1-бөлім	Раздел 1	Section 1
Математика	Математика	Mathematics

A Survey on the Oscillation of Difference Equations with Constant Delays

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In this survey, necessary and sufficient conditions for the oscillation of all solutions of delay difference equations with one or several constant arguments, in terms of the characteristic equation, are presented. Explicit necessary and sufficient conditions (in terms of the constant coefficient and constant argument only) are also presented in the case of one constant argument. In the case of several arguments explicit but sufficient conditions only are given. In this case the results are also extended to equations with variable coefficients.

Key words: Oscillation, Delay, Difference Equations.

Исследование колебаний разностных уравнений с постоянными запаздываниями

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В этом обзоре представлены необходимые и достаточные условия для колебания всех решений дифференциальных уравнений с запаздыванием с одной или несколькими постоянными аргументами, в терминах характеристического уравнения. Явные необходимые и достаточные условия (в терминах постоянного коэффициента и одного постоянного аргумент) также представлены в случае одного постоянного аргумент. В случае нескольких аргументов даются явные, но только достаточные условия. В этом случае результаты также распространяются на уравнения с переменными коэффициентами.

Ключевые слова: Колебание, запаздывание, разностные уравнения.

1 Introduction

Consider the first-order linear difference equation with several delay arguments of the form

$$\Delta x(n) + \sum_{i=1}^{m} p_i x(n - k_i) = 0, \quad n \geq 0, \tag{1}$$

and the special case $(m = 1)$ of the above equation

$$\Delta x(n) + p x(n - k) = 0, \quad n \geq 0, \tag{2}$$

where Δ denotes the forward difference operator, i.e. $\Delta x(n) = x(n + 1) - x(n)$, and for $1 \leq i \leq m$, k_i are nonnegative integers and p_i are real numbers.

By a *solution* of the difference equation (1), we mean a sequence of real numbers $\{x(n)\}_{n=-k_i}^{\infty}$ which satisfies (1) for all $n \geq 0$. (Analogously for Eq. (2)).

A solution $\{x(n)\}_{n=-k_i}^{\infty}$ of the difference equation (1) is said to be *oscillatory,* if the terms of the sequence $\{x(n)\}_{n=-k_i}^{\infty}$ are neither eventually positive nor eventually negative. Otherwise, the solution $\{x(n)\}_{n=-k_i}^{\infty}$ is said to be *nonoscillatory.* (Analogously for Eq.(2)).

In the last few decades, the oscillatory behavior of the solutions to difference equations has been extensively studied. See, for example, [4−8, 11-13,17,19−25] and the references cited therein. For the general theory of difference equations the reader is referred to the monographs [1,2,9,16].

2 Necessary and sufficient conditions

In this section we present necessary and sufficient conditions under which all solutions of the equations under consideration oscillate.

Consider the linear delay difference equation (1) with constant coefficients. In the following theorem a necessary and sufficient condition for the oscillation of all solutions of (1) in terms of the characteristic equation associated with (1) is given.

Theorem 1. ([9]) Consider the difference equation

$$\Delta x(n) + \sum_{i=1}^{m} p_i x(n - k_i) = 0, \quad n \geq 0, \tag{1}$$

where the coefficients p_i are real numbers and the delays k_i are non-negative integers. Then all solutions of (1) oscillate if and only if its characteristic equation

$$\lambda - 1 + \sum_{i=1}^{m} p_i \lambda^{-k_i} = 0 \tag{3}$$

has no positive roots.

In the special case of Eq.(2), we have the following theorem.

Theorem 2.([9]) Consider the difference equation with one constant coefficient and one constant delay

$$\Delta x(n) + px(n - k) = 0, \quad n \geq 0, \tag{2}$$

where p is a real number and k is a non-negative integer. Then the following statements are equivalent.

(i) All solutions of Eq.(2) oscillate.
(ii) The characteristic equation

$$\lambda - 1 + p\lambda^{-k} = 0 \tag{4}$$

has no positive roots.

3 Explicit Oscillation Conditions

In this section we present explicit (in terms of the coefficients and the arguments only) oscillation conditions. In the case of equations with one delay an explicit necessary and sufficient condition is also presented.

3.0.1 Difference equations with constant coefficients

Theorem 3. ([9]) Consider the difference equation with several constant coefficients and retarded arguments

$$\Delta x(n) + \sum_{i=1}^{m} p_i x(n - k_i) = 0, \quad n \geq 0, \tag{1}$$

where p_i, are positive constants and k_i are non-negative integers for $i = 1, 2, ..., m$. Then the following condition

$$\sum_{i=1}^{m} p_i(k_i + 1) > \left(\frac{k_i}{k_i + 1} \right)^{k_i} \tag{5}$$

implies that all solutions of Eq.(1).

