Algorithms of Determination by the Trajectory of Robots in the Conditions of Interval Uncertainty of the Data

Ziyavidin Yuldashev, Alimjan Ibragimov, and Shomurod Shominasov

National University of Uzbekistan,
Vuzgorodok, 100057 Tashkent, Uzbekistan
ziyaut@mail.ru

Abstract. In work the problem of bilateral approximation of possible trajectories of robot in the conditions of interval uncertainty of the data is considered. The problem is reduced to construction of interval splines, for cases of linear and cubic interval splines algorithms of their construction and results of numerical experiments are described, the corresponding graphic interpretations are presented.

Keywords: Mathematical modeling, robot, set of possible trajectories, condition of progressiveness of movement, interval values, generalized intervals, interval splines, error function.

1 Formulation of the Problem

Let's examine the problem of definition of trajectory of certain mechanism or robot moved from starting point $M_0(x_0, y_0)$ to the final point $M_n(x_n, y_n)$. In reality there can happen a whole set of the factors leading to a deviation of robot from the planned or optimal trajectory, especially if the robot is used on insufficiently known district with low possibility of direct supervision or adjustment of its path. In certain cases the possibility of presence of man or his supervision over of the process of robot functioning is totally excluded. The mathematical modeling and controlling in this case assumes the necessity of definition of all set of possible trajectories, namely all functions of condition or all decisions of modeling equations and directions since the coordinates of starting and final points can be known with errors, and robot, having begun movement from any point in the vicinity $M_0(x_0, y_0)$, can get in the result to any point from vicinity of point $M_n(x_n, y_n)$.

Let's assume that the robot due to the reasons which influence accuracy of definition of its coordinates can be arranged in some fixed vicinity of point M_0 and get into only a certain vicinity of point M_n. In work [1] it is supposed that points M_0 and M_n are in parallelotopes, which in two-dimensional case are ordinary rectangles. Thus there is an infinite set of lines connecting initial and final points of possible trajectories of the robot. It is appropriate to assume that desired trajectories are schedules of functions of a certain class. Further the problem on definition of some continuous line (a robot trajectory) is set which connects points $M_0 \in \tilde{M}_0$ and $M_n \in \tilde{M}_n$ where $\tilde{M}_0$, $\tilde{M}_n$ are limited neighborhoods of initial and final points of trajectory. The options of trajectories thus can be very diverse, beginning from a primitive broken line without self-crossings, with the typical slope angles of segments to the axis of abscissa, ending with some sufficed arbitrary curve with self-crossings.

Let the first derivative of function $y = f(x)$ in any point $x_0 < x < x_n$ is positive, i.e. $f'(x > 0)$, $\forall x \in (x_0, x_n)$, what corresponds to the condition of progressiveness of movement and absence of self-crossings of curve of the function $y = f(x)$. Further by some criterion which will be specified further, on an interval $[x_0, x_n]$, we will choose $n - 1$ point: $x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n$. The splitting chosen this way we will designate as $\triangle$. It can be abscisses of barriers or points in which the robot should to stay a certain time in rest, execute some instruction, command and etc. If the choice of these points is not stipulated in advance, it is possible to assume that

$$x_i = \frac{x_n + x_0}{2} + \frac{x_n - x_0}{2} \cdot \cos\frac{\pi(2i+1)}{2n}, \quad i = 0, 1, \ldots, n-1, \tag{1}$$

since at the construction of approximation for the method of interpolation the way of interpolation on knots of Chebyshev's polynomial is optimal in sense of minimality of deviation of function error from zero, for any approached hypothetical function (trajectory) $y = f(x)$ [2].

If values are set $r_i \geq 0$ $(i = 0, 1, 2, \ldots, n)$ then it is possible to assume that points $M_i(x_i, y_i)$ are centers of circles with radiuses r_i in which the robot can be located due to influence of hindrances or casual forces.

