[1]R. Oinarov, [2]B.K. Omarbayeva, [3]A.M. Temirkhanova
[1]Doctor of Physical and Mathematical Sciences, E-mail: o_ryskul@mail.ru
[2]PhD student, E-mail: gaziza.omarbaeva@mail.ru
[3]PhD, E-mail: ainura-t@yandex.kz
L.N. Gumilyov Eurasian National University, Nur-Sultan, Kazakhstan

DISCRETE ITERATED HARDY-TYPE INEQUALITIES WITH THREE WEIGHTS

Discrete, continuous Hardy-type inequalities are of great importance and have numerous applications in harmonic analysis, in the theory of integral, differential and difference operators, in the theory of embeddings of function spaces and in other branches of mathematics. In recent years, weighted estimates for multidimensional Hardy-type operators have been intensively studied, which have an important application in the study of boundedness properties of operators from a Lebesgue weighted space to a local Morrey-type space, as well as in the study of bilinear operators in Lebesgue weighted spaces. The discrete case of Hardy type inequalities with three weights is an open problem. An inequality involving an iteration of the discrete Hardy operator is traditionally considered difficult to estimate because it contains three independent weight sequences and three parameters at their different ratios. In this paper we prove some new discrete iterated Hardy-type inequality involving three weights for the case $0 < p \le \min\{q, \theta\}$.
Key words: inequalities, Hardy-type operator, weight, sequences, discrete Lebesgue spaces.

[1]Р. Ойнаров, [2]Б.Қ. Омарбаева, [3]А.М. Темірханова
[1]Ф.-м.ғ.д., E-mail: o_ryskul@mail.ru
[2]Докторант, E-mail: gaziza.omarbaeva@mail.ru
[3]PhD, E-mail: ainura-t@yandex.kz
Л.Н. Гумилев атындағы Еуразия ұлттық университеті, Нұр-Сұлтан қ., Қазақстан

Үш салмақты дискретті итерацияланған Харди типті теңсіздіктер

Дискретті, үзіліссіз Харди типті теңсіздіктері гармоникалық анализде, интегралдық, диф-ференциалдық және айырымдық операторлар теориясында, функционалдық кеңістіктердегі енгізу теориясында және басқада математиканың бөлімдерінде көптеген қолданыстары бар. Соңғы жылдары көп өлшемді Харди типті операторлардық салмақты бағалаулары қарқын-ды түрде зерттелуде, яғни олар Лебег салмақты кеңістігінен Морри типті локальді кеңістігіне шенелімділік қасиеттерін зерттеуде, сонымен қатар Лебег салмақты кеңістігіндегі бисызықты операторларды зерттеуде қолданылады. Үш салмақты Харди типті теңсіздігінің дискретті жағдайы ашық мәселе болып табылады. Әдетте, дискретті Харди операторының итерация-сымен болатын теңсіздіктерді бағалау қиын деп саналады, өйткені оның құрамында үш тәу-елсіз салмақты тізбек және әр түрлі қатынастағы үш параметр бар. Бұл мақалада біз кейбір жаңа үш салмақты дискретті итерацияланған Харди типті теңсіздікті $0 \le p \le \min\{q, \theta\}$ жағ-дайы үшін дәлелдейміз.
Түйін сөздер: теңсіздік, Харди типті оператор, салмақ, тізбектер, дискретті Лебег кеңістігі.

[1]Р. Ойнаров, [2]Б.К. Омарбаева, [3]А.М. Темирханова
[1]Д.ф.-м.н., E-mail: o_ryskul@mail.ru
[2]Докторант, E-mail: gaziza.omarbaeva@mail.ru
[3]PhD, E-mail: ainura-t@yandex.kz
Евразийский национальный университет имени Л.Н. Гумилева, г. Нур-Султан, Казахстан

Дискретные итерационные неравенства типа Харди с тремя весами

Дискретные, непрерывные неравенства типа Харди имеют большое значение и многочисленные приложения в гармоническом анализе, в теориях интегральных, дифференциальных и разностных операторов, в теории вложений функциональных пространств и в других разделах математики. В последние годы интенсивно исследуются весовые оценки для многомерных операторов типа Харди, которые имеют важное приложение в исследовании свойств ограниченности операторов из весового пространства Лебега в локальное пространство типа Морри, а также в исследовании билинейных операторов в весовых пространствах Лебега. Дискретный случай неравенств типа Харди с тремя весами является открытой проблемой. Неравенство, включающее итерацию дискретного оператора Харди, традиционно считается трудным для оценки, поскольку оно содержит три независимых весовых последовательностей и три параметра, при их различных соотношениях. В этой статье мы доказываем некоторое новое дискретное итерационное неравенство типа Харди с тремя весами для случая $0 < p \leq \min\{q, \theta\}$.

