The Existence of a Generalized Solution Model of Inhomogeneous Fluid in a Magnetic Field

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Abstract. We consider the generalized solutions of the non-homogeneous fluid in a magnetic field. Proved a theorem for a generalized solution of an inhomogeneous liquid in a magnetic field .In this article we examine the method of fictitious areas for the non-liner hyperbolic equations. The estimation of rate of convercence decisions is recaived. In some cases the unimproved estimation of convergence rate of the decision is received.

Keywords: permeability, fluid flow rate, conductivity, boundary value, unimproved estimation.

1 Introduction

The mathematical study of the correctness of boundary problems for incompressible viscous fluid began with work Lere Zh. Various aspects of the theory of the Navier-Stokes equations are detailed in the monograph of LadyzhenskayaA.O.Studying the correctness of the model is dedicated to the work of an inhomogeneous fluid Ladyzhenskaya O.A. and Solonnikova V.A. Then, this method was developed by Lions J.L., Temam R. and Smagulov S.S. Correctness of magnetic gas dynamics for the one-dimensional case well studied in the work of Orunhanov M.K. and Smagulov S.S.

2 Problem formulation

We consider the flow of a viscous incompressible fluid in an inhomogeneous magnetic field, motion is described by the following non-linear system of equations [1]:

$$\rho(\upsilon_t + (\upsilon\nabla)\upsilon - \mu(H\nabla) = \upsilon\nabla\upsilon - \nabla\left(p + \mu\frac{|H|^2}{2}\right) + \rho f, \tag{1}$$

$$\rho_t + (\upsilon\nabla)\rho = 0, \tag{2}$$

$$div\upsilon = 0, \tag{3}$$

$$-\mu H_t = rotE, \tag{4}$$

$$rotH = \sigma(E + \mu[\upsilon, H]) + j_0, \tag{5}$$

$$divH = 0. \tag{6}$$

where

$\upsilon(x,t)$ – speed of fluid flow;

$H(x,t)$, $E(x,t)$ – magnetic vectors and Voltage;
$p(x,t)$ – pressure;
$f(x,t)$ – external hydrodynamic forces;
$j_0(x,t)$ – given current;
μ – magnetic permeability;
σ – conductivity;
$\rho(x,t)$ – density of the liquid;
v – fluid viscosity.

Note that from (4) and (5) follows the equation

$$-\mu H_t - \frac{1}{\sigma} rotrotH + \mu rot[v, H] + \frac{1}{\sigma} rotj_0 = 0. \quad (7)$$

Subsequently, suppose

$$(j_0\tau)|_s = 0. \quad (8)$$

Let the liquid is in the limited area $\Omega \cap R^3$ with border Sand on the border of the condition sticking

$$v|_s = 0. \quad (9)$$

At the border, is an ideal conductor, must be carried out:

$$Hn \equiv H_n = 0, \quad E_\tau = E - nE = 0. \quad (10)$$

Here,

$$(rotH)_\tau|_s = 0, \;\; when \;\; (j_0\tau)|_s = 0.$$

Assume that the initial conditions:

$$v|_{t=0} = v_0(x), \;\; H|_{t=0} = H_0(x), \;\; \rho|_{t=0} = \rho_0(x). \quad (11)$$

Let us define some notations are used:

$\overset{0}{J}(\Omega)$, $\overset{0}{J^1}(\Omega)$ – circuiting infinite differentiable finite solenoidal vector functions in the norms of $L_2(\Omega)$, $W_2^1(\Omega)$ respectively;

$\overset{0}{H}(\Omega)$ – subspace $L_2(\Omega)$, which is the closure of continuously differentiable solenoidal vector functions normally $L_2(\Omega)$, and such that

$$Hn|_s = H_n|_s = 0, \quad (12)$$

$\overset{0}{H}_{1n}(\Omega)$ – subspace $W_2^1(\Omega)$, which is the closure of continuously differentiable solenoidal vector functions normally $W_2^1(\Omega)$, such that

$$Hn|_s \equiv H_n|_s = 0.$$

$\overset{0}{H}_{1\tau}(\Omega)$ – subspace $W_2^1(\Omega)$, which is the closure of continuously differentiable solenoidal vector functions normally $W_2^1(\Omega)$, such that

$$H_\tau|_s = H - Hn|_s = 0. \quad (13)$$

We give the following

Definition 1. A generalized solution of problem (1) – (3), (6) – (11) is the set of functions $\{\upsilon(x,t),\ \rho(x,t),\ H(x,t)\}$:

$$\upsilon(x,t) \in L_\infty(0,T; \overset{0}{J}(\Omega)) \cap L_2(0,T; \overset{0}{J}{}^1(\Omega))$$

$$\rho(x,t) \in L_\infty(0,T; L_\infty(\Omega)),\ \ 0 < m \le \rho(x,t) \le M < \infty,$$

$$H(x,t) \in L_\infty(0,T; \overset{0}{H}(\Omega)) \cap L_2(0,T; \overset{0}{H}_{1n}(\Omega)),$$