Let in points x_i intervals $\boldsymbol{d}_i = [\underline{d}_i, \overline{d}_i]$ are set such that $\omega(\boldsymbol{d}_i) = \overline{d}_i - \underline{d}_i = 2 * r_i$ [3]. Here and everywhere below we for designation of interval sizes and objects will use format "**bold**" without any preliminary stipulations, and also assume existence of acquaintance with basics of interval analysis [3,4]. Then set of points $\boldsymbol{M}_\triangle(x_i, \boldsymbol{d}_i) \in \mathbb{R} \otimes \mathbb{IR}$ be considered as the inputs of problem of interval interpolation, *problem* type ***IIN1*** [5], for certain interval value function $\boldsymbol{y} = \boldsymbol{f}(x) = [f_1(x), f_2(x)]$ [6]. Thus values of r_i are unequivocally defined by admissions of locations of the robot or values of errors, and $\boldsymbol{d}_i$ can be considered as the generalized intervals [3], such as $\boldsymbol{d}_i = y_i + [-r_i, r_i]$. In this case, as distinct from [1], in geometrical aspect it is a matter not about parallelotopes and roundtopes, such circumference that $\forall x \in (x_0, x_n)$ the section $f_2(a) - f_1(a)$ represents diameter of circumference within which a robot can be situated. The problem of interval interpolation of type ***IIN1*** is studied in [7] where interval variants for a number "classical"interpolation formulas of Lagrange, Newton, Ermit are presented. In this case interval analogies of interpolation polynoms can be accepted as natural interval expansions since interpolation conditions will be satisfied without transformations due to properties of generalized interval arithmetics [3].

2 Construction of trajectory of the robot

At search of trajectory of the robot, satisfying to the conditions of interpolation on the points $\boldsymbol{M}_\triangle(x_i, \boldsymbol{d}_i)$, the interval variant of linear spline can be considered as the most natural decision of a problem since geometrically it contains sets of every possible trajectories which are polygonal line on which the corresponding robot can move. Since the cubic splines in real case, namely at the consideration of only dot values possess certain advantages including minimized kinetic energy of object making a trajectory on the given line [8], the interval analogue of the cubic spline constructed on the basis of resulted in [2] arguments is considered as well. At that coincidence of designations should not be treated as copying as all of the objects and statements are understood in interval sense.

Definition: Let's name *I-spline* of multiple m function which is an interval polynomial of degree m on each of segments $[x_i, x_{i+1}]$ $(i = 0, 1, \ldots, n-1)$:

$$\boldsymbol{S}^m_\triangle(\boldsymbol{f}, x) = \boldsymbol{P}_{im}(x) = \boldsymbol{a}_{i0} + \boldsymbol{a}_{i1}x + \ldots + \boldsymbol{a}_{im}x^m, \quad x_i \leq x \leq x_{i+1} \tag{2}$$

and satisfying to the conditions of continuity of derivatives to an $m-1$ multiple in points $x_1, x_2, \ldots, x_{n-1}$:

$$\boldsymbol{P}^{(k)}_{im}(x_i) = \boldsymbol{P}^{(k)}_{i+1,m}(x_i), \tag{3}$$

at $k = 0, 1, \ldots, m-1;\ i = 1, 2, \ldots, n-1$.

Everywhere below derivatives from *I-spline* and approached by it hypothetical interval value function $y = \boldsymbol{f}(x)$ accepting in the points x_i values $\boldsymbol{d}_i = [\underline{d}_i, \overline{d}_i]$ are understood in sense "formal derivative" from [9], and integrals in sense of [10]. Thus in expressions operations over interval sizes are understood in the generalized sense, in the assumption that in practice the mechanism of reboots can be used allowing to implement calculations within the limits of rules of suitable interval arithmetics [3,4] with application of the special software, as for example from [11,12]. It should be noted that conditions (3) are understood in interval sense [3], and components of system matrix for definition of coefficients $\boldsymbol{S}^m_\triangle(\boldsymbol{f}, x)$ also will be interval.