Ключевые слова: неравенство, оператор типа Харди, вес, последовательности, дискретное пространство Лебега.

1 Introduction

Let $0 < p, q, \theta < \infty$ and $\varphi = \{\varphi_k\}_{k=1}^{\infty}$ be a non-negative sequence, $u = \{u_i\}_{i=1}^{\infty}$, $\omega = \{\omega_i\}_{i=1}^{\infty}$ be positive sequences of real numbers, which will be referred to as weight sequences. We denote by $l_{p,u}$ the space of sequences $f = \{f_j\}_{j=1}^{\infty}$ of real numbers such that

$$\|f\|_{p,u} = \Big(\sum_{j=1}^{\infty} |u_j f_j|^p \Big)^{\frac{1}{p}} < +\infty, \quad 1 \leq p < \infty.$$

In this paper we characterize the following discrete Hardy-type inequalities:

$$\Big(\sum_{n=1}^{\infty} \omega_n^{\theta} \Big(\sum_{k=n}^{\infty} \Big| \varphi_k \sum_{i=k}^{\infty} f_i \Big|^q \Big)^{\frac{\theta}{q}} \Big)^{\frac{1}{\theta}} \leq C \Big(\sum_{j=1}^{\infty} |u_j f_j|^p \Big)^{\frac{1}{p}}, \quad \forall f \in l_{p,u}, \tag{1}$$

for the following cases:

a) $1 < p \leq min\{q, \theta\} < \infty$,

b) $p \in (0, 1]$ and $p \leq min\{q, \theta\} < \infty$,

where C is a positive constant independent of f.

Note that our result in case b) is especially interesting, since a continuous analogue of inequality (1) doesn't exist in this case.

2 Literature review

Since last century, one-dimensional discrete, continuous weighted Hardy inequalities have been investigated intensively in various functional spaces. The results of these works can be seen in the books of such authors, as for example B. Opic, A. Kufner, L. Maligranda, L.-E. Persson and N. Samko ([1]-[3]). In recent years, the general cases of the discrete, continuous weighted Hardy inequalities are investigated. For example the papers [4]-[15] have been devoted to the continuous analogue of discrete Hardy-type inequalities (1). An interest

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in this type of inequalities are caused by their applicability to spaces of the Morrey type ([16], [17]). Moreover, the characterizations of these inequalities can be applied to research weighted inequalities for Hardy's bilinear inequalities ([18]-[20]). However, the discrete Hardy-type inequality (1) is study very little. For example, see the papers [21] and [22], where in particular, in [22] a criterion for the fulfilment of inequality (1) was obtained for the case $0 < q < \theta < p < \infty$, $p > 1$.

3 Material and research methods

The research methods are as follows: in this paper a method of partition of the sequence of elements of the Hardy operator on the part in each point is developed, which allows us to effectively estimate the sum on the parts. Note that such "blocking technic" was developed in [4]. During the estimate, various classical inequalities are used, such as Minkowski inequality, Holder inequality and the following elementary inequalities:

If $a_i > 0$, $i = 1, 2, ..., k$, then

$$\Big(\sum_{m=1}^{k} a_i \Big)^{\alpha} \leq \sum_{m=1}^{k} a_i^{\alpha}, \ \ 0 < \alpha \leq 1, \tag{2}$$

and

$$\Big(\sum_{m=1}^{k} a_i \Big)^{\alpha} \geq \sum_{m=1}^{k} a_i^{\alpha}, \ \ \alpha \geq 1. \tag{3}$$

In the proofs of our main results we will need the following well-known version of the discrete Minkowski inequality:

Lemma. Let $\{a_{i,j}\}$, $i = 1, 2, ..., n \leq +\infty$, $j = 1, 2, ..., m$, be a positive matrix. Then the inequalities

$$\Big(\sum_{i=1}^{n} \Big| \sum_{j=1}^{m} a_{i,j} \Big|^{\sigma} \Big)^{\frac{1}{\sigma}} \leq \sum_{j=1}^{m} \Big(\sum_{i=1}^{n} |a_{i,j}|^{\sigma} \Big)^{\frac{1}{\sigma}}, \tag{4}$$

and

$$\Big(\sum_{i=1}^{n} \Big| \sum_{j=1}^{i} a_{i,j} \Big|^{\sigma} \Big)^{\frac{1}{\sigma}} \leq \sum_{j=1}^{n} \Big(\sum_{i=j}^{n} |a_{i,j}|^{\sigma} \Big)^{\frac{1}{\sigma}}, \tag{5}$$

hold, where $\sigma \geq 1$.

Convention: The symbol $M \ll K$ means that $M \leq cK$, where $c > 0$ is a constant depending only on unessential parameters. If $M \ll K \ll M$, then we write $M \approx K$.

4 Results and discussion

4.1 Main result

Our main result reads as follows.

Theorem. *(i) If $1 < p \le min\{q, \theta\} < \infty$, then the inequality (1) holds, if and only if $A_1 < \infty$, where*

$$A_1 := \sup_{r \ge 1} \Big(\sum_{n=1}^{r} \omega_n^\theta \Big(\sum_{k=n}^{r} \varphi_k^q \Big)^{\frac{\theta}{q}} \Big)^{\frac{1}{\theta}} \Big(\sum_{i=r}^{\infty} u_i^{-p'} \Big)^{\frac{1}{p'}}.$$

Moreover, $C \approx A_1$, where C is the best constant in (1).

(ii) If $p \in (0,1]$ and $p \le min\{q, \theta\} < \infty$, then the inequality (1) holds, if and only if $A_2 < \infty$, where

$$A_2 := \sup_{r \ge 1} \Big(\sum_{n=1}^{r} \omega_n^\theta \Big(\sum_{s=n}^{r} \varphi_s^q \Big)^{\frac{\theta}{q}} \Big)^{\frac{1}{\theta}} \sup_{r \le k} u_k^{-1}.$$

Moreover, $C \approx A_2$, where C is the best constant in (1).

Proof. *Necessity:* Suppose that the inequality (1) holds with best constant $C > 0$.
(i) Let us show that $A_1 < \infty$. Let $1 \le r < N < \infty$ and take a test sequence $\widetilde{f} = \{\widetilde{f}_s\}_{s=1}^{\infty}$ such that $\widetilde{f}_s = 0$ for $1 \le s < r$ and $s > N$ and $\widetilde{f}_s = u_s^{-p'}$ for $r \le s \le N < \infty$.

Then

$$\|\widetilde{f}\|_{l_{p,u}} = \Big(\sum_{s=1}^{\infty} |\widetilde{f}_s \cdot u_s|^p \Big)^{\frac{1}{p}} = \Big(\sum_{s=r}^{N} |u_s^{-p'} \cdot u_s|^p \Big)^{\frac{1}{p}} = \Big(\sum_{s=r}^{N} u_s^{-p'} \Big)^{\frac{1}{p}}. \tag{6}$$

By substituting $\widetilde{f}$ in the left hand side of inequality (1), we can deduce that

$$I(\widetilde{f}) := \Big(\sum_{n=1}^{\infty} \omega_n^\theta \Big(\sum_{k=n}^{\infty} \Big| \varphi_k \sum_{i=k}^{\infty} \widetilde{f}_i \Big|^q \Big)^{\frac{\theta}{q}} \Big)^{\frac{1}{\theta}} \ge \Big(\sum_{n=1}^{r} \omega_n^\theta \Big(\sum_{k=n}^{N} \Big| \varphi_k \sum_{i=k}^{N} \widetilde{f}_i \Big|^q \Big)^{\frac{\theta}{q}} \Big)^{\frac{1}{\theta}} \ge$$

$$\ge \Big(\sum_{n=1}^{r} \omega_n^\theta \Big(\sum_{k=n}^{r} \Big| \varphi_k \sum_{i=r}^{N} u_i^{-p'} \Big|^q \Big)^{\frac{\theta}{q}} \Big)^{\frac{1}{\theta}} \ge \Big(\sum_{n=1}^{r} \omega_n^\theta \Big(\sum_{k=n}^{r} \varphi_k^q \Big)^{\frac{\theta}{q}} \Big)^{\frac{1}{\theta}} \Big(\sum_{i=r}^{N} u_i^{-p'} \Big). \tag{7}$$

