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## Necessary and Sufficient Conditions for Oscillations of Functional Differential Equations

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In this survey, necessary and sufficient conditions for the oscillation of solutions of retarded, advanced and neutral differential equations of first and higher order with one or several constant coefficients and constant arguments, in terms of the characteristic equation, are presented. Explicit (in terms of the constant coefficient and constant argument only) necessary and sufficient conditions are also presented in the case of one argument. In the case of  $n$ th order equations necessary and sufficient conditions for the oscillation of all solutions are presented when  $n$  is odd, while necessary and sufficient conditions for the oscillation of all bounded solutions are presented when  $n$  is even. In this case explicit sufficient conditions for the oscillation of all solutions are presented when  $n$  is odd, while explicit sufficient conditions for the oscillation of all bounded solutions for retarded equations and of all unbounded solutions for advanced equations are presented when  $n$  is even. In the case of several arguments explicit but sufficient conditions only are given and the results are also extended to equations with several variable coefficients.

**Key words:** Oscillation; Delay, Necessary and sufficient conditions, Characteristic equation, Difference Equations.

## Необходимые и достаточные условия для колебательных функциональных дифференциальных уравнений

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В этой статье представлены необходимые и достаточные условия для колебаний всех решений запаздывающих, продвинутых и нейтральных дифференциальных уравнений первого и высшего порядка с одним или несколькими постоянными коэффициентами и постоянными аргументами в терминах характеристического уравнения. Явные (только по постоянному коэффициенту и постоянному аргументу) необходимые и достаточные условия также представлены в случае одного аргумента. В случае уравнения  $n$ -го порядка необходимые и достаточные условия для колебаний всех решений представлены когда  $n$  является нечетным, а необходимые и достаточные условия для колебаний всех граничных решений представлены когда  $n$  является четным. В этом случае явные достаточные условия для колебаний всех решений представлены когда  $n$  является нечетным, а явные достаточные условия для колебаний всех граничных решений для уравнений с запаздыванием и всех неграничных решений для продвинутых уравнений представлены когда  $n$  является нечетным. В случае нескольких аргументы явные, но достаточные условия даются, и результаты также распространяется на уравнения с несколькими переменными коэффициентами.

**Ключевые слова:** Колебание; запаздывание, необходимые и достаточные условия, характеристическое уравнение, разностные уравнения.

## 1 Introduction

Consider the first-order linear functional differential equations with several deviating arguments of the retarded

$$x'(t) + \sum_{i=1}^n p_i x(t - \tau_i) = 0 \quad (1)$$

and the advanced type

$$x'(t) - \sum_{i=1}^n p_i x(t + \tau_i) = 0. \quad (1)'$$

In the special case that  $n = 1$  the above equations reduce to the following retarded

$$x'(t) + px(t - \tau) = 0 \quad (2)$$

and advanced differential equation

$$x'(t) - px(t + \tau) = 0. \quad (2)'$$

Equations of higher order and equations with variable coefficients are also studied. Several sufficient and necessary and sufficient conditions under which all solutions oscillate are presented.

As it is customary, a solution is said to be oscillatory if it has arbitrarily large zeros. Otherwise it is called non-oscillatory and in this case it is eventually positive or eventually negative. Solutions are assumed to be defined for all  $t \geq 0$ .

The oscillation theory of differential equations was initiated by Sturm [26] in 1836. Since then many papers have been published on the subject. See, for example, the references [1-26] and the papers cited therein. For the general theory of delay equations the reader is referred to the monographs [8,9,6,4].

## 2 Literature review

The oscillation theory of Ordinary Differential Equations (ODEs) was originated by Sturm in 1836. Since then hundreds of papers have been published studying the oscillation theory of ODEs.

The oscillation theory of Delay Differential Equations (DDEs) was mainly developed after the 2nd world war. It was during the war that the admirals and officers in Navy (Fleet) observed that the ships were vibrating and asked the engineers and the scientists to solve the problem. Investigating the problem of vibrations (oscillations) the scientists found out that the equation which was to be taken into consideration was not an ODE (a usual equation without delays) but it was a differential equation with delays.