For construction of $\boldsymbol{S}^m_\triangle(\boldsymbol{f}, x)$ it is necessary to define $n(m+1)$ of unknown coefficients $\boldsymbol{a}_{im}$. Parities (3) form system from $m(n-1)$ equations. Other equations for coefficients are resulting from condition of affinity to the approached function and from some additional conditions.

Let $m = 1$. It means that the robot from point to point moves on straight line, or to be more exact on one of polygonal line included into $\boldsymbol{S}^m_\triangle(\boldsymbol{f}, x)$ spline. Then the total number of free parameters equals $2n$. Though in reality each interval is set by bottom and top limits and it is necessary for us to define $4n$ of real numbers, we, nevertheless, believe that it is necessary to define $2n$ interval coefficients since algorithms will be constructed within the limits of the chosen structure of interval numbers.

So, the question of construction of $\boldsymbol{S}^1_\triangle(\boldsymbol{f},\ x)$ spline coinciding with $\boldsymbol{f}(x)$ in points $x_0, x_1, \dots, x_n$ is stated.

From conditions of interpolation the system of the equations will turn out as

$$\begin{cases} \boldsymbol{P}_{i1}(x_{i-1}) = \boldsymbol{f}(x_{i-1}), \quad i = 1, 2, \dots, n \\ \boldsymbol{P}_{i1}(x_i) = \boldsymbol{f}(x_i), \quad i = 1, 2, \dots, n \end{cases} \tag{4}$$

which breaks up to the system of equations relative to coefficients of separate polynomials

$$\begin{cases} \boldsymbol{P}_{i1}(x_{i-1}) = \boldsymbol{a}_{n0} + \boldsymbol{a}_{n1}x_{n-1} = \boldsymbol{f}(x_{i-1}) = \boldsymbol{d}_{i-1}, \quad i = 1, 2, \dots, n \\ \boldsymbol{P}_{i1}(x_i) = \boldsymbol{a}_{n0} + \boldsymbol{a}_{n1}x_n = \boldsymbol{f}(x_i) = \boldsymbol{d}_i, \quad i = 1, 2, \dots, n. \end{cases} \tag{5}$$

From here, considering operations with intervals within the framework of generalized interval arithmetics [4] where in particular $\boldsymbol{a} - \boldsymbol{a} = [0, 0]$, we find that

$$\begin{cases} \boldsymbol{a}_{i1} = \dfrac{\boldsymbol{f}(x_i) - \boldsymbol{f}(x_{i-1})}{x_i - x_{i-1}} = \dfrac{\boldsymbol{d}_i - \boldsymbol{d}_{i-1}}{x_i - x_{i-1}}, \quad i = 1, 2, \dots, n \\ \boldsymbol{a}_{i0} = \boldsymbol{f}(x_{i-1}) - \boldsymbol{a}_{i1}x_{i-1} = \boldsymbol{d}_i - \boldsymbol{a}_{i1}x_{i-1}, \quad i = 1, 2, \dots, n. \end{cases} \tag{6}$$

The constructed polynomial $\boldsymbol{P}_{i1}(x)$ is interval interpolation polynomial of the first degree with interpolation knots $x_i,\ i = 0, 1, \dots, n$, in which $\boldsymbol{P}_{i1}(x_i) = \boldsymbol{d}_i$.

Let $m = 3$. Also as in purely real case, the linear spline $S^1_\triangle(\boldsymbol{f},\ x)$ constructed above, possesses extreme properties: any real contraction $Rs\left(S^1_\triangle(\boldsymbol{f},\ x)\right) = q(x)$ realizes the bottom limit $\inf\limits_{q \in Q_1} I_1(q)$, where Q_1 is a set of continuous functions $q(x)$, satisfying to the conditions

$$q(x_i) = f(x_i) \in \boldsymbol{d}_i, \quad i = 0, 1, \dots, n, \quad I_1(q) = \int_{x_0}^{x_n} \left(q'\right)^2 dx < \infty. \tag{7}$$

