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V.I. Burenkov[1], M. Lanza de Cristoforis[2], N.A. Kydyrmina[3]
[1]*L.N. Gumilyov Eurasian National University, Astana, Kazakhstan*
E-mail: burenkov@cf. ac.uk
[2]*University of Padova, Padova, Italy E-mail: mldc@math.unipd.it*
[3]*Institute of Applied Mathematics of CS of MES of RK, Karaganda, Kazakhstan*
E-mail: nurgul-k@mail.ru

Approximation by C^∞ functions in Sobolev-Morrey spaces

In this paper we consider Sobolev spaces built on Morrey spaces, also referred to as Sobolev Morrey spaces, i.e., the spaces of functions which have derivatives up to a certain order in Morrey spaces.
First we characterize the functions in a Morrey space which can be approximated in norm by smooth functions, as the functions which belong to a specific subspace of the Morrey space, which we call the 'little' Morrey space.
Contrary to the classical Sobolev spaces built on the L_p spaces with $p < \infty$, the Sobolev spaces built on Morrey spaces are not separable spaces even if $p < \infty$ and we cannot expect that the set of C^∞ functions of a Sobolev Morrey space be dense in a Sobolev Morrey space. However, we show that the functions in a Sobolev space built on little Morrey spaces can be approximated by C^∞ functions.
Key words: Morrey spaces, Sobolev-Morrey spaces, approximation.

В. И. Буренков, М. Ланца де Кристофорис, Н. А. Кыдырмина

Приближение бесконечно дифференцируемыми функциями в пространствах Соболева-Морри

В данной работе мы рассматриваем пространства Соболева, построенные на основе пространств Морри, также носящих название пространств Соболева-Морри, т.е. пространства функций, имеющих производные вплоть до определенного порядка в пространствах Морри.
Сперва мы описываем функции из пространства Морри, которые можно приблизить по норме бесконечно дифференцируемыми функциями, как функции, принадлежащие определенному подпространству пространства Морри, называемому 'малым' пространством Морри.
В отличие от классических пространств Соболева, построенных на основе пространств L_p, где $p < \infty$, пространства Соболева, построенные на основе пространств Морри, не являются сепарабельными пространствами даже при $p < \infty$ и у нас нет оснований ожидать, что множество бесконечно дифференцируемых функций из пространства Соболева-Морри будет плотно в пространстве Соболева-Морри. Однако, мы показываем, что функции из пространства Соболева, построенного на основе малого пространства Морри, могут быть приближены бесконечно дифференцируемыми функциями.
Ключевые слова: пространства Морри, пространства Соболева, аппроксимация.

В. И. Буренков, М. Ланца де Кристофорис, Н. А. Кыдырмина

Соболев-Морри кеңiстiгiндегi функцияларды шексiз көп дифференциалданатын функциялармен жуықтау

Бұл жұмыста Морри кеңiстiгiнiң негiзiнде құрылған Соболев кеңiстiгi қарастырылды. Алдымен Морри кеңiстiгiнiң сансыз көп рет дифференциалданатын функциялармен жуықталатын функцияларын - "Морридiң iшкi кеңiстiгi" деп аталатын iшкi кеңiстiгiнiң элементi ретiнде бейнеледiк.

Соболев-Морри кеңiстiгi L_p кеңiстiгiнiң негiзiнде құрылған классикалық Соболев кеңiстiгi сияқты сеперабельдi кеңiстiк болмағандықтан, сансыз көп рет дифференциалданатын функциялар жиыны Соболев-Морри кеңiстiгiнiң барлық жерiнде дерлiк тығыз болады деп айта алмаймыз. Бiрақ, бiз осы жұмыста "iшкi Морри кеңiстiгiнiң" негiзiнде құрылған Соболев кеңiстiгiнiң кез келген элементi сансыз көп рет дифференциалданатын функциялармен жуықталатынын көрсеттiк.

Түйiн сөздер: Морри кеңiстiгi, Соболев-Морри кеңiстiгi, жуықтау.