In the decade of 1970 a great number of papers were written extending known results from ODEs to DDEs. Of particular importance, however, has been the study of oscillations which are caused by the delay and which do not appear in the corresponding ODE. In recent years there has been a great deal of interest in the study of oscillatory behavior of the solutions to DDEs and also the discrete analogue Delay Difference Equations (DΔEs). See, for example, [1-26] and the references cited therein.

The problem of establishing sufficient conditions for the oscillation of all solutions to the differential equation

$$x'(t) + p(t)x(\tau(t)) = 0, \quad t \geq t_0, \quad (1)$$

where the functions  $p, \tau \in C([t_0, \infty), \mathbb{R}^+)$  (here  $\mathbb{R}^+ = [0, \infty)$ ),  $\tau(t)$  is non-decreasing,  $\tau(t) < t$  for  $t \geq t_0$ , and  $\lim_{t \rightarrow \infty} \tau(t) = \infty$ , has been the subject of many investigations. See, for example, [4-6, 8-12, 14-17, 19, 21, 22, 24] and the references cited therein.

By a solution of Eq. (1) we understand a continuously differentiable function defined on  $[\tau(T_0), \infty)$  for some  $T_0 \geq t_0$  and such that (1) is satisfied for  $t \geq T_0$ . Such a solution is called *oscillatory* if it has arbitrarily large zeros, and otherwise it is called *nonoscillatory*.

The oscillation theory of the (discrete analogue) delay difference equation

$$\Delta x(n) + p(n)x(\tau(n)) = 0, \quad n = 0, 1, 2, \dots, \quad (1)'$$

where  $p(n)$  is a sequence of nonnegative real numbers and  $\tau(n)$  is a sequence of integers such that  $\tau(n) < n - 1$  for  $n \geq 0$  and  $\lim_{n \rightarrow \infty} \tau(n) = \infty$ , has also attracted growing attention in the recent few years. The reader is referred to [1-3, 7, 13, 18, 20, 23, 25] and the references cited therein.

By a solution of Eq. (1)' we mean a sequence  $x(n)$  which satisfies (1)' for  $n \geq 0$ . A solution  $x(n)$  of (1)' is said to be *oscillatory* if the terms of the solution are not eventually positive or eventually negative. Otherwise the solution is called *nonoscillatory*.

## 3 Materials and methods

### 3.1 Necessary and sufficient conditions

In this section we present necessary and sufficient conditions under which all solutions of the equations under consideration oscillate.

### 3.1.1 First-order Equations

Consider the first-order linear retarded differential equation (1) with constant coefficients. In the following theorem a necessary and sufficient condition for the oscillation of all solutions of (1) in terms of the characteristic equation associated with (1) is given.

**Theorem 1. ([15])** Consider the equation

$$x'(t) + \sum_{i=1}^n p_i x(t - \tau_i) = 0 \quad (1)$$

where the coefficients  $p_i$  are real numbers and the delays  $\tau_i$  are non-negative real numbers. Then all solutions of (1) oscillate if and only if its characteristic equation

$$\lambda + \sum_{i=1}^n p_i e^{-\lambda \tau_i} = 0 \quad (3)$$

has no real roots.

In the special case of Eq. (2) and (2)', we have the following theorem.

**Theorem 2. ([17,8])** Consider the equation with one constant coefficient and one constant delay

$$x'(t) + px(t - \tau) = 0, \quad (2)$$

where  $p, \tau$  are real numbers. Then all solutions of (2) oscillate if and only if its characteristic equation

$$\lambda + pe^{-\lambda \tau} = 0 \quad (4)$$

has no real roots.

Consider now the first-order neutral differential equation

$$\frac{d}{dt}[x(t) + px(t - \tau)] + qx(t - \sigma) = 0. \quad (5)$$

The following theorem holds.

**Theorem 3. ([24,8])** Consider Eq.(5), where  $p, q, \tau$  and  $\sigma$  are real numbers. Then all solutions of Eq. (5) oscillate if and only if its characteristic equation

$$\lambda + p\lambda e^{-\lambda \tau} + qe^{-\lambda \sigma} = 0 \quad (6)$$

has no real roots.

In the general case of the first-order neutral differential equation with several coefficients we have the following.