By analogy to a real case, the approaching function we will be sought in space of functions bigger smoothnesses, namely we will consider set Q_2 of functions $q(x)$ from the continuous first derivative, satisfying to the conditions

$$q(x_i) = f(x_i) \in \boldsymbol{d}_i, \quad i = 0, 1, \dots, n, \quad I_2(q) = \int_{x_0}^{x_n} \left(q''\right)^2 dx < \infty. \tag{8}$$

Let's search for the function $q_2 \in Q_2$ realizing $\inf\limits_{q \in Q_1} I_1(q)$. Euler's formal equation for corresponding functional $\boldsymbol{S}^{(4)}(x) = 0$, i.e.

$$\boldsymbol{q}_2(x) = \boldsymbol{P}_{i3}(x) = \boldsymbol{a}_{i0} + \boldsymbol{a}_{i1}x + \boldsymbol{a}_{i2}x^2 + \boldsymbol{a}_{i3}x^3$$

on intervals $[x_{i-1}, x_i]$. The number of parametres $\boldsymbol{a}_{ik}$, subject to definition equals $4n$, but only when number of conditions equals $3n - 1$:

$2n$ conditions

$$\boldsymbol{P}_{i3}(x_i) = \boldsymbol{P}_{i+1,3}(x_i) = \boldsymbol{d}_i \tag{9}$$

and $n - 1$ condition of continuity of first derivative

$$\boldsymbol{P}'_{i3}(x_i) = \boldsymbol{P}'_{i+1,3}(x_i) \tag{10}$$

in points $x_1, x_2, \ldots, x_{n-1}$. The missing conditions are obtained as natural boundary conditions in points x_i.

Let's assume that minimizing function $\boldsymbol{q}_2(x)$ exists and it is polynomial of the third degree on intervals $[x_{i-1}, x_i]$. Let $\eta(x)$ be infinitely differentiated in sense of formal derivative function, such that,

$$\eta(x_0) = \ldots = \eta(x_n) = 0. \tag{11}$$

Since within the frameworks accepted above agreements, considerations from [1], by definition lacking $n + 1$ conditions can be repeated taking into account (11), then following interval conditions exist

$$\boldsymbol{P}''_{13}(x_0) = \boldsymbol{P}''_{n3}(x_n) = 0, \tag{12}$$

$$\boldsymbol{P}''_{n3}(x_i) - \boldsymbol{P}''_{n+1,3}(x_i) = 0, \tag{13}$$

where $i = 1, 2, \ldots, n - 1$; close the system for construction $\boldsymbol{P}_{i3}(x)$, being a cubic parabola on each of intervals $[x_{i-1}, x_i]$.

Now we will consider the problem on resolvability and about the practical decision of system of equations ((9), (10), (12), (13)). For convenience we will enter into consideration values $\boldsymbol{M}_i = \boldsymbol{q}''_2(x_i)$. Since function $\boldsymbol{q}''_2(x_i)$ is linear on each interval $[x_{i-1}, x_i]$, then

$$\boldsymbol{q}''_2(x) = \frac{\boldsymbol{M}_{i-1}(x_i - x)}{h_i} + \frac{\boldsymbol{M}_i(x - x_{i-1})}{h_i} \quad \texttt{on} \quad [x_{i-1}, x_i], \tag{14}$$

where it is allowed $h_i = x_i - x_{i-1}$. From relation (14) and conditions

$$\boldsymbol{q}_2(x_{i-1}) = \boldsymbol{f}(x_{i-1}), \quad \boldsymbol{q}_2(x_i) = \boldsymbol{f}(x_i),$$

we can get that

$$\begin{aligned}\boldsymbol{q}_2(x) = \boldsymbol{P}_{i3}(x) = \boldsymbol{M}_{i-1}\tfrac{(x_i-x)^3}{6h_i} + \boldsymbol{M}_i\tfrac{(x-x_{i-1})^3}{6h_i} + \left(\boldsymbol{f}(x_{i-1}) - \tfrac{\boldsymbol{M}_{i-1}h_i^2}{6}\right) \times \\ \times \tfrac{x_i-x}{h_i} + \left(\boldsymbol{f}(x_i) - \tfrac{\boldsymbol{M}_i h_i^2}{6}\right)\tfrac{x-x_{i-1}}{h_i} \quad \texttt{on} \quad [x_{i-1}, x_i].\end{aligned} \tag{15}$$