1. Morrey spaces

Let $\mathbb{N}$ denote the set of all natural numbers including 0. Throughout the paper, n is an element of $\mathbb{N}\setminus\{0\}$.

Morrey spaces were introduced by C. Morrey in [1]. Here we consider the following variant of Morrey spaces (coinciding with Morrey spaces for bounded sets).

Definition 1. *Let Ω be a Lebesgue measurable subset of $\mathbb{R}^n$. Let $1 \le p \le +\infty$, $0 \le \lambda \le \frac{n}{p}$ and $w_\lambda(\rho) = \begin{cases} \rho^{-\lambda}, & \rho \in]0,1], \\ 1, & \rho \ge 1, \end{cases}$. Denote by $M_p^\lambda(\Omega)$ the space of all real-valued measurable functions on Ω for which*

$$\|f\|_{M_p^\lambda(\Omega)} = \sup_{x\in\Omega} \|w_\lambda(\rho)\|f\|_{L_p(B(x,\rho)\cap\Omega)}\|_{L_\infty(0,\infty)} < \infty,$$

where $B(x,\rho)$ is the open ball of radius $\rho > 0$ centered at the point $x \in \mathbb{R}^n$.

Clearly, $M_p^0(\Omega) = L_p(\Omega)$. Also, $M_p^{\frac{n}{p}}(\Omega) = L_\infty(\Omega)$.

By results in [2], [3] it follows that for $\lambda < 0$ or $\lambda > \frac{n}{p}$ the space $M_p^\lambda(\Omega)$ is trivial, that is consists only of functions equivalent to 0 on Ω.

We find convenient to set

$$|f|_{\rho,\lambda,p,\Omega} \equiv \sup_{x\in\Omega} \left\| w_\lambda(r)\|f\|_{L_p(B(x,r)\cap\Omega)} \right\|_{L_\infty(0,\rho)} \quad \forall \rho \in]0,+\infty[$$

for all measurable functions f from Ω to $]0,+\infty[$.

Definition 2. *Let Ω be an open subset of $\mathbb{R}^n$. Let $p \in [1,+\infty]$, $\lambda \in [0, n/p[$. Then we define as the little Morrey space with the exponents λ, p is the subspace*

$$M_p^{\lambda,0}(\Omega) \equiv \left\{ f \in M_p^\lambda(\Omega) \,:\, \lim_{\rho\to 0} |f|_{\rho,\lambda,p,\Omega} = 0 \right\}$$

of $M_p^\lambda(\Omega)$.

Lemma 1. *Let Ω be an open subset of $\mathbb{R}^n$. Let $p \in [1,+\infty]$. Let $\lambda \in [0, n/p]$. Then $M_p^{\lambda,0}(\Omega)$ is a closed proper subspace of $M_p^\lambda(\Omega)$.*

Theorem 1. *Let Ω be an open subset of $\mathbb{R}^n$. Let p_1, $p_2 \in [1,+\infty]$ be such that $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p}$. Let $\lambda_1,\lambda_2 \in [0,+\infty[$, $\lambda=\lambda_1+\lambda_2$. Then the pointwise multiplication is bilinear and continuous from $M_{p_1}^{\lambda_1}(\Omega)\times M_{p_2}^{\lambda_2}(\Omega)$ to $M_p^{\lambda}(\Omega)$ and maps $M_{p_1}^{\lambda_1,0}(\Omega)\times M_{p_2}^{\lambda_2}(\Omega)$ to $M_p^{\lambda,0}(\Omega)$ and $M_{p_1}^{\lambda_1}(\Omega)\times M_{p_2}^{\lambda_2,0}(\Omega)$ to $M_p^{\lambda,0}(\Omega)$*

Remark. *This statement proves the Hölder inequality for Morrey space $M_p^{\lambda}(\Omega)$:*

$$\|fg\|_{M_p^{\lambda}(\Omega)} \le \|f\|_{M_{p_1}^{\lambda_1}(\Omega)}\|g\|_{M_{p_2}^{\lambda_2}(\Omega)} \quad \forall\,(f,g)\in M_{p_1}^{\lambda_1}(\Omega)\times M_{p_2}^{\lambda_2}(\Omega).$$