**Theorem 4. ([7])** Consider the neutral differential equation

$$\frac{d}{dt}[x(t) + \sum_{i=1}^n p_i x(t - \tau_i)] + \sum_{i=1}^n q_i x(t - \sigma_i) = 0. \quad (7)$$

where  $p_i, q_i, \tau_i$  and  $\sigma_i$  are real numbers. Then all solutions of Eq.(7) oscillate if and only if the characteristic equation associated with (7)

$$\lambda + \lambda \sum_{i=1}^n p_i e^{-\lambda \tau_i} + \sum_{i=1}^n q_i e^{-\lambda \sigma_i} = 0 \quad (8)$$

has no real roots.

### 3.1.2 Higher-order Equations

Consider now the  $n$ th-order delay equation

$$x^{(n)}(t) + (-1)^{n+1} p x(t - \tau) = 0, \quad p, \tau > 0; \quad n \geq 1. \quad (9)$$

In this case the characteristic equation of Eq. (9) is

$$\lambda^n + (-1)^{n+1} p e^{-\lambda \tau} = 0. \quad (10)$$

We have the following theorem.

**Theorem 5.** ([16]) For  $n$  odd [ $n$  even] the following statements are equivalent.

- (a) All solutions of Eq.(9) oscillate [All bounded solutions of Eq.(9) oscillate].
- (b) The characteristic equation (10) has no real roots [Eq.(10) has no real roots in  $(-\infty, 0]$ ].

In the general case of the  $n$ th-order differential equation with several coefficients of the form

$$x^{(n)}(t) + (-1)^{n+1} \sum_{i=1}^n p_i x(t - \tau_i) = 0, \quad p_i, \tau_i > 0 \text{ and } n \geq 1 \quad (11)$$

the characteristic equation of Eq.(11) is

$$\lambda^n + (-1)^{n+1} \sum_{i=1}^n p_i e^{-\lambda \tau_i} = 0. \quad (12)$$

and we have the following.

**Theorem 6.** ([16]) For  $n$  odd [ $n$  even] the following statements are equivalent.

- (a) All solutions of Eq.(11) oscillate [All bounded solutions of Eq.(11) oscillate].
- (b) The characteristic equation (12) has no real roots [Eq.(12) has no real roots in  $(-\infty, 0]$ ].

Consider now the general case of the  $n$ th-order neutral differential equation with several coefficients

$$\frac{d^n}{dt^n} [x(t) + \sum_{\mathcal{J}} p_i x(t - \tau_i)] + \sum_{\mathcal{K}} q_k x(t - \sigma_k) = 0, \quad n \geq 1 \quad (13)$$

where  $\mathcal{J}, \mathcal{K}$  are initial segments of natural numbers and  $p_i, \tau_i, q_k, \sigma_k \in R$  for  $i \in \mathcal{J}$  and  $k \in \mathcal{K}$ .

**Theorem 7.** ([2]) A necessary and sufficient condition for the oscillation of all solutions of Eq.(13) is that the characteristic equation associated with (13)

$$\lambda^n + \lambda^n \sum_{\mathcal{J}} p_i e^{-\lambda \tau_i} + \sum_{\mathcal{K}} q_k e^{-\lambda \sigma_k} = 0 \quad (14)$$

has no real roots.

### 3.2 Explicit Oscillation Conditions

In this section we present explicit (in terms of the coefficients and the arguments only) oscillation conditions. In the case of equations with one delay an explicit necessary and sufficient condition is also derived.

#### 3.2.1 First-order Equations

**Theorem 8.** ([19,1,10]) Consider the differential equation with several constant retarded arguments

$$x'(t) + \sum_{i=1}^n p_i x(t - \tau_i) = 0 \quad (1)$$

and the differential equation with several constant advanced arguments

$$x'(t) - \sum_{i=1}^n p_i x(t + \tau_i) = 0 \quad (1)'$$

where  $p_i$  and  $\tau_i$ ,  $i = 1, 2, \dots, n$  are positive constants. Then each one of the following conditions

- (i)  $p_i \tau_i > \frac{1}{e}$  for some  $i$ ,  $i = 1, 2, \dots, n$ ,
- (ii)  $(\sum_{i=1}^n p_i) \tau > \frac{1}{e}$ , where  $\tau = \min\{\tau_1, \tau_2, \dots, \tau_n\}$ ,
- (iii)  $\sum_{i=1}^n p_i \tau_i > \frac{1}{e}$ ,
- (iv)  $[\prod_{i=1}^n p_i] (\sum_{i=1}^n \tau_i) > \frac{1}{e}$ ,
- (v)  $\frac{1}{n} (\sum_{i=1}^n (p_i \tau_i)^{1/2})^2 > \frac{1}{e}$

implies that all solutions of (1) and (1)' oscillate.

