Multidimensional Analogues of Gelfand–Levitan, Marchenko and Krein Equations. Theory, Numerics and Applications

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Abstract. *We consider the method of regularization of two dimensional (2D) inverse coefficient problems based on the projection method and the approach of I.M. Gelfand, B.M. Levitan, M.G. Krein and V.A. Marchenko. We propose a method of reconstruction of the potential, density and velocity in 2D inverse coefficient problems. The 2D analogies of the I.M. Gelfand, B.M. Levitan and M.G. Krein method are established. The 2D analog of the V.A. Marchenko equation is considered for the Kadomtsev-Petviashvili equation. This approach can be easily applied to corresponding multidimensional inverse problems. The results of numerical calculations are presented.*

Keywords: Gelfand-Levitan equation, Krein equation, Marchenko equation, inverse coefficient problem, inverse scattering problem

We consider the method of regularization of 2D inverse coefficient problems based on the projection method and the approach of I. M. Gelfand, B. M. Levitan, M. G. Krein and V.A. Marchenko.

In 1951 I. M. Gelfand and B. M. Levitan [8] established a method of reconstructing the Sturm–Liouville operator from a spectral function and gave the sufficient conditions for a given monotonic function to be a spectrum function of the operator. In 1951 and 1954 M. G. Krein [11,12] considered the physical statement of the inverse boundary value problem and proved solvability. In 1950 and 1952 V.A. Marchenko [13,14] applied the transformation operators for investigation of the inverse problems and proved that spectral function of the Sturm–Liouville operator defines the operator uniquely.

In 1967 C. S. Gardner, J. M. Greene, M. D. Kruskal and R. M. Miura [7] developed the Inverse Scattering Transform method. The idea is to solve an initial-value problem for the The Korteweg–de Vries (KdV) equation within a class of initial conditions. Later generalised to many other completely integrable equations such as the nonlinear Schrödinger equation, the Sine-Gordon equation etc. The availability of the travelling wave (and, in particular, solitary wave) solutions for the KdV equation does not constitute its integrability. The practical implication of complete integrability is the ability to integrate the KdV equation for a reasonably broad class of initial or boundary conditions. The Kadomtsev-Petviashvili (KP) equation is the 2D analog of the KdV equation.

One of the advantages of our approach (for 1D inverse coefficient problems see also [27,6,26]) is that it allows one to avoid multiple solution of 2D direct problem (see also the boundary control method proposed by M.I. Belishev [2,3] and the globally convergent method proposed by

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M. V. Klibanov [4,5]). In [16] we proved that boundary control method and the method by M.G. Krein are equivalent in 1D case.

1 2D analogy of Gelfand-Levitan equation

Let us consider the sequence of direct problems ($k = 0, \pm 1, \pm 2, \dots$)

$$u_{tt}^{(k)} = u_{xx}^{(k)} + u_{yy}^{(k)} - q(x,y)u^{(k)}, \qquad x \in \mathrm{R}, \quad y \in \mathrm{R}, \quad t > 0;$$

$$u^{(k)}|_{t=0} = 0, \quad u_t^{(k)}|_{t=0} = \delta(x)\mathrm{e}^{\mathrm{i}ky},$$

$$u^{(k)}|_{y=\pi} = u^{(k)}|_{y=-\pi}.$$

Inverse problem 1: find function $q(x,y)$ using additional information

$$u^{(k)}|_{x=0} = f^{(k)}(y,t), \quad u_x^{(k)}|_{x=0} = 0, \qquad k = 0, \pm 1, \pm 2, \dots$$

The uniqueness of the inverse problem 1 can be proved using the technique in [28,29]
Let us consider the sequence of the auxilary problems ($m = 0, \pm 1, \pm 2, \dots$) [18,21]

$$w_{tt}^{(m)} = w_{xx}^{(m)} + w_{yy}^{(m)} - q(x,y)w^{(m)}, \qquad x > 0, \quad y \in \mathrm{R}, \quad t \in \mathrm{R}; \tag{1}$$

$$w^{(m)}|_{x=0} = \mathrm{e}^{\mathrm{i}my}\delta(t), \qquad w_x^{(m)}|_{x=0} = 0. \tag{2}$$

