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On a boundary value problem for the nonhomogeneous heat equation in an angular domain

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Due to the fact that the results find theoretical and practical applications, great attention is paid to the study of boundary value problems for parabolic equations. Also the relevance of studying such problems is justified by their physical application in the modeling of such processes as the propagation of heat in homogeneous and nonhomogeneous media, the interaction of filtration and channel flows, and other. Therefore, at the present stage of its development, the theory of partial differential equations is one of the important branches of mathematics and is actively developed by various mathematical schools. However, a number of significant problems in the theory of partial differential equations remain, as before, unresolved. In the paper we study a boundary value problem for the nonhomogeneous heat equation in an angular domain. Note that the problem does not have the initial condition. It is caused by the form of the domain. We obtain a boundary condition for the nonhomogeneous heat equation considered in the angular domain. It is proven that the heat potential is a unique classical solution to this problem.

Key words: Heat equation, Green's function, classical solution.

Об одной задаче для неоднородного уравнения теплопроводности в угловой области

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В силу того, что результаты находят теоретические и практические применения исследованию краевых задач для параболических уравнений уделяется огромное внимание. Также актуальность изучения таких задач обоснована их физическим применением в моделировании таких процессов как распространение тепла в однородных и неоднородных средах, взаимодействия фильтрационных и каналовых потоков и другие. Поэтому на сегодняшнем этапе своего развития теория дифференциальных уравнений в частных производных является одним из важных разделов математики и активно разрабатывается различными математическими школами. Однако ряд существенных проблем теории дифференциальных уравнений в частных производных остается по-прежнему не разрешенным. В нашей работе рассматривается граничная задача для неоднородного уравнения теплопроводности в угловой области. Стоит отметить, что поставленная нами граничная задача не имеет начального условия. Это обусловливается формой выбранной области. Нами получено граничное условие для неоднородного уравнения теплопроводности, рассматриваемого в угловой области. Доказан тот факт, что для правой части неоднородного уравнения теплопроводности принадлежащей выбранной нами угловой области, тепловой потенциал является единственным классическим решением данного неоднородного уравнения теплопроводности с найденным граничным условием.

Ключевые слова: Уравнение теплопроводности, функция Грина, классическое решение.

Үшбұрышты облыстағы біртекті емес жылуөткізгіштік теңдеуі үшін қойылған бір есеп туралы

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Алынған нәтижелер теориялық және практикалық қолданылымға ие болғандықтан параболалық теңдеулерге арналған шекаралық есептерді зерттеуге көп көңіл бөлінуде. Сонымен қатар, осындай есептерді зерттеудің өзектілігі олардың біртекті және біртекті емес ортадағы жылудың таралуы, фильтрациялы және каналды ағындардың өзара әсерлесуі сияқты және т.б. үдерістерді модельдеудегі физикалық қолданылысымен түсіндіріледі. Сондықтан дербес туындылы дифференциалдық теңдеулер теориясы бүгінгі таңдағы даму деңгейіне байланысты математика ғылымының маңыздыларының бірі болып табылып, әртүрлі математикалық мектептер оны қарқынды дамытуда. Алайда дербес туындылы дифференциалдық теңдеулер теориясының бірқатар мәселелері шешілмеген түрде қалып отыр. Біз жұмысымызда үшбұрышты облыстағы біртекті емес жылуөткізгіштік теңдеуі үшін қойылған шекаралық есебін қарастырамыз. Қойылған шекаралық есептің бастапқы шартының жоқ екендігін айта кеткен жөн. Бұл таңдалған облыстың пішініне байланысты. Үшбұрышты облыста қарастырылған біртекті емес жылуөткізгіштік теңдеуі үшін шекаралық шарт алынды. Жұмысымызда біртекті емес жылуөткізгіштік теңдеуінің оң жағындағы және біз таңдаған үшбұрышты облысына тиісті функциясы үшін жылу потенциалы қойылған біртекті емес жылуөткізгіштік теңдеуі үшін табылған шекаралық шарты бар есептің жалғыз классикалық шешімі болатындығы дәлелденді.
Түйін сөздер: Жылуөткізгіштік теңдеуі, Грин функциясы, классикалық шешім.