Conditions

$$\boldsymbol{P}'_{i3}(x_n) = \boldsymbol{P}'_{i+1,3}(x_i), \quad i = 1, 2, \ldots, n - 1,$$

form equation

$$\frac{h_i}{6}\boldsymbol{M}_{i-1} + \frac{h_i + h_{i+1}}{3}\boldsymbol{M}_i + \frac{h_{i+1}}{6}\boldsymbol{M}_{i+1} = \frac{\boldsymbol{f}(x_{i+1}) - \boldsymbol{f}(x_i)}{h_{i+1}} - \frac{\boldsymbol{f}(x_i) - \boldsymbol{f}(x_{i-1})}{h_i}. \tag{16}$$

Besides that we have conditions

$$\boldsymbol{P}'_{13}(x_0) = 0, \quad \boldsymbol{P}'_{n3}(x_n) = 0,$$

other wise $\boldsymbol{M}_0 = 0$, $\boldsymbol{M}_n = 0$. After substitution $\boldsymbol{M}_0 = 0$, $\boldsymbol{M}_n = 0$, correspondingly, into first and last equation (15) we will receive the system

$$C\boldsymbol{M} = \boldsymbol{b} \tag{17}$$

$n-1$-th equation with $n-1$-th unknown:

$$\boldsymbol{M} = (\boldsymbol{M}_1, \boldsymbol{M}_2, \ldots, \boldsymbol{M}_{n-1})^\top, \quad \boldsymbol{b} = (\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_{n-1})^\top.$$

Elements of c_{ij} $(i, j = 1, \ldots, n-1)$ matrix , according to (16), are set by relations

$$c_{ij} = \begin{cases} \frac{h_i}{6} & j = i-1, \\ \frac{h_i + h_{i+1}}{3} & j = i, \\ \frac{h_{i+1}}{6} & j = i+1, \\ 0 & |i-j| > 1, \end{cases}$$

and $\boldsymbol{b}_i$ elements of $\boldsymbol{b}$ column by relations

$$\boldsymbol{b}_i = \frac{\boldsymbol{f}(x_{i+1}) - \boldsymbol{f}(x_i)}{h_{i+1}} - \frac{\boldsymbol{f}(x_i) - \boldsymbol{f}(x_{i-1})}{h_i} = \frac{\boldsymbol{d}_{i+1} - \boldsymbol{d}_i}{h_{i+1}} - \frac{\boldsymbol{d}_i - \boldsymbol{d}_{i-1}}{h_i}.$$

The system (17) can be resolved by the "method of interval marching"[13] by $O(n)$ interval arithmetic operations. After finding $\boldsymbol{M}_j$ by formula (15) we will define polynomials $\boldsymbol{P}_{i3}(x)$, and function $\boldsymbol{\Phi}(\mathbf{x}) = \boldsymbol{q}_2(\mathbf{x}) + [\underline{\mathbf{r}}(\mathbf{x}), \overline{\mathbf{r}}(\mathbf{x})]$ allows to determine in any point $x \in [x_{i-1}, x_i]$ location of robot, namely: $r = \frac{|\boldsymbol{\Phi}(\mathbf{x})|}{2}$ - the radius of circumference within which it will be situated. At that

$$\max_{1 \le i \le n-1} |[\underline{r}(x), \overline{r}(x)]| \le \frac{3}{\min\limits_{1 \le i \le n-1} h_i} \max_{1 \le i \le n-1} |\boldsymbol{d}_i|,$$

what estimated the maximum expandability of the strip $\boldsymbol{q}_2(x)$ within each local interval of a grid $\triangle$ at $x \in (x_{i-1}, x_i)$.