It was proved in [18,21] that the solution to the problem (1), (2) has the form

$$\tilde{w}^{(m)}(x,y,t) = \frac{1}{4}\mathrm{e}^{\mathrm{i}my}\theta(x-|t|)\left[xm^2 + \int_0^{\frac{x+t}{2}} q(\xi,y)\mathrm{d}\xi + \int_0^{\frac{x-t}{2}} q(\xi,y)\mathrm{d}\xi\right] +$$

$$+\frac{1}{2}\int_0^t \int_{x-t+\tau}^{x+t-\tau} [-\tilde{w}_{yy}^{(m)} + q(x,y)\tilde{w}^{(m)}](\xi,y,\tau)\mathrm{d}\xi\mathrm{d}\tau.$$

Therefore

$$\tilde{w}^{(m)}(x,y,x-0) = \frac{1}{4}\mathrm{e}^{\mathrm{i}my}\left[xm^2 + \int_0^x q(\xi,y)\mathrm{d}\xi\right]. \tag{3}$$

The inverse problem 1 can be reduced to the system of integral equations ($k = 0, \pm 1, \pm 2, \dots$) of the first

$$\int_{-x}^{x} \sum_m f_m^{(k)}(t-s)\tilde{w}^{(m)}(x,y,s)\mathrm{d}s = -\frac{1}{2}\left[f^{(k)}(y,t-x) + f^{(k)}(y,t+x)\right], \tag{4}$$

or the second kind

$$\tilde{w}^{(k)}(x,y,t) + \int_{-x}^{x} \sum_m f_m^{(k)\prime}(t-s)\tilde{w}^{(m)}(x,y,s)\mathrm{d}s =$$

$$= -\frac{1}{2}\left[f_t^{(k)}(y,t-x) + f_t^{(k)}(y,t+x)\right]. \tag{5}$$

Here $|t| < x$, $y \in \mathrm{R}$. The system of equations (4) and (5) are 2D analogy of the Gelfand-Levitan equation.

Note that according to (3) $q(x,y)$ can be calculated as follows

$$q(x,y) = 4\frac{\mathrm{d}}{\mathrm{d}x}\tilde{w}^{(0)}(x,y,x-0).$$

2 2D analogy of M.G. Krein equation

Let us consider the sequence of direct problems ($k = 0, \pm 1, \pm 2, \ldots$):

$$u_{tt}^{(k)} = u_{xx}^{(k)} + u_{yy}^{(k)} - \nabla \ln \rho(x,y)\, \nabla u^{(k)}, \quad x > 0,\ y \in \mathrm{R},\ t > 0;$$

$$u^{(k)}|_{t<0} \equiv 0, \qquad u_x^{(k)}(+0,y,t) = \mathrm{e}^{\mathrm{i}ky}\,\delta(t);$$
$$u^{(k)}|_{y=\pi} = u^{(k)}|_{y=-\pi}.$$

Inverse problem 2: find function $\rho(x,y)$ using additional information

$$u^{(k)}(+0,y,t) = f^{(k)}(y,t), \qquad k = 0, \pm 1, \pm 2, \ldots$$

The inverse problem 2 can be reduced to the 2D analogy of M.G. Krein equation [18,21] $k = 0, \pm 1, \pm 2, \ldots$:

$$2\Phi^k(x,t) + \sum_m \int_{-x}^{x} f_m^{(k)\prime}(t-s)\Phi^{(m)}(x,s)\mathrm{d}s = -\int_{-\pi}^{\pi} \frac{\mathrm{e}^{\mathrm{i}ky}}{\rho(0,y)}\mathrm{d}y, \quad |t| < x. \tag{6}$$

The inverse problem solution $\rho(x,y)$ can be calculated by the formula

$$\rho(x,y) = \frac{\pi^2}{\rho(0,y)}\Big[\sum_{m=-\infty}^{\infty} \Phi^{(m)}(x,x-0)\,\mathrm{e}^{-\mathrm{i}my}\Big]^{-2}. \tag{7}$$

For finding inverse problem solution $\rho(x,y)$ in point $x_0 > 0$ we have to solve the system (6) with $x = x_0$ and calculate $\rho(x_0,y)$ by formula (7). For numerical calculations (see figures 1–4) we use N–approximation [17,22] of M.G. Krein equation [21] e.g. we cut the system (6) putting $\Phi^k(x,t) \equiv 0$ for all $N < |k|$ [23].