that satisfies the integral identity

$$\int\limits_0^\tau [(-\rho\upsilon, \varphi_t + (\upsilon\nabla)\varphi)_\Omega - \mu((H\nabla)H,\varphi)_\Omega + \upsilon(\nabla\upsilon,\nabla\varphi)_\Omega - \\ -(\rho f,\varphi)_\Omega]dt - \int\limits_\Omega \rho_0(x)\upsilon_0(x)\varphi(x,0)dx = 0, \tag{14}$$

$$-\int\limits_0^T (\rho, \eta_t + (\upsilon\nabla)\eta)_\Omega dt - \int\limits_\Omega \rho_0(x)\eta(x,0)dx = 0, \tag{15}$$

$$\int\limits_0^T\int\limits_\Omega \left(\mu H\psi_t + \frac{1}{\sigma} rotH \cdot rot\psi - \mu[\upsilon,H]rot\psi - j_0 rot\psi\right)dxdt - \\ -\mu\int\limits_\Omega H_0(x)\psi(x,0)dx = 0, \tag{16}$$

for any $\eta,\varphi,\psi \in W_2^1(Q)$, $Q = (0,T)\times\Omega$, satisfying the conditions $\varphi(x,T) = 0$, $\psi(x,T) = 0$, $\eta(x,T) = 0$, $\varphi \in \overset{0}{J}{}^1(\Omega)$, $\psi \in \overset{0}{H}_{1n}(\Omega)$, $\eta \in W_2^1(Q)$ for all $t \in [0,T]$.

Theorem 1. *Let*

$$f(x,t) \in L_2(0,T; L_{6/5}(\Omega)),\ \ j_0(x,t) \in L_2(0,T,L_2(\Omega)),\ \ \Omega \subset R^3,$$

$$0 \le m \le \rho_0(x) \le M < \infty,\ \ \|\rho_0(x)\|_{L_\infty(Q_T)} < \infty,$$

$\upsilon_0(x) \in \overset{0}{J}(\Omega)$, $H_0(x) \in \overset{0}{H}(\Omega)$, $(j_0,\tau)|_s = 0$, $\tau = (\tau_1,\tau_2)_\rho$, $divj_0'$, τ_1, τ_2 *– tangent vectors at the border.*

Then there exists at least one generalized solution of (1) – (3), (6) – (11).

For proof, first we obtain some a priori estimates.

From equation (2) follows:

$$0 < m \le \rho(x,t) \le M < \infty. \tag{17}$$

In (14) and (16)

$$\varphi(x,t) = \upsilon(x,t), \psi(x,t) = H(x,t),$$

And using (2), we obtain

$$\frac{1}{2}\int_0^T\int_\Omega \frac{\partial}{\partial t}(\rho|\upsilon|^2)dxdt + v\|\nabla\upsilon\|^2_{L_2(0,T;L_2(\Omega))} =$$

$$= \mu\int_0^T\int_\Omega (H\nabla)H\upsilon dxdt + \int_0^T\int_\Omega \rho f\upsilon dxdt + C,$$

$$\frac{\mu}{2}\int_0^T\int_\Omega \frac{\partial}{\partial t}(|H|^2)dxdt + \frac{1}{\sigma}\|H\|^2_{L_2(0,T;\overset{0}{H^1}(\Omega))} =$$

$$= \mu\int_0^T\int_\Omega [\upsilon, H]rotHdxdt + \int_0^T\int_\Omega j_0 rotHdxdt + C,$$

$C = const.$

Next, we estimate some terms, applying the Holder and Jung inequality:

$$\int_0^T\int_\Omega \rho f\upsilon dxdt \le CM\|f\|_{L_2(0,T;L_{6/5})}\|\upsilon\|_{L_2(0,T;\overset{0}{J^1}(\Omega))} \le \frac{\upsilon^2}{2}\int_0^T\int_\Omega \|\upsilon\|_{L_2(0,T;\overset{0}{J^1}(\Omega))} + C,$$

$$\int_0^T\int_\Omega j_0 rotHdxdt \le \|j_0\|_{L_2(0,T;L_2(\Omega))}\|H\|_{L_2(0,T;\overset{0}{H^1}(\Omega))} \le \frac{1}{2\sigma}\|H\|_{L_2(0,T;\overset{0}{H^1}(\Omega))} + C.$$

Then the sum of the original integral equation and obtain

$$\|\upsilon(t)\|_{L_\infty(0,T;\overset{0}{J}(\Omega))} + \|\upsilon(t)\|_{L_\infty(0,T;\overset{0}{J^1}(\Omega))} +$$
$$+\|H(t)\|_{L_\infty(0,T;\overset{0}{H}(\Omega))} + \|H(t)\|_{L_2(0,T;\overset{0}{H^1}(\Omega))} \le C < \infty. \tag{18}$$

We have the following

Lemma 1. *For a generalized solution of the problem (1) – (3), (6) – (11) a fair assessment:*

$$\int_0^T (\|\upsilon(t+\tau) - \upsilon(t)\|^2 + \|H(t+\delta) - H(t)\|^2)dt \le C\delta^{1/2}, \quad 0 < \delta < T - \delta. \tag{19}$$

Proof. Fix the value of δ and τ, $0 < t \le T - \delta$, and we consider the equation (8) in the interval $\tau \in (t, t+\delta)$.