For the delay differential equation

$$x'(t) + \sum_{i=1}^{m} p_i x(t - \tau_i) = 0 \tag{1'}$$

where p_i, τ_i are positive constants for $i = 1, 2, ..., m$, it is known [15,3,10,18] that every solution oscillates if

$$\sum_{i=1}^{m} p_i \tau_i > \frac{1}{e}. \tag{5'}$$

Observe that

$$\left(\frac{k_i}{k_i + 1} \right)^{k_i} = \left(\frac{1}{1 + \frac{1}{k_i}} \right)^{k_i} \downarrow \frac{1}{e} \quad \text{as} \quad k_i \to \infty,$$

and therefore condition (5) can be interpreted as the discrete analogue of $(5)'$.

Remark 1. ([9]) It is noteworthy to observe that when $m = 1$, that is, in the case of a difference equation with one delay argument, condition (5) reduces to

$$p(k + 1) > \left(\frac{k}{k + 1} \right)^{k} \tag{6}$$

which is a *necessary and sufficient condition* for all solutions of the delay difference equation

$$\Delta x(n) + p x(n - k) = 0, \quad n \geq 0, \tag{2}$$

to be oscillatory.

We present the proof of this fact.

Proof. The characteristic equation associated with Eq.(2) is

$$F(\lambda) = \lambda - 1 + p\lambda^{-k} = 0.$$

It is easy to compute the critical points of $F(\lambda)$ and evaluate the extreme values. The first derivative $F'(\lambda) = 1 - pk\lambda^{-k-1}$ and the only critical point of $F(\lambda)$ in $(0,\infty)$ is $\lambda_0 = (pk)^{\frac{1}{k+1}}$. The second derivative

$$F''(\lambda) = pk(k + 1)\lambda^{-(k+1)} > 0 \quad \text{for} \quad \lambda > 0.$$

Therefore at the critical point λ_0 the function $F(\lambda)$ has a minimum value

$$F(\lambda_0) = \lambda_0 - 1 + p\lambda_0^{-k} = \lambda_0 \left[1 - \frac{1}{\lambda_0} + \frac{1}{k}\right] = \lambda_0 \left[\frac{k+1}{k} - \frac{1}{\lambda_0}\right].$$

The minimum value $F(\lambda_0)$ would be positive if and only if $\lambda_0 > \frac{k}{k+1}$ that is, if and only if $pk = \lambda_0^{k+1} > \left(\frac{k}{k+1}\right)^{k+1}$ if and only if

$$p(k+1) > \left(\frac{k}{k+1}\right)^k$$

which completes the proof.

It is also known [14,9] that

$$p\tau > \frac{1}{e} \tag{6$'$}$$

is a *necessary and sufficient condition* for all solutions of the delay differential equation

$$x'(t) + px(t - \tau) = 0, \quad p, \tau > 0, \tag{2$'$}$$

to be oscillatory. As before, observe that

$$\left(\frac{k}{k+1}\right)^k = \left(\frac{1}{1 + \frac{1}{k}}\right)^k \downarrow \frac{1}{e} \quad \text{as} \quad k \to \infty,$$

and therefore condition (6) can be interpreted as the discrete analogue of (6)$'$.

3.0.2 Difference equations with one variable coefficient

Here we present explicit oscillation conditions for difference equations with one variable co-efficient.

Consider the difference equation

$$\Delta x(n) + p(n)x(n-k) = 0, \quad n \geq 0, \tag{7}$$

where $\{p(n)\}_{n=0}^{\infty}$ is a nonnegative sequence of reals and k is a nonnegative integer.

In 1981, Domshlak [7] considered the case where $k = 1$. In 1989, Erbe and Zhang [8] proved that all solutions of (7) oscillate if

$$\beta := \liminf_{n \to \infty} p(n) > 0 \quad and \quad \limsup_{n \to \infty} p(n) > 1 - \beta \tag{8}$$

or

$$\liminf_{n \to \infty} p(n) > \frac{k^k}{(k+1)^{k+1}} \tag{9}$$

or

$$A := \limsup_{n \to \infty} \sum_{i=n-k}^{n} p(i) > 1. \tag{10}$$

while Ladas, Philos and Sficas [13] improved the above condition (9) as follows

$$\alpha := \liminf_{n \to \infty} \sum_{i=n-k}^{n-1} p(i) > \left(\frac{k}{k+1}\right)^{k+1}. \tag{11}$$

Note that this condition is sharp in the sense that the fraction on the right hand side can not be improved, since when $p(n)$ is a constant, say $p(n) = p$, then this condition reduces to

$$p > \frac{k^k}{(k+1)^{k+1}},$$

which is a *necessary and sufficient condition* for the oscillation of all solutions to Eq.(2). Moreover, concerning the constant $\frac{k^k}{(k+1)^{k+1}}$ in (9), it should be emphasized that, as it is shown in [8], if

$$\sup p(n) < \frac{k^k}{(k+1)^{k+1}}, \tag{N_1}$$

then (7) has a nonoscillatory solution.