Discrete analogies of the Gelfand–Levitan equation were considered in [9,25,19,20].

3 2D analogy of M.G. Marchenko equation

The Kadomtsev-Petviashvili equation (the KP equation) is a nonlinear partial differential equation in two spatial and one temporal coordinate which describes the evolution of nonlinear, long waves of small amplitude with slow dependence on the transverse coordinate. There are two distinct versions of the KP equation, which can be written in normalized form as follows:

$$(u_t + 6uu_x + u_{xxx})_x + 3\sigma^2 u_{yy} = 0. \tag{8}$$

Here $u = u(x,y,t)$ is a scalar function, x and y are respectively the longitudinal and transverse spatial coordinates, and $\sigma^2 = \pm 1$.

The case $\sigma = 1$ is known as the KPII equation, and models, for instance, water waves with small surface tension. The case $\sigma = \mathrm{i}$ is known as the KPI equation, and may be used to model waves in thin films with high surface tension. The equation is often written with different coefficients in front of the various terms, but the particular values are inessential, since they can be modified by appropriately rescaling the dependent and independent variables.

The KP equation is a universal integrable system in two spatial dimensions in the same way that the Korteweg-de Vries (the KdV) equation can be regarded as a universal integrable system in one spatial dimension, since many other integrable systems can be obtained as reductions. As such, the KP equation has been extensively studied in the mathematical community in the last forty years. The KP equation is also one of the most universal models in nonlinear wave theory, which arises as a reduction of system with quadratic nonlinearity which admit weakly dispersive waves, in a paraxial wave approximation. The equation naturally emerges as a distinguished limit in the asymptotic description of such systems in which only the leading order terms are retained and an asymptotic balance between weak dispersion, quadratic nonlinearity and diffraction is assumed. The different role played by the two spatial variables accounts for the asymmetric way in which they appear in the equation.

The KP equation originates from a 1970 paper by B.B. Kadomtsev and V.I. Petviashvili [24]. They derived the equation as a model to study the evolution of long ion-acoustic waves of small amplitude propagating in plasmas under the effect of long transverse perturbations. In the absence of transverse dynamics, this problem is described by the KdV equation. The KP equation was soon widely accepted as a natural extension of the classical KdV equation to two spatial dimensions, and was later derived as a model for surface and internal water waves [1], and in nonlinear optics [15], as well as in other physical settings.

As shown in [10], the KP equation

$$u_t + 6uu_x + u_{xxx} + 3\sigma^2 w_y = 0, \tag{9}$$

$$w_x = u_y, \tag{10}$$

on the half-plane $x \in \mathbb{R}$, $y > 0$ with the boundary condition

$$u_x + \sigma w|_{y=0} = 0, \tag{11}$$

is compatible with such characteristic signs of integrability as higher symmetries and the B$\ddot{a}$cklund transformation.

The problem can be reduced to the following Gelfand–Levitan–Marchenko equation

$$K(x,z,y,t) + F(x,z,y,t) + \int\limits_{-\infty}^{x} K(x,\xi,y,t)F(\xi,z,y,t)\mathrm{d}\xi = 0, \tag{12}$$

where the kernel $F(x,z,y,t)$ solves the system of partial-differential equations

$$\sigma F_y - F_{xx} + F_{zz} = 0,$$

$$F_t + 4(F_{xxx} + F_{zzz}) = 0.$$

Therefore, the solution of nonlinear equation can be found by formula

$$u(x,y,t) = 2\frac{\partial}{\partial x}K(x,x,y,t).$$

4 Reconstruction of the velocity $c(x,y)$

Inverse problem 3: find the velocity $c(x,y)$ from the sequence of relations ($k = 0, \pm1, \pm2, \ldots$):