Multiply (8) on $\psi \in \overset{0}{H^1}(\Omega)$ scalar:

$$-\mu\frac{\partial}{\partial t}(H,\psi)_\Omega - \frac{1}{\sigma}(rotH, rot\psi)_\Omega + \mu([\upsilon, H], rot\psi)_\Omega + \frac{1}{\sigma}(j_0, rot\psi)_\Omega = 0.$$

Further, integrating over τ in $(t+\delta)$

$$-\mu(H(t+\delta)-H(t),\psi)_{\Omega}-\frac{1}{\sigma}\int\limits_{t}^{t+\delta}(rotH,rot\psi)_{\Omega}d\tau+$$

$$+\mu\int\limits_{t}^{t+\tau}([\upsilon,H],rot\psi)_{\Omega}d\tau+\frac{1}{\sigma}\int\limits_{t}^{t+\tau}(j_0,rot\psi)_{\Omega}d\tau=0.$$

Now take $\psi=H(t+\delta)-H(t)$ and find out that

$$-\mu\|H(t+\delta)-H(t)\|^2_{L_2(\Omega)}+\frac{1}{\sigma}\int\limits_{t}^{t+\delta}(rotH(\tau),rotH(t+\delta)-rotH(t))_{\Omega}d\tau+$$
$$+\mu\int\limits_{t}^{t+\tau}([\upsilon(\tau),H(\tau)],rotH(t+\tau)-H(t))_{\Omega}d\tau+ \tag{20}$$
$$+\frac{1}{\sigma}\int\limits_{t}^{t+\tau}(j_0,rotH(t+\delta)-H(t)\_Omega)d\tau=0.$$

Now, (27) integrable $t\in(0,T-\delta)$, and estimate some terms:

$$\int\limits_{0}^{T-\delta}\int\limits_{\Omega}\int\limits_{t}^{t+\delta}rotH(\tau)(rotH(t+\delta)-rotH(t))d\tau dxdt=$$

$$=\int\limits_{0}^{T-\delta}\int\limits_{\Omega}\int\limits_{t}^{t+\delta}rotH(\tau)\frac{\partial}{\partial t}\int\limits_{t}^{t+\delta}rotH(\tau)d\tau d\tau dxdt=$$

$$=\int\limits_{0}^{T-\delta}\frac{1}{2}\frac{d}{dt}\int\limits_{\Omega}\left(\int\limits_{t}^{t+\delta}rotH(\tau)d\tau\right)^2dxdt=$$

$$=-\frac{1}{2}\left(\int\limits_{\Omega}\left(\int\limits_{0}^{\delta}rotH(\tau)d\tau\right)^2dx-\int\limits_{\Omega}\left(\int\limits_{T-\delta}^{T}rotH(\tau)d\tau\right)^2dx\right).$$

Here, using the Holder inequality, we have

$$\int\limits_{\Omega}\left(\int\limits_{0}^{\delta}rotH(\tau)d\tau\right)^2dx\le\delta\int\limits_{\Omega}\int\limits_{0}^{\delta}|rotH(\tau)|^2d\tau dx\le$$

$$\le\delta\int\limits_{0}^{T}\int\limits_{\Omega}|rotH(\tau)|^2dxd\tau\le C\delta,$$

$$\int\limits_{\Omega}\left(\int\limits_{T-\delta}^{T}rotH(\tau)d\tau\right)^2dx\le\delta\int\limits_{\Omega}\int\limits_{T-\delta}^{T}|rotH(\tau)|^2d\tau dx\le$$

$$\leq \delta \int\limits_{\Omega} \int\limits_{0}^{T} |rotH(\tau)|^2 d\tau dx \leq C\delta.$$

To have the following integral

$$\left| \int\limits_{0}^{T-\delta} \int\limits_{t}^{t+\delta} \int\limits_{\Omega} j_0 rot(H(t+\delta) - H(t)) dx d\tau dt \right| =$$

$$= \left| \int\limits_{0}^{T-\delta} \int\limits_{t}^{t+\delta} \int\limits_{\Omega} j_0 rot \frac{d}{dt} \left(\int\limits_{t}^{t+\delta} H(\tau) d\tau \right) dx d\tau dt \right| =$$

$$= \left| \int\limits_{0}^{T-\delta} \int\limits_{\Omega} \frac{d}{dt} \left(\int\limits_{t}^{t+\delta} rotH(\tau) d\tau \right) \int\limits_{t}^{t+\delta} j_0(\tau) d\tau dx dt \right| \leq$$

$$\leq C \|j_0(x,t)\|_{L_2(0,T;L_2)} \left(\int\limits_{\Omega} \left(\int\limits_{T-\delta}^{T} rotH dt \right)^2 dx \right)^{1/2} \leq$$

$$\leq C\sqrt{\delta} \left(\int\limits_{\Omega} \int\limits_{T-\delta}^{T} |rotH dt|^2 dt dx \right)^{1/2} \leq C\sqrt{\delta} \|H\|_{L_2(0,T;\overset{0}{H}{}^1(\Omega))} \leq C\sqrt{\delta}.$$