From (1), (6) and (7), it follows that

$$C \ge \Big(\sum_{n=1}^{r} \omega_n^\theta \Big(\sum_{k=n}^{r} \varphi_k^q \Big)^{\frac{\theta}{q}} \Big)^{\frac{1}{\theta}} \Big(\sum_{i=r}^{N} u_i^{-p'} \Big)^{\frac{1}{p'}}, \quad \text{for all } 1 \le r < N.$$

Since $r \ge 1$ is arbitrary, passing to the limit as $N \to \infty$, we have that

$$A_1 = \sup_{r \ge 1} \Big(\sum_{n=1}^{r} \omega_n^\theta \Big(\sum_{k=n}^{r} \varphi_k^q \Big)^{\frac{\theta}{q}} \Big)^{\frac{1}{\theta}} \Big(\sum_{i=r}^{\infty} u_i^{-p'} \Big)^{\frac{1}{p'}} \le C. \tag{8}$$

(ii) Let us show that $A_2 < \infty$. Now for $1 < r \le k < \infty$ we assume that $\widetilde{g} = \{\widetilde{g}_s\}_{s=1}^{\infty}$, where $\widetilde{g}_s = 0$ for $s \ne k$ and $\widetilde{g}_s = u_k^{-1}$ for $s = k$, where $u_k \ne 0$. Then

$$\|\widetilde{g}\|_{l_{p,u}} = u_k^{-1} \cdot u_k = 1. \tag{9}$$

Substituting $\widetilde{g}$ in the left hand side of inequality (1), we find that

$$I(\widetilde{g}) := \Big(\sum_{n=1}^{\infty} \omega_n^\theta \Big(\sum_{s=n}^{\infty} \Big| \varphi_s \sum_{i=s}^{\infty} \widetilde{g}_i \Big|^q \Big)^{\frac{\theta}{q}} \Big)^{\frac{1}{\theta}} \ge \Big(\sum_{n=1}^{r} \omega_n^\theta \Big(\sum_{s=n}^{r} \varphi_s^q \Big)^{\frac{\theta}{q}} \Big)^{\frac{1}{\theta}} u_k^{-1}, \quad \forall k \ge r,$$

,

$$I(\widetilde{g}) \geq \Big(\sum_{n=1}^{r} \omega_n^{\theta} \Big(\sum_{s=n}^{r} \varphi_s^q \Big)^{\frac{\theta}{q}} \Big)^{\frac{1}{\theta}} \sup_{r \leq k} u_k^{-1}, \ \ \forall r \geq 1.$$

Therefore

$$I(\widetilde{g}) \geq \sup_{r \geq 1} \Big(\sum_{n=1}^{r} \omega_n^{\theta} \Big(\sum_{s=n}^{r} \varphi_s^q \Big)^{\frac{\theta}{q}} \Big)^{\frac{1}{\theta}} \sup_{r \leq k} u_k^{-1} = A_2. \tag{10}$$

From (1), (9) and (10), we have that

$$A_2 = \sup_{r \geq 1} \Big(\sum_{n=1}^{r} \omega_n^{\theta} \Big(\sum_{s=n}^{r} \varphi_s^q \Big)^{\frac{\theta}{q}} \Big)^{\frac{1}{\theta}} \sup_{r \leq k} u_k^{-1} \leq C. \tag{11}$$