**Remark 1.** ([17,19]) It is noteworthy to observe that when  $n = 1$ , that is, in the case of a differential equation with one deviating argument, each one of the conditions (i), (ii), (iii), (iv), (v) reduces to

$$p\tau > \frac{1}{e} \quad (15)$$

which is a *necessary and sufficient condition* for all solutions of the retarded

$$x'(t) + px(t - \tau) = 0, \quad p, \tau > 0, \quad (2)$$

and the advanced differential equation

$$x'(t) - px(t + \tau) = 0, \quad p, \tau > 0. \quad (2)'$$

to be oscillatory.

We present the proof of this fact in the case of Eq.(2). [The proof in the case of Eq.(2)' is similar].

**Proof.** The characteristic equation associated with Eq.(2) is

$$F(\lambda) \equiv \lambda + pe^{-\lambda\tau} = 0.$$

It is easy to compute the critical points of  $F(\lambda)$  and evaluate the extreme values. The first derivative  $F'(\lambda) = 1 - p\tau e^{-\lambda\tau}$  and therefore the critical point is  $\lambda_0 = \frac{1}{\tau} \ln(p\tau)$ . The second derivative  $F''(\lambda) = p\tau^2 e^{-\lambda\tau} > 0$ . Therefore at the critical point  $\lambda_0$  the function  $F(\lambda)$  has a minimum value  $F(\lambda_0) = \frac{\ln(p\tau)+1}{\tau}$ . The minimum value would be positive if and only if  $\ln(p\tau) + 1 > 0$ , that is, if and only if  $p\tau > \frac{1}{e}$ , which completes the proof.

Next we consider neutral differential equations of the retarded and advanced type as well as neutral equations of the mixed type and present explicit sufficient oscillation conditions.

**Theorem 9. ([25])** Consider the neutral differential equation with several constant retarded arguments

$$\frac{d}{dt}[x(t) + cx(t-r)] + \sum_{i=1}^k p_i x(t - \tau_i) = 0. \quad (16)$$

and the neutral equation with several constant advanced arguments

$$\frac{d}{dt}[x(t) + cx(t+r)] - \sum_{i=1}^k p_i x(t + \tau_i) = 0. \quad (16)'$$

and the neutral equations of mixed type

$$\frac{d}{dt}[x(t) + cx(t-r)] + \sum_{i=1}^k p_i x(t - \tau_i) + \sum_{j=1}^{\ell} q_j x(t + \sigma_j) = 0, \quad (17)$$

and

$$\frac{d}{dt}[x(t) + cx(t+r)] - \sum_{i=1}^k p_i x(t + \tau_i) - \sum_{j=1}^{\ell} q_j x(t - \sigma_j) = 0, \quad (17)'$$

where  $c \in R$ ,  $r \in (0, \infty)$ ,  $p_i, q_j \in (0, \infty)$  and  $\tau_i, \sigma_j \in [0, \infty)$  for  $i = 1, 2, \dots, k$ ;  $j = 1, 2, \dots, \ell$ . Then in any of the following cases all solutions of the equations (16), (16)', (17) and (17)' oscillate:

- (i)  $c = -1$
- (ii)  $-1 < c$ ,  $r < \tau_1$  or  $c < -1$ ,  $r > \tau_k$  and furthermore

$$\frac{1}{1+c} \sum_{i=1}^k p_i (\tau_i - r) > \frac{1}{e}$$

or

$$\frac{1}{1+c} \left[ \prod_{i=1}^k p_i \right]^{1/k} \left( \sum_{i=1}^k p_i (\tau_i - r) \right) > \frac{1}{e}$$

is satisfied;

(iii)  $-1 < c < 0$  and

$$\sum_{i=1}^k p_i \tau_i > \frac{1}{e}$$

or

$$\left[ \prod_{i=1}^k p_i \right]^{1/k} \left( \sum_{i=1}^k \tau_i \right) > \frac{1}{e}$$

is satisfied.