Now turn to the equation (1). Multiply it by $\Phi \in \overset{0}{J}{}^1(\Omega)$ scalar $L_2(\Omega)$ and obtain

$$\frac{\partial}{\partial \tau}(\rho(\tau)\upsilon(\tau), \Phi)_\Omega - (\frac{\partial \rho}{\partial \tau}\upsilon(\tau), \Phi)_\Omega - ((\upsilon\nabla)\rho\upsilon, \Phi)_\Omega - ((\rho(\tau)\upsilon(\tau)\nabla)\Phi, \upsilon(\tau))_\Omega +$$

$$+\mu((H(\tau)\nabla\Phi, H(\tau))_\Omega + v(\upsilon_x(\tau), \Phi_x)_\Omega) = (\rho(\tau)f(\tau), \Phi)_\Omega.$$

Using (2) we have

$$\frac{\partial}{\partial \tau}(\rho\upsilon, \Phi)_\Omega - ((\rho\upsilon\nabla)\Phi, \upsilon)_\Omega + \mu(H\nabla\Phi, H)_\Omega + v(H_x, \Phi_x)_\Omega = (\rho f, \Phi)_\Omega. \quad (21)$$

Next, we use the expression

$$\rho(t+\delta)\upsilon(t+\delta) - \rho(t)\upsilon(t) = \rho(t+\delta)(\upsilon(t+\delta) - \upsilon(t)) + (\rho(t+\delta) - \rho(t))\upsilon t). \quad (22)$$

From (2), integrating $\tau \in [t, t+\delta]$, we find

$$\rho(t+\delta) - \rho(t) = -\int\limits_{t}^{1+\delta} (\upsilon * \nabla)\rho d\tau. \quad (23)$$

Now we integrate (28) over $\tau \in [t, t+\delta]$ and let $\Phi = \upsilon(t+\delta) - \upsilon(t)$, we have

$$\begin{gathered}
\|\sqrt{\rho(t+\delta)}[\upsilon(t+\delta)-\upsilon(t)]\|^2_{L_2(\Omega)}+ \\
+((\rho(t+\delta)-\rho(t))-\upsilon(t),\upsilon(t+\delta)-\upsilon(t))_\Omega+ \\
+\mu\int\limits_t^{t+\delta}\int\limits_\Omega((\rho(\tau)\upsilon(\tau)\nabla)[\upsilon(t+\delta)-\upsilon(t)]\upsilon(\tau))dxd\tau+ \\
+\mu\int\limits_t^{t+\delta}\int\limits_\Omega((H(\tau)\nabla)[\upsilon(t+\delta)-\upsilon(t)]H(\tau))dxd\tau+ \\
+\upsilon\int\limits_t^{t+\delta}\int\limits_\Omega(\upsilon_x(\tau)[\upsilon_x(t+\delta)+\upsilon_x(t)])dxd\tau= \\
=\int\limits_t^{t+\delta}\int\limits_\Omega((\rho(\tau)f(\tau))[\upsilon(t+\delta)-\upsilon(t)])dxd\tau.
\end{gathered} \tag{24}$$

Next we will consider some integral terms (24). Firstly, using (23) we have

$$\int\limits_\Omega(\rho(t+\delta)-\rho(t))\upsilon(t)(\upsilon(t+\delta)-\upsilon(t))dx=$$

$$=\int\limits_\Omega\int\limits_t^{t+\delta}(\upsilon(\tau)\nabla)\rho(\tau)\upsilon(t)(\upsilon(t+\delta)-\upsilon(t))d\tau dx.$$

Integrate it with respect to t from 0 to $T-\delta$ and estimate

$$\int\limits_0^{T-\delta}\int\limits_t^{t+\delta}\int\limits_\Omega(\upsilon(\tau)\nabla)\upsilon(t)(\upsilon(t+\delta)-\upsilon(t))\rho(\tau)dxd\tau dt\leq$$

$$\leq M\int\limits_0^{T-\delta}\int\limits_t^{t+\delta}\int\limits_\Omega(|\upsilon(t)||\upsilon_x(t)|(|\upsilon(t+\delta)|-|\upsilon(t)|)dx)d\tau dt\leq$$

$$\leq M\int\limits_0^{T-\delta}\|\upsilon_x(t)\|_{L_2}\|\upsilon(t+\delta)\|_{L_4}\int\limits_t^{t+\delta}\|\upsilon(\tau)\|_{L_4}d\tau dt\leq$$

$$CM\int\limits_0^{T-\delta}(|\upsilon_x(t)\|_{L_2}|\upsilon_x(t+\delta)\|_{L_4}(|\upsilon(\tau)|_{L_2(0,T;L_4)}\sqrt(\upsilon)dt\leq$$

$$\leq C\sqrt{\delta}|\upsilon_x(t)\|_{L_2(0,T;L_2(\Omega)}\||\upsilon\|_{L_2(0,T;L_4(\Omega)}\|\leq C\sqrt{\delta}.$$