In 1990, Ladas [12] conjectured that Eq. (7) has a nonoscillatory solution if

$$\frac{1}{k} \sum_{i=n-k}^{n-1} p(i) \leq \frac{k^k}{(k+1)^{k+1}}$$

holds eventually. However this conjecture is not correct and a counter-example was given in 1994 by Yu, Zhang and Wang [25]. Moreover, in 1999 Tang and Yu [23], using a different technique, showed that Eq.(7) has a nonoscillatory solution if the so-called "corrected Ladas conjecture"

$$\sum_{i=n-k}^{n} p(i) \leq \left(\frac{k}{k+1}\right)^{k+1} \quad \text{for all large } n, \tag{N_2}$$

is satisfied.

In 2017 Karpuz [11] studied this problem and derived the following conditions. If

$$\liminf_{n \to \infty} \inf_{\lambda \geq 1} \left[\frac{1^n}{\lambda_{i=n-k}} [1 + \lambda p(i)] \right] > 1,$$

then every solution of Eq.(7) oscillates, while if there exists $\lambda_0 \geq 1$ such that

$$\frac{1^n}{\lambda_{0i=n-k}} [1 + \lambda_0 p(i)] \leq 1 \quad \text{for all large } n,$$

then Eq.(7) has a nonoscillatory solution. From the above conditions, using the Arithmetic-Geometric mean, it follows that if

$$\sum_{i=n-k}^{n} p(i) \leq \left(\frac{k}{k+1}\right)^{k} \quad \text{for all large } n, \tag{N_3}$$

then all solutions of Eq. (17) oscillate. That is, Karpuz [11] replaced condition (N_2) by (N_3), which is a weaker condition.

It is interesting to establish sufficient conditions for the oscillation of all solutions to Eq.(7) when both (10) and (11) are not satisfied.

Stavroulakis [20] established the following.

Theorem 4.([20]) *Assume that*

$$0 < \alpha \le \left(\frac{k}{k+1}\right)^{k+1}$$

and

$$\limsup_{n\to\infty} p(n) > 1 - \frac{\alpha^2}{4}, \tag{12}$$

that all solutions of (7) *oscillate.*

Then, Stavroulakis [21] and Chatzarakis and Stavroulakis [5] improved the above result as follows.

Theorem 5.([21,5]) *Assume that* $0 < \alpha \le \left(\frac{k}{k+1}\right)^{k+1}$. *Then either one of the conditions*

$$\limsup_{n\to\infty} \sum_{i=n-k}^{n-1} p(i) > 1 - \frac{\alpha^2}{4}, \tag{13}$$

$$\limsup_{n\to\infty} \sum_{i=n-k}^{n-1} p(i) > 1 - \alpha^k \tag{14}$$

or

$$\limsup_{n\to\infty} \sum_{i=n-k}^{n-1} p(i) > 1 - \frac{\alpha^2}{2(2-\alpha)} \tag{15}$$

implies that all solutions of (7) *oscillate.*

Also Chen and Yu [6], following the above mentioned direction, derived the following oscillation condition

$$A > 1 - \frac{1 - \alpha - \sqrt{1 - 2\alpha - \alpha^2}}{2}. \tag{16}$$

In 2000, Shen and Stavroulakis [19], using new techniques, improved the previous results as follows.

Theorem 6. ([19]) *Assume that* $0 \le \alpha \le k^{k+1}/(k+1)^{k+1}$ *and that there exists an integer* $l \ge 1$ *such that*

$$\limsup_{n\to\infty} \left\{ \begin{array}{l} \sum_{i=1}^{k} p(n-i) + [\overline{d}(\alpha)]^{-k} \prod_{i=1}^{k} \sum_{j=1}^{k} p(n-i+j) + \\ \sum_{m=0}^{l-1} [d(\alpha/k)]^{-(m+1)k} \sum_{i=1}^{k} \prod_{j=0}^{m+1} p(n-kj+i) \end{array} \right\} > 1, \tag{17}$$

where $\overline{d}(\alpha)$ *and* $d(\alpha/k)$ *are the greater real roots of the equations*