Proof. Note that

$$w_\lambda(\rho)=\begin{cases}\rho^{-\lambda}, & \rho\in]0,1],\\ 1, & \rho\ge 1,\end{cases} = \begin{cases}\rho^{-\lambda_1-\lambda_2}, & \rho\in]0,1],\\ 1, & \rho\ge 1,\end{cases} = w_{\lambda_1}(\rho)w_{\lambda_2}(\rho).$$

Then, by Hölder inequality, we have

$$|fg|_{\rho,\lambda,p,\Omega} = \sup_{(x,r)\in\Omega\times]0,\rho[} w_\lambda(r)\|fg\|_{L_p(B(x,r)\cap\Omega)} \le$$
$$\le \sup_{(x,r)\in\Omega\times]0,\rho[} w_\lambda(r)\|f\|_{L_{p_1}(B(x,r)\cap\Omega)}\|g\|_{L_{p_2}(B(x,r)\cap\Omega)} \le$$
$$\le \sup_{(x,r)\in\Omega\times]0,\rho[} w_{\lambda_1}(r)\|f\|_{L_{p_1}(B(x,r)\cap\Omega)} \sup_{(x,r)\in\Omega\times]0,\rho[} w_{\lambda_2}(r)\|g\|_{L_{p_2}(B(x,r)\cap\Omega)} =$$
$$= |f|_{\rho,\lambda_1,p_1,\Omega}|g|_{\rho,\lambda_2,p_2,\Omega} \quad \text{for all} \quad \rho\in]0,+\infty].$$

Therefore, by taking $\rho=+\infty$, we deduce that $fg\in M_p^{\lambda}(\Omega)$ when $(f,g)\in M_{p_1}^{\lambda_1}(\Omega)\times M_{p_2}^{\lambda_2}(\Omega)$.

By letting $\rho\to 0$, we deduce that $fg\in M_p^{\lambda,0}(\Omega)$ when $(f,g)\in M_{p_1}^{\lambda_1,0}(\Omega)\times M_{p_2}^{\lambda_2}(\Omega)$.

The case when $f\in M_{p_1}^{\lambda_1}(\Omega)$ and $g\in M_{p_2}^{\lambda_2,0}(\Omega)$ can be analyzed in the same way.

Theorem 2. *Let Ω be an open subset of $\mathbb{R}^n$. Let $p\in[1,+\infty]$. Let $\lambda\in[0,n/p]$. Then the pointwise multiplication is bilinear and continuous from $M_p^{\lambda}(\Omega)\times L_\infty(\Omega)$ to $M_p^{\lambda}(\Omega)$ and maps $M_p^{\lambda,0}(\Omega)\times L_\infty(\Omega)$ to $M_p^{\lambda,0}(\Omega)$.*

Proof. We show that if $f\in M_p^{\lambda}(\Omega)$, $g\in L_\infty(\Omega)$, then

$$|fg|_{\rho,\lambda,p,\Omega} = \sup_{(x,r)\in\Omega\times]0,\rho[} w_\lambda(r)\|fg\|_{L_p(B(x,r)\cap\Omega)} \le$$
$$\le \|g\|_{L_\infty(\Omega)} \sup_{(x,r)\in\Omega\times]0,\rho[} w_\lambda(\rho)\|f\|_{L_p(B(x,r)\cap\Omega)} =$$
$$= \|g\|_{L_\infty(\Omega)}|f|_{\rho,\lambda,p,\Omega} \quad \text{for all} \quad \rho\in]0,+\infty].$$

Hence, by taking $\rho=+\infty$, we deduce that $fg\in M_p^{\lambda}(\Omega)$ when $(f,g)\in M_p^{\lambda}(\Omega)\times L_\infty(\Omega)$.
By letting $\rho\to 0$, we deduce that $fg\in M_p^{\lambda,0}(\Omega)$ when $(f,g)\in M_p^{\lambda,0}(\Omega)\times L_\infty(\Omega)$.