$$c^{-2}(x,y)u^{(k)}_{tt} = u^{(k)}_{xx} + u^{(k)}_{yy}, \qquad x \in \mathrm{R}, \quad y \in \mathrm{R}, \quad t > 0;$$

$$u^{(k)}|_{t=0} = 0, \qquad u^{(k)}_t|_{t=0} = \mathrm{e}^{\mathrm{i}ky}\,\delta(x).$$

$$u^{(k)}(0,y,t) = f^{(k)}(y,t), \qquad u^{(k)}_x(+0,y,t) = 0.$$

Let $\tau(x,y)$ be a solution of Cauchy problem for the eikonal equation

$$\tau_x^2 + \tau_y^2 = c^{-2}(x,y), \qquad x>0, \quad y \in \mathrm{R}; \tag{13}$$

$$\tau|_{x=0} = 0, \qquad \tau_x|_{x=0} = c^{-1}(0,y), \qquad y \in \mathrm{R}. \tag{14}$$

Let us introduce new variables $z = \tau(x,y)$, $y = y$ and new functions

$$v^{(k)}(z,y,t) = u^{(k)}(x,y,t), \qquad b(z,y) = c(x,y). \tag{15}$$

Since the velocity is supposed to be strictly positive this change of variables is not degenerate at least in some interval $x \in (0,h)$.

Let us consider the sequence of the auxiliary problems ($m = 0, \pm 1, \pm 2, \ldots$) [18,**?**]:

$$w_{tt}^{(m)} = w_{zz}^{(m)} + b^2 w_{yy}^{(m)} + q w_{yz}^{(m)} + p w_z^{(m)}, \qquad z>0, \quad y \in \mathrm{R}, \quad t \in \mathrm{R}; \tag{16}$$

$$w^{(m)}(0,y,t) = \mathrm{e}^{\mathrm{i}my}\,\delta(t), \qquad w_z^{(m)}(0,y,t) = 0. \tag{17}$$

Here

$$q(z,y) = 2b^2\tau_y, \qquad p(z,y) = b^2(z,y)(\tau_{xx} + \tau_{zz}). \tag{18}$$

We suppose that $c(0,y) = b(0,y)$ is known and for simplicity $b(0,y) \equiv 1$ for $y \in \mathrm{R}$.

In the neighborhood of the plane $t = z$ the solution of the direct problem (16), (17) has the form [18,**?**]:

$$w^{(m)}(z,y,t) = S^{(m)}(t,y)\delta(z-t) + Q^{(m)}(t,y)\theta(z-t) + \tilde{w}^{(m)}(z,y,t). \tag{19}$$

Here $\tilde{w}^{(m)}$ is continuous function and functions $S^{(m)}$ and $Q^{(m)}$ solve the following problems:

$$2S_t^{(m)} + qS_y^{(m)} + pS^{(m)} = 0, \qquad t>0, \quad y \in \mathrm{R}; \tag{20}$$

$$S^{(m)}|_{t=0} = \frac{1}{2}\mathrm{e}^{\mathrm{i}my}. \tag{21}$$

$$2Q_{tt}^{(m)} = S_{tt}^{(m)} - \Big[qQ_y^{(m)} + b^2 S_{yy}^{(m)} + pQ^{(m)}\Big], \qquad t>0, \quad y \in \mathrm{R}; \tag{22}$$

$$Q^{(m)}|_{t=0} = 0. \tag{23}$$

The 2D analogy of M.G. Krein equation follows from (19) ($m = 0, \pm 1, \pm 2, \ldots$):

$$\sum_m S^{(m)}(z,y) f_m^{(k)\prime}(t-z) + \tilde{w}^{(k)}(z,y,t) + \sum_m \int_{-z}^{z} f_m^{(k)\prime}(t-s)\tilde{w}^{(m)}(z,y,s)\mathrm{d}s = 0, \quad |t|<z. \tag{24}$$

So for solving the inverse problem 3 we can solve the system (20)–(24), using the projection method and then find $c(x,y)$ from the following iterative algorithm.