Similarly, evaluated and integral

$$\mu\int\limits_0^{T-\delta}\int\limits_t^{t+\delta}\int\limits_\Omega(\rho(\tau)\upsilon(\tau)\nabla)(\upsilon(t+\delta)-\upsilon(t))\upsilon(\tau)dxd\tau dt\leq C\sqrt{\delta}.$$

Next, consider the following integral in (24), pre-integrated with respect to t from 0 to $-\delta$

$$\int_0^{T-\delta}\int_t^{t+\delta}\int_\Omega \upsilon_x(\tau)[\upsilon_x(t+\delta)-\upsilon_x(t)]dxd\tau dt =$$

$$= \int_0^{T-\delta}\int_\Omega\int_t^{t+\delta} \upsilon_x(\tau)\frac{\partial}{\partial t}\int_t^{t+\delta}\upsilon_x(\tau)d\tau dxdt = \frac{1}{2}\int_\Omega\int_0^{T-\delta}\frac{\partial}{\partial t}\left(\int_t^{t+\delta}\upsilon_x(\tau)d\tau\right)dtdx =$$

$$= \frac{1}{2}\int_\Omega\left(\int_{T-\delta}^{T}\upsilon_x dt - \int_0^\delta \upsilon_x dt\right)dx \le$$

$$\le \frac{1}{2}\int_\Omega\left[\left(\int_{T-\delta}^{T}|\upsilon_x|^2 dt\right)^{1/2} - \left(\int_0^\delta |\upsilon_x|^2 dt\right)^{1/2}\sqrt{\delta}\right]dx \le$$

$$\le C\sqrt{\delta}\|\upsilon_x\|_{L_2(0,T;J_2(\Omega))} \le C\sqrt{\delta}.$$

Now we estimate the integral

$$\int_0^{T-\delta}\int_t^{\delta}\int_\Omega p(\tau)f(\tau)[\upsilon(t+\delta)-\upsilon(t)]dxd\tau dt \le M\int_0^{T-\delta}\int_t^{t+\delta}\|f\|_{L_{6/5}(\Omega)}\times$$

$$\times[\|\upsilon(t+\delta)\|_{L_6}+\|\upsilon(t)\|_{L_6}]d\tau dt \le CM\int_0^{T-\delta}\|\upsilon(t)\|_{L_6}\int_t^{t+\delta}\|f\|_{L_{6/5}(\Omega)}d\tau dt \le$$

$$\le CM\int_0^{T-\delta}\|\upsilon\|_{L_6}\sqrt{\delta}\left(\int_t^{t+\delta}\|f\|_{L_{6/5}(\Omega)}d\tau\right)dt \le$$

$$\le C\sqrt{\delta}\|f\|_{L_2(0,T;L_{6/5}(\Omega))}\|\upsilon\|_{L_2(0,T;\overset{0}{J}{}^1(\Omega))} \le C\sqrt{\delta}.$$

Now, with all the resulting integral inequalities developed (24) and (27). As a result, we obtain

$$\int_0^{T-\delta}\|\sqrt{p(t+\delta)}(\upsilon(t+\delta)-\upsilon(t))\|^2+\mu\|H(t+\delta)-H(t)\|^2 dt \le C\sqrt{\delta},$$

that is

$$\int_0^{T}(\|\upsilon(t+\delta)-\upsilon(t)\|^2+\|H(t+\delta)-H(t)\|^2)dt \le C\sqrt{\delta},\ \ 0<\delta<T-\delta.$$

Lemma 1 is proved.

Now turn to the proof of Theorem 1. To do this, use the method of Galerkin. The solution of (1) – (3), (6) – (11) will be sought in the form

$$\upsilon^N(t) = \sum_{j=1}^N \alpha_j^N(t)\omega_j, \quad H^N(t) = \sum_{j=1}^N \beta_j^N(t)l_j, \tag{25}$$

where $\{\alpha_j^N(t)\}_{j=1}^N$, $\{\beta_j^N(t)\}_{j=1}^N$ – is found by solving a system of ordinary differential equations:

$$(p^n)(\upsilon_t^N(t) + ((\upsilon^N\nabla)\upsilon^N), \omega_j)_\Omega - \mu((H^N\nabla)H^N, \omega_j)_\Omega + \upsilon(\nabla\upsilon^N, \nabla\omega_j)_\Omega = \\ = (p^N f^N(t), \omega_j)_\Omega, \quad j = 1, \ldots, N; \tag{26}$$

$$\mu(H_t^N(t), l_j)_\Omega + \frac{1}{\sigma}(rotH^N, rotl_j)_\Omega - \mu([\upsilon^N, H^N], rotl_j)_\Omega - \\ -(j_0 rotl_j)_\Omega = 0, \quad j = 1, \ldots, N; \tag{27}$$