2. Sobolev-Morrey spaces

Definition 3. *Let $\Omega\subset\mathbb{R}^n$ be an open set. Let $l\in\mathbb{N}$, $1\le p\le+\infty$, and $0\le\lambda\le\frac{n}{p}$. Then we define the Sobolev space of order l built on the Morrey space $M_p^{\lambda}(\Omega)$, as the set*

$$W_p^{l,\lambda}(\Omega)\equiv\left\{f\in M_p^{\lambda}(\Omega) : D_w^{\alpha}f\in M_p^{\lambda}(\Omega)\ \forall\,\alpha\in\mathbb{N}^n, |\alpha|\le l\right\},$$

where $D_w^\alpha f$ is the weak derivative of f.

Then we set

$$\|f\|_{W_p^{l,\lambda}(\Omega)} = \sum_{|\alpha|\le l} \|D_w^\alpha f\|_{M_p^\lambda(\Omega)} \quad \forall\, f \in W_p^{l,\lambda}(\Omega).$$

In particular, $W_p^{0,\lambda}(\Omega) = M_p^\lambda(\Omega)$ and $W_p^{l,0}(\Omega) = W_p^l(\Omega)$, where $W_p^l(\Omega)$ denotes the classical Sobolev space with exponents l, p in Ω. It is obvious that $W_p^{l,\lambda}(\Omega) \subset W_p^l(\Omega)$.

Definition 4. *Let $\Omega \subset \mathbb{R}^n$ be an open set. Let $l \in \mathbb{N}$, $p \in [1, +\infty]$ and $\lambda \in \left[0, \frac{n}{p}\right]$. Then we define the Sobolev space of order l built on the little Morrey space $M_p^{\lambda,0}(\Omega)$, as the set*

$$W_p^{l,\lambda,0}(\Omega) \equiv \left\{ f \in M_p^{\lambda,0}(\Omega) : D_w^\alpha f \in M_p^{\lambda,0}(\Omega)\; \forall\, \alpha \in \mathbb{N}^n, |\alpha| \le l \right\}.$$

Since $M_p^{\lambda,0}(\Omega)$ is a closed subspace of $M_p^\lambda(\Omega)$, we can easily deduce the validity of

Lemma 2. *Let Ω be an open subset of $\mathbb{R}^n$. Let $l \in \mathbb{N}$, $1 \le p \le +\infty$, and $0 \le \lambda \le \frac{n}{p}$. Then $W_p^{l,\lambda,0}(\Omega)$ is a closed proper subspace of $W_p^{l,\lambda}(\Omega)$.*

Proof. Let $u \in W_p^{l,\lambda}(\Omega)$. Let $\{u_k\}_{k\in\mathbb{N}}$ be a sequence in $W_p^{l,\lambda,0}(\Omega)$ which converges to u in $W_p^{l,\lambda}(\Omega)$. We want to show that $u \in W_p^{l,\lambda,0}(\Omega)$.

Since $u_k \to u$ in $W_p^{l,\lambda}(\Omega)$ as $k \to \infty$, we have

$$D_w^\alpha u_k \to D_w^\alpha u \quad \forall\, |\alpha| \le l \quad \text{in} \;\; M_p^\lambda(\Omega)$$

as $k \to \infty$.

We know that $M_p^{\lambda,0}(\Omega)$ is a closed subspace of $M_p^\lambda(\Omega)$. Therefore,

$$D_w^\alpha u \in M_p^{\lambda,0}(\Omega) \quad \forall\, |\alpha| \le l,$$

and, thus, $u \in W_p^{l,\lambda,0}(\Omega)$.

3. The main result

Denote by $C^\infty(\Omega)$ the space of infinitely continuously differentiable functions on Ω and by $C_c^\infty(\Omega)$ the space of functions in $C^\infty(\Omega)$ with compact support.