3 Numerical experiments

Numerical experiments were implemented in the local package named provisionally "Robot"since at the solution of the assigned task by the means of the package [12] the necessity for dynamic visualization of the trajectories emerging depending on the interval initial data has arisen. Besides that the necessity for full set of interval arithmetics [11] included into [12] has disappeared.

In *figure 1* the interface on definition of dispersion of interval values with usage of chosen interval arithmetics is presented: by Moore, Markov and Kahan.

In *figure 2* the diagram of cubic *I-spline* (red strip) when the tabular data, in points of interval $[0, 10]$ are by a randomization method is presented. Within this strip there is a possible trajectory of the robot when for function $y = \boldsymbol{f}(x)$ in points x_i dot values are chosen $y_i = \mathtt{mid}(\boldsymbol{d}_i) = \dfrac{\underline{d}_i + \overline{d}_i}{2}$ and the real cubic spline is constructed.

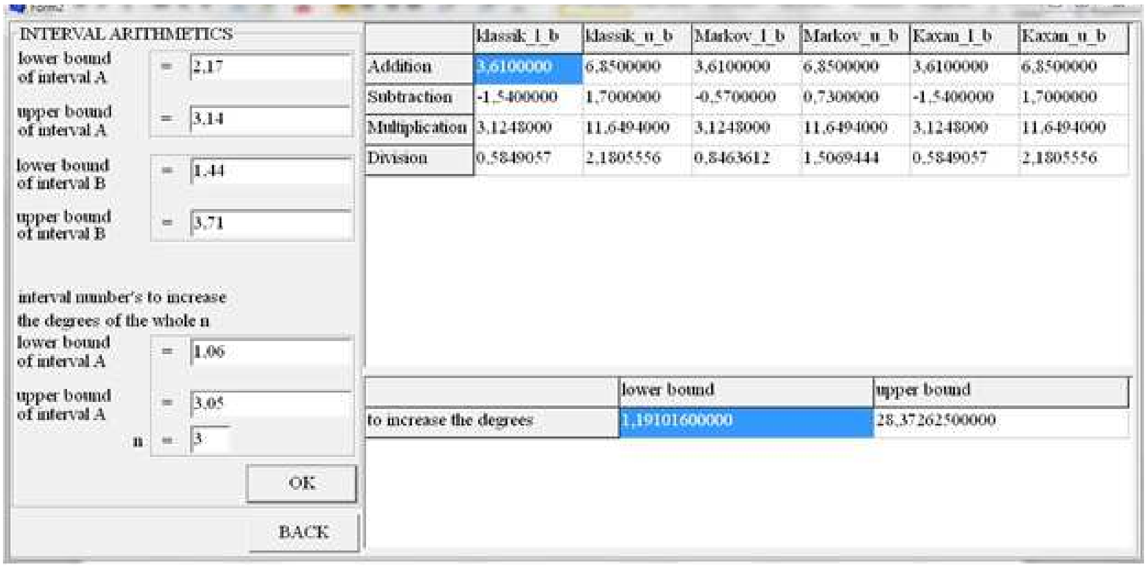
INTERVAL ARITHMETICS
lower bound of interval A = 2,17
upper bound of interval A = 3,14
lower bound of interval B = 1,44
upper bound of interval B = 3,71
interval number's to increase the degrees of the whole n
lower bound of interval A = 1,06
upper bound of interval A = 3,05
n = 3
OK
BACK
klassik_l_b klassik_u_b Markov_l_b Markov_u_b Kaxan_l_b Kaxan_u_b
Addition 3,6100000 6,8500000 3,6100000 6,8500000 3,6100000 6,8500000
Subtraction -1,5400000 1,7000000 -0,5700000 0,7300000 -1,5400000 1,7000000
Multiplication 3,1248000 11,6494000 3,1248000 11,6494000 3,1248000 11,6494000
Division 0,5849057 2,1805556 0,8463612 1,5069444 0,5849057 2,1805556
lower bound upper bound
to increase the degrees 1,19101600000 28,37262500000

Fig. 1.