$p^n(t)$ – these are the solutions of differential equations of the first order:

$$p_t^N(t) + (\upsilon^N(t)\cdot\nabla)p^N(t) = 0. \tag{28}$$

The problem (26) – (28) is solved with the initial conditions

$$\upsilon^N(t)|_{t=0} = \upsilon_0^N(x), \;\; H^N(t)|_{t=0} = H_0^N(x), \;\; p^N|_{t=0} = p_0^N(x). \tag{29}$$

$$\begin{array}{l} \upsilon_0^N(x) \to \upsilon_0(x) \;\; in \;\; \overset{0}{J}(\Omega), \\ H_0^N(x) \to H_0(x) \;\; in \;\; \overset{0}{H}(\Omega), \\ p_0^N(x) \to p_0(x) \;\; in \;\; C(\Omega), \;\; when \;\; N \to \infty, \\ p_0^N(x) \in C^2(\Omega). \\ \upsilon_0^N(t) = \sum\limits_{j=1}^N (\upsilon_0(x), \omega_j)_\Omega \omega_j, \; H_0^N(t) = \sum\limits_{j=1}^N (H_0(x), l_j)_\Omega l_j \end{array} \tag{30}$$

this implies the

$$a_j(t)|_{t=0} = (\upsilon_0(x), \omega_j)_\Omega, \quad \beta_j(t)|_{t=0} = (H_0(x), l_j)_\Omega.$$

Reduce the problem (25) – (29) to an operator equation and on the basis of the theorem of Schauder prove its determination. In the space $C[0,T]$ we take a limited closed space

$$K = \{\psi(t) \mid \psi(t) \in C[0,T], \;\; \|\psi\| \leq \overline{C}\},$$

$$a_j^N(0) = (\upsilon_0(x), \omega_j),$$

$$\beta_j^N(0) = (u_0(x), l_j),$$

where

$$\psi(t) = (a_1^N(t), a_2^N, \ldots, a_N^N(t), \beta_1^N(t), \beta_2^N(t), \ldots, \beta_N^N(t)),$$

$$\|\psi(t)\| = max_t(\sum_{i=1}^N([a_i^N(t)]^2 + [\beta_i^N(t)]^2))^{1/2}.$$

$\overline{C}$ – inequality

$$\|\upsilon(t)\|^2_{L_\infty(0,T;L_2(\Omega))} + \|H(t)\|^2_{L_\infty(0,T;\overset{0}{H}(\Omega))} \leq \overline{C},$$

which, in turn, determined from (18).

Take some element of K :

$$\psi^0 = (a_{1,0}^N, a_{2,0}^N, \ldots, a_{N,0}^N, \beta_{1,0}^N, \ldots, \beta_{N,0}^N).$$

We form vectors

$$u^0 = \sum_{j=1}^{N} a_{j,0}^N(t)\omega_j,\ G^0 = \sum_{j=1}^{N} \beta_{j,0}^N(t)l_j$$

and on the set $u^0(x,t)$ and $G^0(x,t)$ we find $r(x,t)$ of the following task

$$\begin{aligned} &r_l + (u_0\nabla)r = 0, \\ &r|_{l=0} = \rho_0^N(x). \end{aligned} \tag{31}$$

Problem (31) is uniquely solvable. Indeed, let $y(\tau,t,x)$ is the solution of the famous problem

$$\begin{aligned} &\frac{\partial y}{\partial \tau} = u^0, \\ &y|_{t=\tau} = x. \end{aligned} \tag{32}$$

Then the solution of problem (32) in the form

$$r = p_0^N(y(\tau,t,x)t)|_{\tau=0}, \tag{33}$$

From this we have:

$$0 < m \le r(x,t) \le M < \infty, \tag{34}$$

when $0 < m \le p_0(x) \le M < \infty$.

Then $u^0(t)$ and $G^0(t)$ we substitute in (26) and (27) and find

$$u^1(x,t) = \sum_{j=1}^{N} \alpha_{j,1}^N(t)\omega_j \ \ and\ \ G^1(x,t) = \sum_{j=1}^{N} \beta_{j=1}^N(t)l_j,$$

from a system of ordinary differential equations

$$\begin{aligned} &(ru_t^1 + (ru^0\nabla)u^1,\omega_j)_\Omega - \mu((G^0\nabla)G^1,\omega_j) + v(u_x^1,\omega_{jx})_\Omega,\ \ j=\overline{1,N}, \\ &\mu(G_t^1,l_j)_\Omega - \tfrac{1}{\sigma}(rotG^1,rotl_j)_\Omega - \mu([u^1,G^0],rotl_j)_\Omega - (j_0,rotl_j)_\Omega,\ \ j=\overline{1,N}. \end{aligned} \tag{35}$$

The solvability of (35) follows from the theory of ordinary differential equations.