Definition 5. *If $\phi \in L^1(\mathbb{R}^n)$ and $t \in]0, +\infty[$, we denote by $\phi_t(\cdot)$ the function from $\mathbb{R}^n$ to $\mathbb{R}$ defined by $\phi_t(x) \equiv t^{-n}\phi(x/t) \quad \forall\, x \in \mathbb{R}^n$.*

Definition 6. *Let V, Ω be open subsets of $\mathbb{R}^n$. We write $V \subset\subset \Omega$ if $V \subset \bar{V} \subset \Omega$ and $\bar{V}$ is compact, and say that V is compactly embedded in Ω.*

Contrary to the classical Sobolev spaces built on the L_p spaces with $p < \infty$, the Sobolev spaces built on Morrey spaces are not separable spaces even if $p < \infty$ and we cannot expect that the set of C^∞ functions of a Sobolev Morrey space be dense in a Sobolev Morrey space. However, we show that the functions in a Sobolev space built on little Morrey spaces can be approximated by C^∞ functions.

First we state a known Leibnitz formula for Sobolev spaces. For the proof one can see for example [4].

Theorem 3. *Let $l \in \mathbb{N}$. Let $1 \le p < +\infty$. Let Ω be a bounded open subset of $\mathbb{R}^n$. Let $u \in W_p^l(\Omega)$ and $|\alpha| \le l$. If $\zeta \in C_c^l(\Omega)$, then $\zeta u \in W_p^l(\Omega)$ and*

$$D_w^\alpha(\zeta u) = \sum_{\beta\le\alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} D^\beta \zeta D_w^{\alpha-\beta} u,$$

where $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \dfrac{\alpha!}{\beta!(\alpha-\beta)!}$

Lemma 3. *Let $\Omega \subset \mathbb{R}^n$ be an open set, $l \in \mathbb{N}$. Let $u, v \in L_1^{\mathrm{loc}}(\Omega)$. Moreover, assume that for any $\beta \in \mathbb{N}^n$ satisfying $|\beta| \le l$ there exists $1 \le p_\beta < \infty$ such that $D_w^\beta u \in L_{p_\beta}^{\mathrm{loc}}(\Omega)$ and $D_w^\gamma v \in L_{p'_\beta}^{\mathrm{loc}}(\Omega)$ for all $\gamma \in \mathbb{N}^n : |\gamma| \le l - |\beta|$. Then for any $\alpha \in \mathbb{N}^n$ satisfying $|\alpha| \le l$ the weak derivative $D_w^\alpha(uv)$ exists and the Leibnitz formula holds:*

$$D_w^\alpha(uv) = \sum_{0 \le \beta \le \alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} D_w^\beta u D_w^{\alpha-\beta} v.$$

Proof. Let $u \in C^\infty(\Omega)$ and $v \in L_1^{\mathrm{loc}}(\Omega)$ be such that $D_w^\gamma v \in L_1^{\mathrm{loc}}(\Omega)$. Then

$$D_w^\alpha(uv) = \sum_{0 \le \beta \le \alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} D^\beta u D_w^{\alpha-\beta} v.$$

Now let u be as in formulation, *i.e.* $D_w^\beta u \in L_{p_\beta}^{\mathrm{loc}}(\Omega)$. Let also

$$u_k(x) = u(x) * \varphi_{\frac{1}{k}}(x) \quad \forall\, k \in \mathbb{N},$$

where $\varphi_{\frac{1}{k}}(x)$ as in definition 5 with $t = \dfrac{1}{k}$.

Then, by Theorem 3, we have

$$D_w^\alpha(u_k v) = \sum_{0 \le \beta \le \alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} D^\beta u_k D_w^{\alpha-\beta} v = \sum_{0 \le \beta \le \alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} D^\beta [u * \varphi_{\frac{1}{k}}] D_w^{\alpha-\beta} v.$$

Properties of mollifiers imply that

$$u_k \to u \quad \text{in } L_{p_\beta}^{\mathrm{loc}}(\Omega),$$

$$D^\beta u_k \to D_w^\beta u \quad \text{in } L_{p_\beta}^{\mathrm{loc}}(\Omega),$$

as $k \to \infty$. Thus,

$$D_w^\alpha(uv) = \sum_{0 \le \beta \le \alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} D_w^\beta u D_w^{\alpha-\beta} v.$$