Sufficiency: Now, we prove that the inequality (1) holds. Let $0 \leq f \in l_{p,u}$ be such that

$$\sum_{i=1}^{\infty} f_i < \infty. \tag{12}$$

Let

$$k_1 := \inf\{k \in Z : \sum_{i=1}^{\infty} f_i \leq 2^k\},$$

then

$$2^{k_1 - 1} \leq \sum_{i=1}^{\infty} f_i < 2^{k_1}.$$

We consider the sequence $\{j_k\}$, where j_k are defined by

$$j_k := min\{j \geq 1 : \sum_{i=j}^{\infty} f_i \leq 2^{k_1 - k + 1}\}.$$

We note that

$$j_1 = min\{j \geq 1 : \sum_{i=j}^{\infty} f_i \leq 2^{k_1}\} = 1.$$

For all $k \geq 1$ it yields that

$$\sum_{i=j_k}^{\infty} f_i \leq 2^{k_1 - k + 1} < \sum_{i=j_k - 1}^{\infty} f_i. \tag{13}$$

Therefore the set of natural numbers $\mathbb{N}$ can be written

$$\mathbb{N} = \bigcup_{k \geq 1} [j_k, j_{k+1} - 1].$$

Moreover,

$$2^{k_1 - m + 1} < \sum_{i=j_m - 1}^{\infty} f_i = \sum_{i=j_m - 1}^{j_{m+1} - 1} f_i + \sum_{i=j_{m+1}}^{\infty} f_i < \sum_{i=j_m - 1}^{j_{m+1} - 1} f_i + 2^{k_1 - (m+1) + 1}, \ \ m \geq 2.$$

Hence,

$$2^{k_1-m} < \sum_{i=j_m-1}^{j_{m+1}-1} f_i, \quad m \ge 2.$$

$$2^{k_1-m} = 2^{k_1-(m+1)+1} = 2 \cdot 2^{k_1-(m+1)} < 2 \sum_{i=j_{m+1}-1}^{j_{m+2}-1} f_i, \quad m \ge 2.$$

We have that

$$2^{k_1-1+1} = 2^{k_1} = 4 \cdot 2^{k_1-2} < 4 \sum_{i=j_2-1}^{j_{2+1}-1} f_i.$$

Then we obtain that

$$2^{k_1-m+1} < 4 \sum_{i=j_{m+1}-1}^{j_{m+2}-1} f_i, \quad m \ge 1. \tag{14}$$

Therefore, in view of (13),

$$I^\theta(f) := \sum_{n=1}^\infty \omega_n^\theta \left(\sum_{s=n}^\infty \left| \varphi_s \sum_{i=s}^\infty f_i \right|^q \right)^{\frac{\theta}{q}} = \sum_{k=1}^\infty \sum_{n=j_k}^{j_{k+1}-1} \omega_n^\theta \left(\sum_{s=n}^\infty \left| \varphi_s \sum_{i=s}^\infty f_i \right|^q \right)^{\frac{\theta}{q}} \le$$

$$\le \sum_{k=1}^\infty \sum_{n=j_k}^{j_{k+1}-1} \omega_n^\theta \left(\sum_{m=k}^\infty \sum_{s=\max(n,j_m)}^{j_{m+1}-1} \varphi_s^q \left(\sum_{i=j_m}^\infty f_i \right)^q \right)^{\frac{\theta}{q}} \le$$

$$\le \sum_{k=1}^\infty \sum_{n=j_k}^{j_{k+1}-1} \omega_n^\theta \left(\sum_{m=k}^\infty \sum_{s=\max(n,j_m)}^{j_{m+1}-1} \varphi_s^q \left(2^{k_1-m+1} \right)^q \right)^{\frac{\theta}{q}}.$$

Hence, by applying (14) we have that

$$I^\theta(f) \le 4^\theta \sum_{k=1}^\infty \sum_{n=j_k}^{j_{k+1}-1} \omega_n^\theta \left(\sum_{m=k}^\infty \sum_{s=\max(n,j_m)}^{j_{m+1}-1} \varphi_s^q \left(\sum_{i=j_{m+1}-1}^{j_{m+2}-1} f_i \right)^q \right)^{\frac{\theta}{q}}. \tag{15}$$

We must now consider the cases $\theta \le q$ and $\theta > q$ separately.