That is,uniquely defines the vector

$$\psi^1 = (\alpha_{1,1}^N,\alpha_{2,1}^N,\ldots,\alpha_{N,1}^N,\beta_{1,1}^N,\beta_{2,1}^N,\ldots,\beta_{N,1}^N).$$

Multiplying the first and the second identity in (35) on $\alpha_{j,1}^N$ and $\beta_{j,1}^N$ respectively and summing over $j=\overline{1,N}$, obtain the estimate

$$\|u^1\|_{L_\infty(0,T;\overset{0}{J}(\Omega))} + \|G^1\|_{L_\infty(0,T;\overset{0}{H}(\Omega))} \le \bar{c},$$

e.i. $\psi^1 \in K$.

Thus, we are structured operator $A : K \to K$. The fixed point of $\widetilde{A}$ along with the proper function of $r(t)$ defines the solution of the problem (26) and (27). The continuity of $\widetilde{A}$. It follows from a general theorem of stability of solutions of ordinary differential equations for the coefficient from right-hand side. So, Schauder's theorem there is a fixed element $\psi \in K : A\psi = \psi$, that is, from (25) – (29) is uniquely v^N, ρ^N, H^N. For them, we have the estimates:

$$\begin{aligned} &\|v^N(t)\|_{L_\infty(0,T;\overset{0}{J}(\Omega))} + \|v^N(t)\|_{L_\infty(0,T;\overset{0}{J^1}(\Omega))} \le C < \infty, \\ &\|H^N(t)\|_{L_\infty(0,T;\overset{0}{H}(\Omega))} + \|H^N(t)\|_{L_\infty(0,T;\overset{0}{H^1}(\Omega))} \le C < \infty, \\ &0 < m < \rho^N(t) \le M < \infty. \end{aligned} \tag{36}$$

Further, for the approximate solutions υ^N, ρ^N, H^N, as in Lemma 1, we can prove rating:

$$\|\upsilon^N(t+\delta)-\upsilon^N(t)\|_{L_2(0,T-\delta;L_2(\Omega))}+\|H^N(t+\delta)-H^N(t)\|_{L_2(0,T-\delta;L_2)}\leq C\delta^{1/4}.$$

Here the constants C do not depend on N. Therefore,from the sequences $\{\upsilon^N\}$, $\{H^{N^N}\}$, $\{\rho^N\}$ can select a subsequence for which the following relations:

$$\begin{array}{l}\upsilon^N\to\upsilon^*\ \ weakly\ in\ \ L_\infty(0,T;\overset{0}{J}(\Omega)),\\ \upsilon^N\to\upsilon\ \ weakly\ in\ \ L_2(0,T;\overset{0}{J^1}(\Omega)),\\ \upsilon^N\to\upsilon\ \ strongly\ in\ \ L_2(0,T;L_2(\Omega)),\\ p^N\to\upsilon\ \ weakly\ in\ \ L_\infty(Q_r),\\ H^N\to H^*\ \ weakly\ in\ \ L_\infty(0,T;\overset{0}{H}(\Omega)),\\ H^N\to H\ \ weakly\ in\ \ L_2(0,T;\overset{0}{H^1}(\Omega)),\\ H^N\to H\ \ strongly\ in\ \ L_2(0,T;\overset{0}{H}(\Omega)).\end{array}\tag{37}$$

Then, taking the limit as $N\to\infty$ in the identities (26) – (28), we find that the limit function $\upsilon(x,t)$, $H(x,t)$, $p(x,t)$ is a generalized solution of (1) – (3), (6) – (11). Indeed, for the functions ρ^N, υ^N, H^N will show the limit in terms of the integral multiple (14).

At first

$$\int_0^T\int_\Omega p^N\upsilon^N\varphi_t dxdt=\int_0^T\int_\Omega(p^N-p)\upsilon\varphi_t dxdt+$$

$$+\int_0^T\int_\Omega p^N(\upsilon^N-\upsilon)\varphi_t dxdt+\int_0^T\int_\Omega p\upsilon\varphi_t dxdt,$$

where

$$\lim_{n\to\infty}\int_0^T\int_\Omega(p^N-p)\upsilon\varphi_t dxdt=0,$$

by

$$p^N\to p\ \ weakly\ in\ \ L_\infty(Q_T)$$

and

$$\int_0^T\int_\Omega\upsilon\varphi_t dxdt\leq\int_0^T\|\upsilon\|_{L_2}(\Omega)\|\varphi_t\|_{L_2(\Omega)}dt\leq\|\upsilon\|_{L_2(0,T;\overset{0}{J}(\Omega))}\|\varphi_t\|_{L_2(Q_T)}\leq C,$$

Also

$$\lim_{N\to\infty}\int_0^T\int_\Omega p^N(\upsilon^N-\upsilon)\varphi_t dxdt\leq M\lim_{N\to\infty}\|\upsilon^N-\upsilon\|\|\varphi_t\|dt\leq$$

$$\leq M\lim_{N\to\infty}\|\upsilon^N-\upsilon\|_{L_2(0,T;\overset{0}{J}(\Omega))}\|\varphi_t\|_{L_2(Q_T)}=0,$$

by

$$\upsilon^N\to\upsilon,\ \ \upsilon^N\to\upsilon\ \ strongly\ in\ \ L_2(Q_T).$$

That is, there is a

$$\lim_{N\to\infty}\int_0^T\int_\Omega p^N v^N \varphi_t dxdt = \int_0^T\int_\Omega pv\varphi_t dxdt.$$