1 Introduction

For the first time, the boundary condition of the volume potential for an arbitrary bounded domain $\Omega \subset R^n$ with a sufficiently smooth boundary S was found in the article (Kal'menov, 2009,646-649) by T. Sh. Kal'menov and D. Suragan, i.e., in this article it was proven that for a given $f \in L_2(\Omega)$ the volume potential

$$u(x) = \int_\Omega \varepsilon(x,y) f(y) dy$$

is a unique solution to the equation

$$\Delta u(x) = f(x),\ x \in \Omega,$$

satisfying the nonlocal boundary condition

$$-\frac{u(x)}{2} + \int_S \left[\frac{\partial \varepsilon(x,y)}{\partial n_y} u(y) - \varepsilon(x,y) \frac{\partial u(y)}{\partial n_y} \right] dS = 0,\ x \in S,$$

where $\varepsilon(x,y)$ is the fundamental solution of the Laplacian and n is an outer normal. In the case of a two-dimensional disk or a three-dimensional ball, the eigenvalues and eigenfunctions of the volume potential, i.e., the eigenvalues and eigenfunctions of the spectral problem

$$\Delta u(x) = \lambda u(x),\ x \in \Omega,$$

with the nonlocal boundary condition

$$-\frac{u(x)}{2} + \int_S \left[\frac{\partial \varepsilon(x,y)}{\partial n_y} u(y) - \varepsilon(x,y) \frac{\partial u(y)}{\partial n_y} \right] dS = 0,\ x \in S,$$

are determined in (Kal'menov,2011,189). The corresponding problem is also discussed in (Kal'menov,2011,188).

2 Literature review

For the bounded simply-connected domain and the polyharmonic equation, the volume potential boundary conditions are obtained in (Kal'menov,2012a,604-608). We refer also to the article (Suragan,2013,141-149), where the same question is studied for the polyparabolic equation in cylindrical domain. To solve the nonhomogeneous Helmholtz equation in a bounded domain with sufficiently smooth boundary, the authors in (Kal'menov,2012b,164-1065) propose new boundary conditions possessing the property to suppress waves reflected from the boundary. It is demonstrated that in a bounded domain this solution coincides with that in an unbounded domain satisfying the Sommerfeld radiation condition. In (Kal'menov,2014,1-6) the authors considered an initial-boundary value problem for the one-dimensional wave equation and proved the uniqueness of the solution and showed that the solution coincides with the wave potential. In (Kal'menov,2013,1024) authors generalized the results of (Kal'menov, 2009,646-649) to the case of a parabolic operator in a noncylindrical domain and studied a nonlocal boundary value problem for a space-multidimensional parabolic equation in this domain. In the article (Kal'menov,2015,1062) authors studied the so-called permeable potential boundary conditions for the Laplace-Beltrami operator defined in a domain Ω on the unit sphere S in R^3. A model case of the problem of heat diffusion in a homogeneous body with a special initial state was considered in (Kal'menov,2016,126). In that paper we consider volume heat potential in an angular domain. The boundary condition are found for him. Herewith by reason of angular form of the domain the problem does not have initial conditions.

3 Material and methods

To obtain the main results in the work, we use both the previously invented methods for solving boundary value problems for the heat equation, and new methods, which will be described in more detail in the following sections.