4.2 The case $\theta \le q$

(i) Let $1 < p \le \theta \le q$. By applying (2) in (15), we find that

$$I^\theta(f) \le 4^\theta \sum_{k=1}^\infty \sum_{n=j_k}^{j_{k+1}-1} \omega_n^\theta \sum_{m=k}^\infty \left(\sum_{s=\max(n,j_m)}^{j_{m+1}-1} \varphi_s^q \right)^{\frac{\theta}{q}} \left(\sum_{i=j_{m+1}-1}^{j_{m+2}-1} f_i \right)^\theta.$$

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Now, by changing the orders of sums, we get that

$$I^\theta(f) \leq 4^\theta \sum_{m=1}^{\infty} \left(\sum_{i=j_{m+1}-1}^{j_{m+2}-1} f_i \right)^\theta \sum_{k=1}^{m} \sum_{n=j_k}^{j_{k+1}-1} \omega_n^\theta \left(\sum_{s=\max(n,j_m)}^{j_{m+1}-1} \varphi_s^q \right)^{\frac{\theta}{q}} \leq$$

$$\leq 4^\theta \sum_{m=1}^{\infty} \left(\sum_{i=j_{m+1}-1}^{j_{m+2}-1} f_i \right)^\theta \left(\sum_{n=j_1}^{j_2-1} \omega_n^\theta \left(\sum_{s=n}^{j_{m+1}-1} \varphi_s^q \right)^{\frac{\theta}{q}} + \sum_{k=2}^{m} \sum_{n=j_k}^{j_{k+1}-1} \omega_n^\theta \left(\sum_{s=j_m}^{j_{m+1}-1} \varphi_s^q \right)^{\frac{\theta}{q}} \right) \leq$$

$$\leq 4^\theta \sum_{m=1}^{\infty} \left(\sum_{i=j_{m+1}-1}^{j_{m+2}-1} f_i \right)^\theta \left(\sum_{n=j_1}^{j_2-1} \omega_n^\theta \left(\sum_{s=n}^{j_{m+1}-1} \varphi_s^q \right)^{\frac{\theta}{q}} + \sum_{n=j_2}^{j_{m+1}-1} \omega_n^\theta \left(\sum_{s=j_m}^{j_{m+1}-1} \varphi_s^q \right)^{\frac{\theta}{q}} \right) \leq$$

$$\leq 4^\theta \sum_{m=1}^{\infty} \left(\sum_{i=j_{m+1}-1}^{j_{m+2}-1} f_i \right)^\theta \left(\sum_{n=j_1}^{j_2-1} \omega_n^\theta \left(\sum_{s=n}^{j_{m+1}-1} \varphi_s^q \right)^{\frac{\theta}{q}} + \sum_{n=j_2}^{j_{m+1}-1} \omega_n^\theta \left(\sum_{s=n}^{j_{m+1}-1} \varphi_s^q \right)^{\frac{\theta}{q}} \right) \leq$$

$$\leq 4^\theta \sum_{m=1}^{\infty} \left(\sum_{i=j_{m+1}-1}^{j_{m+2}-1} f_i \right)^\theta \sum_{n=j_1}^{j_{m+1}-1} \omega_n^\theta \left(\sum_{s=n}^{j_{m+1}-1} \varphi_s^q \right)^{\frac{\theta}{q}},$$

so that

$$I^\theta(f) \ll \sum_{m=1}^{\infty} \left(\sum_{i=j_{m+1}-1}^{j_{m+2}-1} f_i \right)^\theta \sum_{n=j_1}^{j_{m+1}-1} \omega_n^\theta \left(\sum_{s=n}^{j_{m+1}-1} \varphi_s^q \right)^{\frac{\theta}{q}}. \tag{16}$$

Therefore, by using Holder's inequality and (3) in (16), we obtain that

$$I^\theta(f) \leq \sum_{m=1}^{\infty} \left(\sum_{i=j_{m+1}-1}^{j_{m+2}-1} |f_i \cdot u_i|^p \right)^{\frac{\theta}{p}} \sum_{n=j_1}^{j_{m+1}-1} \omega_n^\theta \left(\sum_{s=n}^{j_{m+1}-1} \varphi_s^q \right)^{\frac{\theta}{q}} \left(\sum_{i=j_{m+1}-1}^{j_{m+2}-1} u_i^{-p'} \right)^{\frac{\theta}{p'}} \leq$$