### 3.2.2 Higher-order equations and inequalities

Consider the  $n$ -th order delay differential inequalities

$$x^{(n)}(t) + (-1)^{n+1} p^n x(t - n\tau) \leq 0, \quad (\text{I})$$

and

$$x^{(n)}(t) + (-1)^{n+1} p^n x(t - n\tau) \geq 0, \quad (\text{II})$$

and the delay differential equation

$$x^{(n)}(t) + (-1)^{n+1} p^n x(t - n\tau) = 0, \quad (\text{III})$$

where  $p, \tau > 0$  and  $n \geq 1$ . A *necessary and sufficient condition* for the behavior of the solutions to the above inequalities and equation is given in the following theorem.

**Theorem 10.** ([18]) The condition

$$p\tau > \frac{1}{e} \quad (15)$$

is necessary and sufficient so that:

(i) When  $n$  is odd: (I) has no eventually positive solutions, (II) has no eventually negative solutions, and (III) has only oscillatory solutions.

(ii) When  $n$  is even: (I) has no eventually negative bounded solutions, (II) has no eventually positive bounded solutions, and every bounded solution of (III) is oscillatory.

In the general case of the  $n$ th-order ( $n \geq 1$ ) differential equation with several retarded arguments

$$x^{(n)}(t) + (-1)^{n+1} \sum_{i=1}^k p_i^n x(t - n\tau_i) = 0, \quad (18)$$

and the  $n$ th-order differential equation with several advanced arguments

$$x^{(n)}(t) - \sum_{i=1}^k p_i^n x(t + n\tau_i) = 0, \quad (18)'$$

where  $p_i, \tau_i > 0, i = 1, 2, \dots, k$ , we present the following sufficient oscillation conditions,

**Theorem 11. ([20])** Each one of the following conditions

(i)  $p_i \tau_i > \frac{1}{e}$  for some  $i, i = 1, 2, \dots, k$ ,

(ii)  $\left(\sum_{i=1}^k p_i^n\right)^{1/n} \tau > \frac{1}{e}$ , where  $\tau = \min\{\tau_1, \tau_2, \dots, \tau_k\}$ ,

implies:

(a) For  $n$  odd, every solution of (18) and (18)' oscillates.

(b) For  $n$  even every bounded solution of (18) and every unbounded solution of (18)' oscillates.

### 3.2.3 First-order equations with variable coefficients

In this section we present a generalization of the results of Theorem 8 to differential equations with several variable coefficients of the retarded

$$x'(t) + \sum_{i=1}^n p_i(t)x(t - \tau_i) = 0 \quad (19)$$

and the advanced type

$$x'(t) - \sum_{i=1}^n p_i(t)x(t + \tau_i) = 0 \quad (19)'$$

where  $\tau_i, i = 1, 2, \dots, n$  are positive constants and  $p_i(t), i = 1, 2, \dots, n$  are positive and continuous functions.

**Theorem 12. ([19])** Consider the differential equations (19) [(19)'] and assume that

$$\liminf_{t \rightarrow \infty} \int_{t-(\tau_i/2)}^t p(s)ds > 0, \quad \left[ \liminf_{t \rightarrow \infty} \int_t^{t+(\tau_i/2)} p(s)ds > 0 \right], \quad i = 1, 2, \dots, n.$$

Then each one of the following conditions

$$\liminf_{t \rightarrow \infty} \int_{t-\tau_i}^t p_i(s)ds > \frac{1}{e}, \quad \left[ \liminf_{t \rightarrow \infty} \int_t^{t+\tau_i} p_i(s)ds > \frac{1}{e} \right], \quad \text{for some } i, i = 1, 2, \dots, n,$$

$$\liminf_{t \rightarrow \infty} \int_{t-\tau}^t \sum_{i=1}^n p_i(s)ds > \frac{1}{e} \quad \left[ \liminf_{t \rightarrow \infty} \int_t^{t+\tau} \sum_{i=1}^n p_i(s)ds > \frac{1}{e} \right], \quad \text{where } \tau = \min\{\tau_1, \dots, \tau_n\},$$