First, we introduce N-approximation of the system (20)–(24), e.g. let $\tilde{w}^{(m)}$, $S^{(m)}$ and $Q^{(m)}$ be equal to 0 for all $|m| > N$. Let us suppose that $c_n(x,y)$ is known. Then we calculate $\tau_n(x,y)$ from (13), (14), $b_n(z,y)$ from (15) and $q_n(z,y)$ and $p_n(z,y)$ from (18). Function $S_n^{(m)}(t,y)$ is calculated from (20), (21). Then solving the 2D analogy of M.G. Krein equation (24) we find $\tilde{w}_n^{(m)}(z,y,t)$ for $|m| \leq N$. It follows from (19) that $Q_n^{(m)}(t,y) = \tilde{w}_n^{(m)}(t+0,y,t)$. Then from equations (20) and (22) we find function $b_{n+1}(z,y)$ and after that new value $c_{n+1}(x,y) = b_{n+1}(z,y)$ is calculated.

In numerical experiments (see figures 1–4) 2D inverse problem 2 is approximated by the finite system of one dimensional inverse acoustic problems [21,22,23]. The inverse problem 2 is solved in the domain $x \in (0,1)$, $y \in (-\pi,\pi)$ and $t \in (0,2)$. The number N is equal to 5 for figure 2 and the number N is equal to 10 for figures 3 and 4. The noisy data is taken as

$$f^{\varepsilon}(t) = f(t) + \varepsilon\alpha(t)(f_{\max} - f_{\min}).$$

Here ε is the level of noise, $\alpha(t)$ is white noise for fixed t, $f_{\max}$ and $f_{\min}$ are maximum and minimum values of exact data. The dimension of the space grid is equal to 100×100.

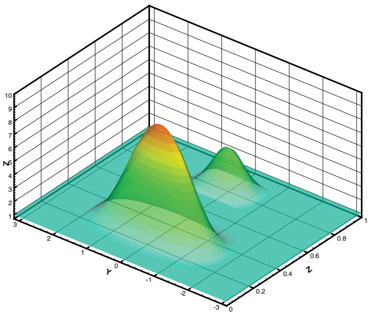

Fig. 1. The exact solution of the inverse problem 2.

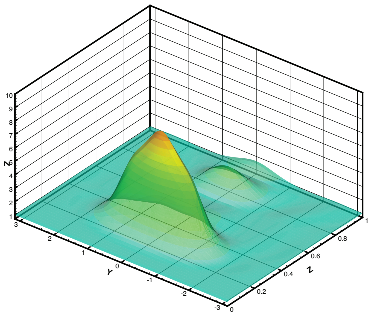

Fig. 2. The approximate solution of the inverse problem 2, $N = 5$, $\varepsilon = 0$.

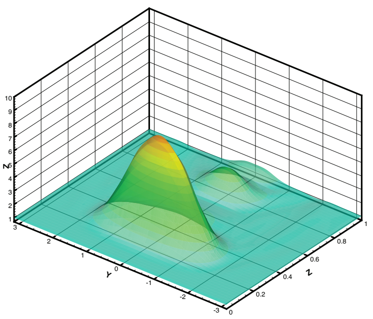

Fig. 3. The approximate solution of the inverse problem 2, $N = 10$, $\varepsilon = 0$.

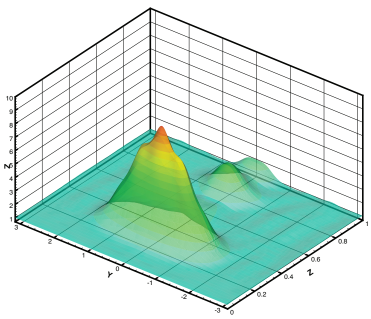

Fig. 4. The approximate solution of the inverse problem 2, $N = 10$, $\varepsilon = 0.05$.

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