$$d^{k+1} - d^k + \alpha^k = 0$$

and

$$d^{k+1} - d^k + \alpha/k = 0,$$

respectively. Then all solutions of (7) *oscillate.*

Notice that when $k = 1$, $d(\alpha) = \overline{d}(\alpha) = (1 + \sqrt{1 - 4\alpha})/2$ (see [19]), and so condition (C_{10}) reduces to

$$\limsup_{n \to \infty} \left\{ Cp(n) + p(n-1) + \sum_{m=0}^{l-1} C^{m+1} \prod_{j=0}^{m+1} p(n-j-1) \right\} > 1, \qquad (18)$$

where $C = 2/(1 + \sqrt{1 - 4\alpha})$, $\alpha = \liminf_{n \to \infty} p_n$. Therefore, from Theorem 6, we have the following corollary.

Corollary 1. ([19]) *Assume that* $0 \leq \alpha \leq 1/4$ *and that* (18) *holds. Then all solutions of the equation*

$$x(n+1) - x(n) + p(n)x(n-1) = 0 \qquad (19)$$

oscillate.

A condition derived from (18), which can be easier verified, is given in the next corollary.

Corollary 2. ([19]) *Assume that* $0 \leq \alpha \leq 1/4$ *and that*

$$\limsup_{n \to \infty} p(n) > \left(\frac{1 + \sqrt{1 - 4\alpha}}{2} \right)^2. \qquad (20)$$

Then all solutions of (19) *oscillate.*

Remark 2. ([19]) Observe that when $\alpha = 1/4$, condition (20) reduces to

$$\limsup_{n \to \infty} p(n) > 1/4$$

which can not be improved in the sense that the lower bound $1/4$ can not be replaced by a smaller number. Indeed, by condition (N_1) (Theorem 2.3 in [8]), we see that (19) has a nonoscillatory solution if

$$\sup p(n) < 1/4.$$

Note, however, that even in the critical state where

$$\lim_{n \to \infty} p(n) = 1/4,$$

(19) can be either oscillatory or nonoscillatory. For example, if $p(n) = \frac{1}{4} + \frac{c}{n^2}$ then (19) will be oscillatory in case $c > 1/4$ and nonoscillatory in case $c < 1/4$ (the Kneser-like theorem, [7]).

Example 1. ([19]) Consider the equation

$$x(n-1) - x(n) + \left(\frac{1}{4} + a \sin^4 \frac{n\pi}{8} \right) x(n-1) = 0,$$

where $a > 0$ is a constant. It is easy to see that

$$\liminf_{n\to\infty} p(n) = \liminf_{n\to\infty} \left(\frac{1}{4} + a \sin^4 \frac{n\pi}{8} \right) = \frac{1}{4},$$

$$\limsup_{n\to\infty} p(n) = \limsup_{n\to\infty} \left(\frac{1}{4} + a \sin^4 \frac{n\pi}{8} \right) = \frac{1}{4} + a.$$

Therefore, by Corollary 2, all solutions oscillate. However, none of the conditions $(8) - (16)$ is satisfied.

3.0.3 Difference equations with several variable coefficients

In this subsection we present explicit oscillation conditions for difference equations with several variable coefficients and with several constant retarded arguments of the form

$$\Delta x(n) + \sum_{i=1}^{m} p_i(n) x(n - k_i) = 0, \quad n \geq 0, \tag{21}$$

where $\{p_i(n)\}_{n=0}^{\infty}$, is a nonnegative sequence of real numbers and k_i are non-negative integers for $i = 1, 2, ..., m$.

In 1989, Erbe and Zhang [8], and Tang and Deng [22] derived the following oscillation conditions for the difference equation (21)

$$\sum_{i=1}^{m} \left(\liminf_{n\to\infty} p_i(n) \right) \frac{(k_i + 1)^{k_i+1}}{(k_i)^{k_i}} > 1, \tag{22}$$

$$\liminf_{n\to\infty} \sum_{i=1}^{m} \frac{(k_i + 1)^{k_i+1}}{k_i^{k_i}} p_i(n) > 1, \tag{23}$$

respectively.

In 1999, Tang and Yu [23] replaced the coefficients with their arithmetic means and improved (23) as follows

$$\liminf_{n\to\infty} \sum_{i=1}^{m} \left(\frac{k_i + 1}{k_i} \right)^{k_i+1} \sum_{j=n+1}^{n+k_i} p_i(j) > 1. \tag{24}$$

while in 2001, Tang and Zhang [24] derived the following upper limit condition

$$\limsup_{n\to\infty} \sum_{i=1}^{m} \sum_{j=n}^{n+k_i} p_i(j) > 1, \tag{25}$$

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