$$\leq \left(\sum_{m=1}^{\infty} \sum_{i=j_{m+1}-1}^{j_{m+2}-1} |f_i \cdot u_i|^p \right)^{\frac{\theta}{p}} \left(\sup_{m \geq 1} \left(\sum_{n=j_1}^{j_{m+1}-1} \omega_n^\theta \left(\sum_{s=n}^{j_{m+1}-1} \varphi_s^q \right)^{\frac{\theta}{q}} \right)^{\frac{1}{\theta}} \left(\sum_{i=j_{m+1}-1}^{\infty} u_i^{-p'} \right)^{\frac{1}{p'}} \right)^\theta \leq$$

$$\leq \left(\sum_{i=1}^{\infty} |f_i u_i|^p \right)^{\frac{\theta}{p}} \left(\sup_{r \geq 1} \left(\sum_{n=1}^{r} \omega_n^\theta \left(\sum_{s=n}^{r} \varphi_s^q \right)^{\frac{\theta}{q}} \right)^{\frac{1}{\theta}} \left(\sum_{i=r}^{\infty} u_i^{-p'} \right)^{\frac{1}{p'}} \right)^\theta = \left(A_1 \|f\|_{p,u} \right)^\theta.$$

Hence,

$$I(f) \ll A_1 \|f\|_{p,u}, \quad if \quad 1 < p \leq \theta \leq q. \tag{17}$$

(ii) Let $0 < p \le 1$. We start with the inequality (16):

$$I^\theta(f) \ll \sum_{m=1}^{\infty} \left(\sum_{i=j_{m+1}-1}^{j_{m+2}-1} f_i \cdot u_i \cdot u_i^{-1} \right)^{p \cdot \frac{\theta}{p}} \sum_{n=1}^{j_{m+1}-1} \omega_n^\theta \left(\sum_{s=n}^{j_{m+1}-1} \varphi_s^q \right)^{\frac{\theta}{q}}.$$

By applying (2) with $0 < p \le 1$, we obtain that

$$I^\theta(f) \le \sum_{m=1}^{\infty} \left(\sum_{i=j_{m+1}-1}^{j_{m+2}-1} |f_i \cdot u_i|^p \right)^{\frac{\theta}{p}} \left(\max_{j_{m+1}-1 \le i \le j_{m+2}-1} u_i^{-1} \right)^\theta \sum_{n=j_1}^{j_{m+1}-1} \omega_n^\theta \left(\sum_{s=n}^{j_{m+1}-1} \varphi_s^q \right)^{\frac{\theta}{q}} \le$$

$$\le \sum_{m=1}^{\infty} \left(\sum_{i=j_{m+1}-1}^{j_{m+2}-1} |f_i \cdot u_i|^p \right)^{\frac{\theta}{p}} \left(\sup_{j_{m+1}-1 \le k} u_k^{-1} \right)^\theta \sum_{n=1}^{j_{m+1}-1} \omega_n^\theta \left(\sum_{s=n}^{j_{m+1}-1} \varphi_s^q \right)^{\frac{\theta}{q}}.$$

By using (3), we get

$$I^\theta(f) \le \left(\sum_{m=1}^{\infty} \sum_{i=j_{m+1}-1}^{j_{m+2}-1} |f_i \cdot u_i|^p \right)^{\frac{\theta}{p}} \left[\sup_{m \ge 1} \left(\sum_{n=j_1}^{j_{m+1}-1} \omega_n^\theta \left(\sum_{s=n}^{j_{m+1}-1} \varphi_s^q \right)^{\frac{\theta}{q}} \right)^{\frac{1}{\theta}} \sup_{j_{m+1}-1 \le k} u_k^{-1} \right]^\theta \le$$

$$\le \left(\sum_{i=1}^{\infty} |f_i \cdot u_i|^p \right)^{\frac{\theta}{p}} \left[\sup_{r \ge 1} \left(\sum_{n=1}^{r} \omega_n^\theta \left(\sum_{s=n}^{r} \varphi_s^q \right)^{\frac{\theta}{q}} \right)^{\frac{1}{\theta}} \sup_{r \le k} u_k^{-1} \right]^\theta = \left(A_2 \|f\|_{p,u} \right)^\theta,$$

so that

$$I(f) \ll A_2 \|f\|_{p,u}, \quad if \quad p \le \theta \le q, \tag{18}$$

where $p \in (0,1]$.

