} \quad (4.4)$$

where $\alpha \in (1, \infty)$, $\beta \in (0, \infty)$, and $k \in \mathbb{N}$ was considered by Pielou [9, p.194] as the discrete analogue of the delay logistic equation (4.2).

5. Kalman Filter-Solar Station

Several real world PVC (Photovoltaic) parks use various mechanisms, including micro-controllers and autonomous robots to rotate the panels to the sun. The obvious advantage of rotating (as the sunflower) over stationary panels is that the first produce 40% more energy than the second.

There can be several approaches to track the sun. Since the sun's trajectory is fairly regular, the panel's current position – angle $\theta(n)$ and the sequence $\Delta\theta(n-1), \dots, \Delta\theta(n-k)$, of previous rotation to the sun provide an accurate estimate of the rotation to the sun's current location. This idea has a simple formal description in the following equation:

$$y(n) = \sum_{i=1}^N a_i \Delta\theta(n-i) + \sum_{i=0}^{N-1} b_i \theta(n-i) = 0, \quad (4.5)$$

meaning that the rotation to bring the panel to an optimal orientation to the sun's current location is a function of the preceding N rotations and the panel's angle in each of the panel's N most recent positions, along its trajectory.

5 Note

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