4 Results and discussion

Let an angular domain $Q \equiv \{0 < x < t < 1\}$ be given. Consider the boundary problem:

$$u_t(x,t) - u_{xx}(x,t) = f(x,t), \quad (x,t) \in Q, \tag{1}$$

$$u(0,t) = 0, \tag{2}$$

$$-\frac{u(t,t)}{2} + \int_0^t \left(\varepsilon_{1_\xi}(t,\xi,t-s)\Big|_{\xi=s} - \varepsilon_1(t,s,t-s) \right) u(s,s)ds - \tag{3}$$

$$-\int_0^t \varepsilon_1(t,s,t-s) u_\xi(\xi,s)\Big|_{\xi=s} ds = 0,$$

where

$$\varepsilon_1(x,\xi,t-s) = \varepsilon(x-\xi,t-s) - \varepsilon(x+\xi,t-s),$$

and

$$\varepsilon(x,t) = \frac{\theta(t)}{2\sqrt{\pi t}} \exp\left\{-\frac{x^2}{4t}\right\}$$

is the fundamental solution to the Cauchy problem for the heat equation (Friedman,1964,3-347). Note that the problem (1)-(3) does not have the initial condition. It is caused by the form of the domain Q.

THEOREM 1 *The volume heat potential*

$$u(x,t) = \int_0^t ds \int_0^s \varepsilon_1(x,\xi,t-s) f(\xi,s) d\xi \tag{4}$$

is a unique classical solution to the boundary problem (1)-(3).

LEMMA 1 *For all* $u \in C^{2,1}(\overline{Q}), u|_{x=0} = 0$ *we have the following equality*

$$\int_0^t ds \int_0^s \varepsilon_1(x,\xi,t-s)\left(u_s - u_{\xi\xi}\right) d\xi = u(x,t) + \tag{5}$$

$$+ \int_0^t \varepsilon_{1_\xi}(x,\xi,t-s)\bigg|_{\xi=s} u(s,s)ds - \int_0^t \varepsilon_1(x,s,t-s)u(s,s)ds -$$

$$- \int_0^t \varepsilon_1(x,s,t-s)u_\xi(\xi,s)\bigg|_{\xi=s} ds, \;\; (x,t) \in Q.$$

PROOF. Since integral at the left side of (5) is an improper integral due to the singularity, we understood it to be as the limit

$$\lim_{\alpha\to 0} \int_0^{t-\alpha} ds \int_0^s \varepsilon_1(x,\xi,t-s)\left(u_s - u_{\xi\xi}\right) d\xi.$$

The last integral exists in the usual sense and we can make all actions including integrating by parts, for instance. Then for all $(x,t) \in Q$, we have

$$\int_0^t ds \int_0^s \varepsilon_1(x,\xi,t-s)\left(u_s - u_{\xi\xi}\right) d\xi =$$

$$= \lim_{\alpha\to 0} \int_0^{t-\alpha} \varepsilon_1(x,\xi,\alpha)u(\xi,t-\alpha)d\xi - \lim_{\alpha\to 0} \int_0^{t-\alpha} \varepsilon_1(x,\xi,t-\xi)u(\xi,\xi)d\xi -$$

$$- \lim_{\alpha\to 0} \int_0^{t-\alpha} ds \int_0^s \varepsilon_{1_s}(x,\xi,t-s)u(\xi,s)d\xi - \lim_{\alpha\to 0} \int_0^{t-\alpha} \varepsilon_1(x,s,t-s)\cdot$$

$$\cdot u_\xi(\xi,s)\bigg|_{\xi=s} ds + \lim_{\alpha\to 0} \int_0^{t-\alpha} \varepsilon_{1_\xi}(x,\xi,t-s)\bigg|_{\xi=s} u(s,s)ds +$$

$$+ \lim_{\alpha\to 0} \int_0^{t-\alpha} ds \int_0^s \varepsilon_{1_{\xi\xi}}(x,\xi,t-s)u(\xi,s)d\xi.$$