Next, consider another integral

$$-\int_0^T\int_\Omega (H^N\nabla)H^N\varphi dxdt = -\int_0^T\int_\Omega (H^N\nabla)(H^N\varphi)dxdt + \int_0^T\int_\Omega (H^N\nabla)\varphi H^N dxdt =$$

$$= \int_0^T\int_\Omega (H^N\nabla)\varphi H^N dxdt.$$

Now we show that

$$\forall\varphi \in W_2^1(0,T;\overset{0}{J}{}^1(\Omega)),$$

$$\lim_{N\to\infty}\int_0^T\int_\Omega (H^N\nabla)\varphi H^N dxdt = \int_0^T\int_\Omega (H\nabla)\varphi H dxdt.$$

To do this, expand the integral

$$\int_0^T\int_\Omega (H^N\nabla)\varphi H^N dxdt =$$

$$= \int_0^T\int_\Omega ((H^N-H)\nabla)\varphi H^N dxdt + \int_0^T\int_\Omega (H\nabla)\varphi(H^N-H)dxdt +$$

$$+ \int_0^T\int_\Omega (H\nabla)\varphi H dxdt,$$

and evaluate the following integrals

$$\lim_{N\to\infty}\int_0^T\int_\Omega ((H^N-H)\nabla)\varphi H^N dxdt \le$$

$$\le \lim_{N\to\infty}\int_0^T\int_\Omega \|H^N-H\|_{L_4}\|H^N\|_{L_4}\|\nabla\varphi\|_{L_2}dt \le$$

$$+ \lim_{N\to\infty}\|H^N-H\|_{L_2(0,T;L_4)}\|H^N\|_{L_4(0,T;L_4)}\|\varphi\|_{W_2^1(0,T;\overset{0}{J}{}^1(\Omega))} = 0,$$

by $H^N \to H$ strongly in $L_2(0,T;L_4(\Omega))$, which follows from the embedding theorems

$$\int_0^T\int_\Omega (H\nabla)\varphi(H^N-H)dxdt = 0,$$

by $H^N \to H$ weakly in $L_2(0,T;\overset{0}{H}(\Omega))$,

$$\|H\nabla\varphi\|_{L_1(0,T;L_2(\Omega))} \le \|H\|_{L_2(0,T;L_4(\Omega))}\|\nabla\varphi\|_{L_2(0,T;L_4(\Omega))} \le C,$$

for all $\varphi \in W_2^1(0,T;\overset{0}{J}(\Omega) \cap W_2^2(\Omega))$.

So,

$$\lim_{N\to\infty}\int_0^T\int_\Omega (H^N\nabla)\varphi H^N dxdt = \int_0^T\int_\Omega (H\nabla)\varphi H dxdt.$$

Now we study the following integral

$$\int_0^T\int_\Omega (\nabla v^N, \nabla\varphi)dxdt = \int_0^T\int_\Omega (\nabla v^N - \nabla v, \nabla\varphi)dxdt + \int_0^T\int_\Omega (\nabla v, \nabla\varphi)dxdt,$$

where

$$\lim_{N\to\infty}\int_0^T\int_\Omega (\nabla v^N - \nabla v, \nabla\varphi)dxdt = 0,$$

by $v^N \to v$ weakly in $L_2(0,T;\overset{0}{H}{}^1(\Omega))$ and $\nabla\varphi \in L_2(0,T;\overset{0}{J}(\Omega))$.

Limit transition in the rest of member (6) is not much difficult. Now we show the limit in several Member States of the identity (27). For example,

$$\int_0^T\int_\Omega [v^N H^N] rot\psi dxdt = \int_0^T\int_\Omega [v^N - v, H^N] rot\psi dxdt +$$

$$+\int_0^T\int_\Omega [v, H^N - H] rot\psi dxdt + \int_0^T\int_\Omega [v, H] rot\psi dxdt.$$

where

$$\int_0^T\int_\Omega [v, H^N - H] rot\psi dxdt \le$$

$$\le C\|v^N - v\|_{L_2(0,T;L_4(\Omega))}\|H^N\|_{L_2(0,T;L_4)}\|\psi_x\|_{L_2(0,T;L_4(\Omega))} \to 0,$$

when $N \to \infty$.

Similarly,

$$\lim_{N\to\infty}\int_0^T\int_\Omega [v, H^N - H] rot\psi dxdt = 0,$$

then

$$\lim_{N\to\infty}\int_0^T\int_\Omega (rot(H^N - H), rot\psi)dxdt = 0,$$

by $H^N \to H$ weakly in $L_2(0,T;\overset{0}{H}{}^1(\Omega))$ and $rot\psi \in L_2(0,T;L_2(\Omega))$.

Also go to the limit of the rest of Member States (27). So, we have fully justified limit in the corresponding integral identities. Then the limit functions v, H, ρ – generalized solution of (1) – (3), (6) – (11). Theorem 1 is proved.

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