$$\left[ \prod_{i=1}^k \left( \sum_{j=1}^n \liminf_{t \rightarrow \infty} \int_{t-\tau_j}^t p_i(s)ds \right) \right]^{\frac{1}{n}} > \frac{1}{e} \quad \left[ \prod_{i=1}^k \left( \sum_{j=1}^n \liminf_{t \rightarrow \infty} \int_t^{t+\tau_j} p_i(s)ds \right) \right]^{\frac{1}{n}} > \frac{1}{e}$$

or

$$\frac{1}{n} \sum_{i=1}^n \left( \liminf_{t \rightarrow \infty} \int_{t-\tau_i}^t p_i(s)ds \right) + \frac{2}{n} \sum_{i < j, i, j=1}^n \left[ \left( \liminf_{t \rightarrow \infty} \int_{t-\tau_j}^t p_i(s)ds \right) \times \left( \liminf_{t \rightarrow \infty} \int_{t-\tau_i}^t p_j(s)ds \right) \right]^{1/2} > \frac{1}{e}$$

$$\left[ \begin{array}{c} \frac{1}{n} \sum_{i=1}^n \left( \liminf_{t \rightarrow \infty} \int_t^{t+\tau_i} p_i(s) ds \right) + \\ \frac{2}{n} \sum_{i < j, i, j=1}^n \left[ \left( \liminf_{t \rightarrow \infty} \int_t^{t+\tau_j} p_i(s) ds \right) \times \left( \liminf_{t \rightarrow \infty} \int_t^{t+\tau_i} p_j(s) ds \right) \right]^{1/2} > \frac{1}{e} \end{array} \right]$$

implies that every solution of (19) [(19)'] oscillates.

Next we present a further extension to the following differential equation with variable delay of the form

$$x'(t) + \sum_{i=1}^n p_i(t)x(t - \tau_i(t)) = 0 \quad (20)$$

where  $\tau_i$  are continuous and positive valued on  $[0, \infty)$ .

**Theorem 13.** ([10]) If there is a uniform upper bound  $\tau_0$  on the  $\tau_i$ 's and

$$\liminf_{t \rightarrow \infty} \sum_{i=1}^n p_i(t)\tau_i(t) > \frac{1}{e}$$

then all solutions of Eq.(20) oscillate.

## 4 Results and Discussion

In this survey paper we present necessary and sufficient conditions for the oscillation of solutions of retarded, advanced and neutral differential equations of first and higher order with one or several constant coefficients and constant arguments, in terms of the characteristic equation. Explicit (in terms of the constant coefficient and constant argument only) necessary and sufficient conditions are also presented in the case of one argument only. In the case of  $n$ th order equations necessary and sufficient conditions for the oscillation of all solutions are presented when  $n$  is odd, while necessary and sufficient conditions for the oscillation of all bounded solutions only are presented when  $n$  is even. In this case explicit sufficient conditions for the oscillation of all solutions are presented when  $n$  is odd, while explicit sufficient conditions for the oscillation of all bounded solutions for retarded equations and of all unbounded solutions for advanced equations are presented when  $n$  is even. It is to be pointed out that in the case of several arguments explicit but sufficient conditions only are given. The results are also extended to equations with several variable coefficients where sufficient conditions only are given.

## 5 Conclusion

We conclude that necessary and sufficient conditions for oscillation of all solutions have been given in the case of differential equations with several constant coefficients and constant arguments in terms of the characteristic equation only. While explicit (in terms of the constant coefficient and constant argument only) necessary and sufficient conditions have been given in the case of one argument only. It is to be pointed out that in the case of  $n$ th order equations necessary and sufficient conditions for the oscillation of all solutions are presented when  $n$  is odd, while necessary and sufficient conditions for the oscillation of all bounded solutions

only are presented when  $n$  is even. In this case of  $n$ th order equations explicit sufficient conditions for the oscillation of all solutions are presented when  $n$  is odd, while explicit sufficient conditions for the oscillation of all bounded solutions for retarded equations and of all unbounded solutions for advanced equations are presented when  $n$  is even. Furthermore, in the case of several arguments explicit but sufficient conditions only are given.

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