Theorem 4. *Let $l \in \mathbb{N}$. Let $1 \le p < +\infty$, $0 \le \lambda \le \frac{n}{p}$. Let Ω be a bounded open subset of $\mathbb{R}^n$. Then for $u \in W_p^{l,\lambda,0}(\Omega)$ there exist functions $u_m \in C^\infty(\Omega) \cap W_p^{l,\lambda}(\Omega)$ such that*

$$u_m \to u \quad \text{in } W_p^{l,\lambda}(\Omega).$$

Proof. We have $\Omega = \bigcup\limits_{i=1}^{\infty} \Omega_i$, where

$$\Omega_i := \left\{ x \in \Omega \colon \operatorname{dist}(x, \partial\Omega) > \frac{1}{i} \right\} \quad (i = 1, 2, \ldots).$$

Write $V_i := \Omega_{i+3} - \bar{\Omega}_{i+1}$.

Choose also any open set $V_0 \subset\subset \Omega$ so that $\Omega = \bigcup_{i=0}^{\infty} V_i$. Now let $\{\zeta_i\}_{i=1}^{\infty}$ be a smooth partition of unity subordinate to the open sets $\{V_i\}_{i=0}^{\infty}$; that is, suppose

$$\begin{cases} 0 \le \zeta_i \le 1, \quad \zeta_i \in C_c^\infty(V_i) \\ \sum_{i=0}^{\infty} \zeta_i = 1 \qquad \text{on } \Omega. \end{cases}$$

Next, choose any function $u \in W_p^{l,\lambda,0}(\Omega)$. By Theorem 3 we know that

$$D_w^\alpha(u\zeta_i) = \sum_{0\le\beta\le\alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} D_w^\beta u D_w^{\alpha-\beta}\zeta_i.$$

Since $D_w^\beta u \in M_p^{\lambda,0}(\Omega)$, $D_w^{\alpha-\beta}\zeta_i \in L_\infty(\Omega)$, by Theorem 2 $D_w^\beta u D_w^{\alpha-\beta}\zeta_i \in M_p^{\lambda,0}(\Omega)$. Therefore, $D_w^\alpha(u\zeta_i) \in M_p^{\lambda,0}(\Omega)$ for all $|\alpha| \le l$. Since $W_p^{l,\lambda,0}(\Omega)$ is a closed subspace of $W_p^{l,\lambda}(\Omega)$, we have $\zeta_i u \in W_p^{l,\lambda,0}(\mathbb{R}^n)$ and $\operatorname{supp}(\zeta_i u) \subset V_i$.

Fix $\delta > 0$. Choose then $\varepsilon_i > 0$ so small that $\phi_{\varepsilon_i} * (\zeta_i u)$ satisfies

$$\begin{cases} \|\phi_{\varepsilon_i} * (\zeta_i u) - \zeta_i u\|_{W_p^{l,\lambda}(\Omega)} \le \frac{\delta}{2^{i+1}}, & (i = 0, 1, \dots) \\ \operatorname{supp}[\phi_{\varepsilon_i} * (\zeta_i u)] \subset W_i & (i = 1, \dots), \end{cases}$$

for $W_i := \Omega_{i+4} - \bar{\Omega}_i \supset V_i$ $(i = 1, \dots)$.

Write $v := \sum_{i=0}^{\infty} \phi_{\varepsilon_i} * (\zeta_i u)$. This function belongs to $C^\infty(\Omega)$, since for each open set $V \subset\subset \Omega$ there are at most finitely many nonzero terms in the sum. Since $u = \sum_{i=0}^{\infty} \zeta_i u$, we have for each $V \subset\subset \Omega$

$$\|v - u\|_{W_p^{l,\lambda}(V)} \le \sum_{i=0}^{\infty} \|\phi_{\varepsilon_i} * (\zeta_i u) - \zeta_i u\|_{W_p^{l,\lambda}(\Omega)} \le \delta \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} = \delta.$$

Take the supremum over sets $V \subset\subset \Omega$, to conclude $\|v - u\|_{W_p^{l,\lambda}(\Omega)} \le \delta$.

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