The properties of the fundamental solution ensure the equality (Friedman,1964,3-347)

$$\int_0^t ds \int_0^s \Diamond^+_{\xi,s} \varepsilon_1(x,\xi,t-s)u(\xi,s)d\xi = 0,$$

for $(x,t)\in Q$, where

$$\Diamond^{+}_{\xi,s}\varepsilon(x,\xi,t-s)=-\frac{\partial\varepsilon(x,\xi,t-s)}{\partial s}-\Delta_{\xi}\varepsilon(x,\xi,t-s)=0$$

and so

$$\int_0^t ds\int_0^s \varepsilon_1(x,\xi,t-s)\,(u_s-u_{\xi\xi})\,d\xi= \tag{6}$$

$$=\lim_{\alpha\to 0}\int_0^{t-\alpha}\varepsilon_1(x,\xi,\alpha)u(\xi,t-\alpha)d\xi-\lim_{\alpha\to 0}\int_0^{t-\alpha}\varepsilon_1(x,\xi,t-\xi)u(\xi,\xi)d\xi-$$

$$-\lim_{\alpha\to 0}\int_0^{t-\alpha}\varepsilon_1(x,s,t-s)u_{\xi}(\xi,s)\Big|_{\xi=s}ds+$$

$$+\lim_{\alpha\to 0}\int_0^{t-\alpha}\varepsilon_{1_\xi}(x,\xi,t-s)\Big|_{\xi=s}u(s,s)ds.$$

Since

$$\lim_{\alpha\to 0}\int_0^{t-\alpha}\varepsilon_1(x,\xi,\alpha)u(\xi,t-\alpha)d\xi=u(x,t)$$

in formula (6) the second integral converges absolutely, the third and forth integrals are the potentials of a double and a simple layer, respectively, and we have (5) for all $(x,t)\in Q$. The lemma is proved.

Proof. Proof of the theorem. Let $u(x,t)$ be represented in the form of (4). It's obvious that $u(x,t)$ is a solution to the equation (1) and satisfies the condition (2). Then from (5) we have

$$\int_0^t \varepsilon_{1_\xi}(x,\xi,t-s)\Big|_{\xi=s}u(s,s)ds-\int_0^t\varepsilon_1(x,s,t-s)u(s,s)ds- \tag{7}$$

$$-\int_0^t\varepsilon_1(x,s,t-s)u_{\xi}(\xi,s)\Big|_{\xi=s}ds=0.$$

Passing to the limit as $x\to t$ and recalling the properties of potentials (see [10]), we arrive at (3). Thus, the volume heat potential (4) meets the boundary condition (3).
Conversely, let us show that the solution to the problem (1)-(3) is unique. Suppose there are two solutions $u_1,\ u_2\in C^{2,1}(\overline{Q})$. Let $u=u_1-u_2$. It is known that u is the solution to the homogeneous equation $u_t-u_{xx}=0$ and satisfies the condition (2). Then from (5) we infer

$$\frac{u(t,t)}{2}+\int_0^t\varepsilon_{1_\xi}(t,\xi,t-s)\Big|_{\xi=s}u(s,s)ds-$$

$$-\int_0^t\varepsilon_1(t,s,t-s)u(s,s)ds-\int_0^t\varepsilon_1(t,s,t-s)u_{\xi}(\xi,s)\Big|_{\xi=s}ds=0.$$

Hence and from (3), we obtain $u(t,t)=0, 0\le t\le 1$. Since $u\in C^{2,1}(\overline{Q})$, then by uniqueness of the solution of the problem with the Dirichlet conditions (Amangalieva,2015,981-995) $u(x,t)=0$ in Q. The theorem is proved.

5 Conclusion

In the paper we obtain a boundary condition for the nonhomogeneous heat equation considered in the angular domain. It is proven that the heat potential is a unique classical solution to this problem. The authors express their gratitude to T.Sh. Kalmenov and B.E. Kanguzhin and to all participants of the Citywide Scientific Seminar "Differential Operators and Their Applications" for the beneficial discussion of the results. This research is financially supported by a grant from the Ministry of Science and Education of the Republic of Kazakhstan, (Grant No. 0824/GF4).

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