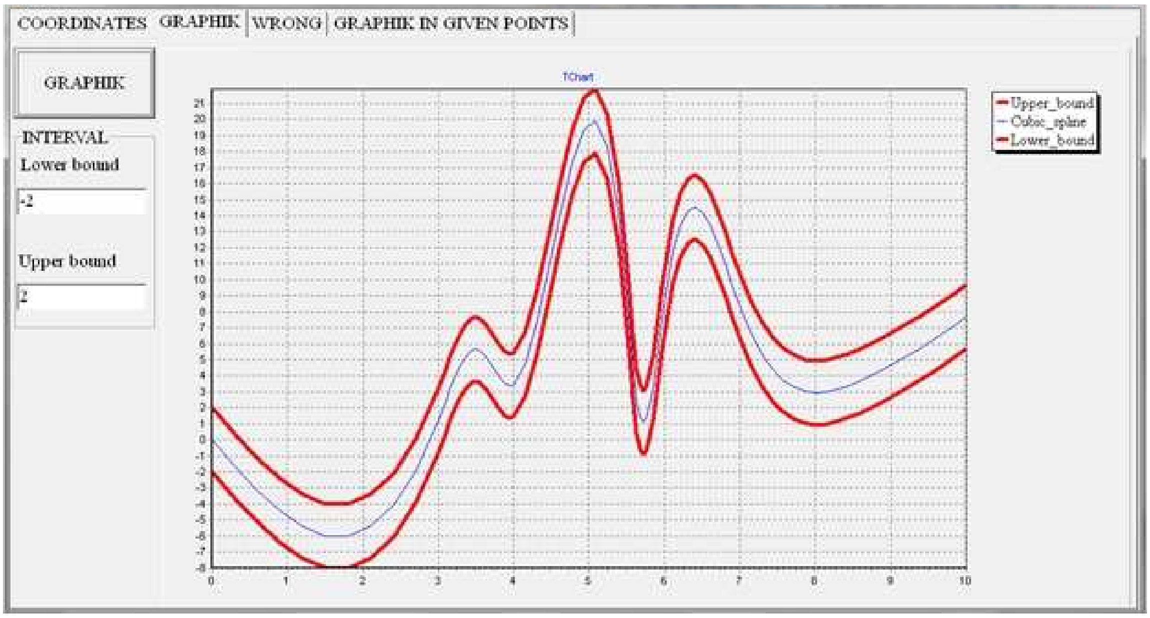
COORDINATES GRAPHIK WRONG GRAPHIK IN GIVEN POINTS
GRAPHIK
INTERVAL
Lower bound
-2
Upper bound
2
TChart
Upper_bound
Cubic_spline
Lower_bound

Fig. 2.

During experiments in the capacity of possible trajectory the certain functions were chosen and approbation of method on so-called "the point solution"was conducted. In *figure 3* graphs are illustrated when function

$$y = 10^{-1} x \cos x + \exp(-x)$$

was chosen. Within the strip in this case the hypothetical curve and corresponding cubic spline are indicated and quantity of control points $n = 24$, in the assumption of possible observations taken with interval of one hour within 24 hours.

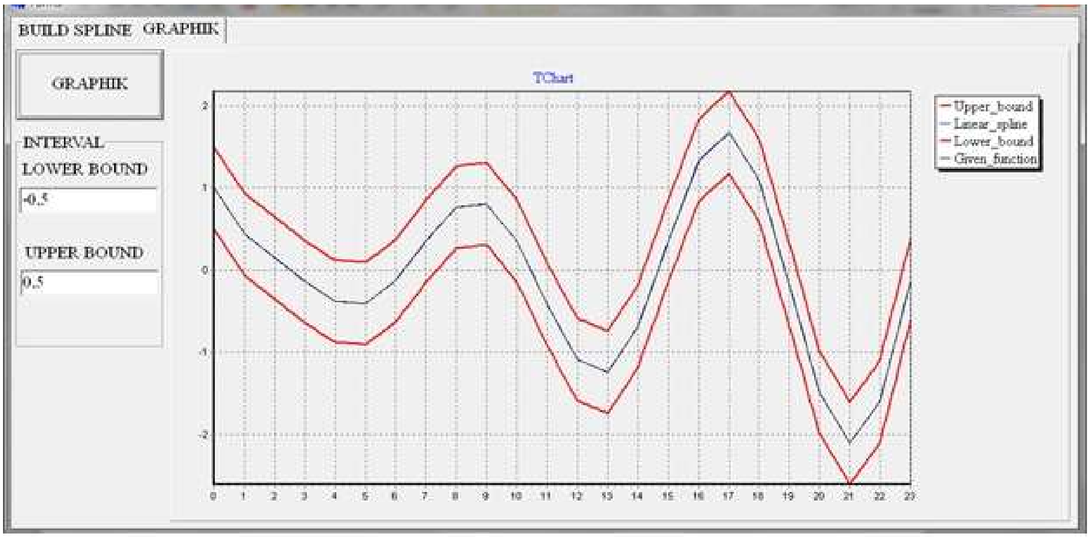

Fig. 3.

References

1. Jaulin L., Kieffer M., Didrit O., Walter E. Applied Interval Analysis. Springer-Verlag London Limited, (2001)
2. Bahvalov N.S. Numerical methods. - Moskva: Science, (1973)
3. Kalmykov S.A., Shokin Yu.I. and Yuldashev Z.KH. Metods of interval analysis. Novosibirsk: Nauka, (1986)
4. Shary S.P. Finite-dimensional interval analysis. Publishing house "XYZ (2010)
5. Yuldashev Z.Kh., Ibragimov A.A., Tadjibaev Sh.Kh. Interval polynomial interpolation for bounded-error data, 15th GAMM-IMACS International Symposium on Scientific Computing, Computer Arithmetic and Verifed Numerical Computations, Novosibirsk, pp. 190-191, (2012)
6. Shokin Yu.I., Yuldashev Z.Kh. Representability of interval value functions with real limited functions. – Collection. "Numerical methods of continuum mechanics vol.**4**, №5, pp. 134-146, (1973)
7. Dobronets B.S., Shokin Yu.I., Yuldashev Z.Kh. Task of interpolation of the interval analysis. AN UzSSR. "Questions of calculus mathematics Vol. **31**, (1975)
8. Zavyalov Yu.S., Kvasov B.I., Miroshnichenko V.L. The methods of spline functions. – Moskva : Science, (1980)
9. Schroder G. Differentiation of interval functions. Proc. Amer. Math. Soc., Vol. **36**, pp. 485-490, (1972)
10. Moore R.E. Interval Analysis. – Englewood Cliffs. N.J.: Prentice-Hall, (1966)
11. Yuldashev Z.Kh., Ibragimov A.A., Kalhanov P.J. The package of interval algorithms for wide usage. Registered in state register of programs for PC of the Republic of Uzbekistan, Certificate of official registration of programs for PC N DGU 02201, Tashkent city, (2011)
12. Yuldashev Z.Kh., Ibragimov A.A., Kalhanov P.J. The complex of programs for calculation of values of interval algebraically admissible expressions within the limits of various interval arithmetics. Registered in state register of programs for PC of the Republic of Uzbekistan, Certificate of official registration of programs for PC N DGU 02202, Tashkent city, (2011)
13. Yuldashev Z.Kh., Ibragimov A.A. About analysis of full error in the method of interval marching and problem of inversion of interval matrixes. Computational technologies. Novosibirsk, Russia. N 9(2), pp.235-240, (2004)