4.3 The case $\theta > q$

(i) Let $1 < p \le q < \theta$. We start with the inequality (15). First we raise both sides in (15) to the power $\frac{q}{\theta} \le 1$, i.e.,

$$I^q(f) \le 4^q \left[\sum_{k=1}^{\infty} \sum_{n=j_k}^{j_{k+1}-1} \omega_n^\theta \left(\sum_{m=k}^{\infty} \sum_{s=\max(n,j_m)}^{j_{m+1}-1} \varphi_s^q \left(\sum_{i=j_{m+1}-1}^{j_{m+2}-1} f_i \right)^q \right)^{\frac{\theta}{q}} \right]^{\frac{q}{\theta}}.$$

Next we apply (4) in the inner sum with $\sigma = \frac{\theta}{q}$ and obtain that

$$I^q(f) \le 4^q \left[\sum_{k=1}^{\infty} \left\{ \sum_{m=k}^{\infty} \left(\sum_{n=j_k}^{j_{k+1}-1} \omega_n^\theta \left(\sum_{s=\max(n,j_m)}^{j_{m+1}-1} \varphi_s^q \right)^{\frac{\theta}{q}} \left(\sum_{i=j_{m+1}-1}^{j_{m+2}-1} f_i \right)^\theta \right)^{\frac{q}{\theta}} \right\}^{\frac{\theta}{q}} \right]^{\frac{q}{\theta}},$$

,

Using (5), we get that

$$I^q(f) \leq 4^q \sum_{m=1}^{\infty} \left[\sum_{k=1}^{m} \sum_{n=j_k}^{j_{k+1}-1} \omega_n^\theta \left(\sum_{s=\max(n,j_m)}^{j_{m+1}-1} \varphi_s^q \right)^{\frac{\theta}{q}} \left(\sum_{i=j_{m+1}-1}^{j_{m+2}-1} f_i \right)^\theta \right]^{\frac{q}{\theta}} \leq$$

$$\leq 4^q \sum_{m=1}^{\infty} \left(\sum_{i=j_{m+1}-1}^{j_{m+2}-1} f_i \right)^q \left[\sum_{k=1}^{m} \sum_{n=j_k}^{j_{k+1}-1} \omega_n^\theta \left(\sum_{s=\max(n,j_m)}^{j_{m+1}-1} \varphi_s^q \right)^{\frac{\theta}{q}} \right]^{\frac{q}{\theta}} \leq$$

$$\leq 4^q \sum_{m=1}^{\infty} \left(\sum_{i=j_{m+1}-1}^{j_{m+2}-1} f_i \right)^q \left[\sum_{n=j_1}^{j_{m+1}-1} \omega_n^\theta \left(\sum_{s=n}^{j_{m+1}-1} \varphi_s^q \right)^{\frac{\theta}{q}} \right]^{\frac{q}{\theta}},$$

so that

$$I^q(f) \ll \sum_{m=1}^{\infty} \left(\sum_{i=j_{m+1}-1}^{j_{m+2}-1} f_i \right)^q \left[\sum_{n=j_1}^{j_{m+1}-1} \omega_n^\theta \left(\sum_{s=n}^{j_{m+1}-1} \varphi_s^q \right)^{\frac{\theta}{q}} \right]^{\frac{q}{\theta}}. \tag{19}$$

Hence, by using Holder's inequality,

$$I^q(f) \leq \sum_{m=1}^{\infty} \left(\sum_{i=j_{m+1}-1}^{j_{m+2}-1} |f_i \cdot u_i|^p \right)^{\frac{q}{p}} \left(\sum_{i=j_{m+1}-1}^{j_{m+2}-1} u_i^{-p'} \right)^{\frac{q}{p'}} \left[\sum_{n=j_1}^{j_{m+1}-1} \omega_n^\theta \left(\sum_{s=n}^{j_{m+1}-1} \varphi_s^q \right)^{\frac{\theta}{q}} \right]^{\frac{q}{\theta}}.$$

Therefore, by applying (3) with $\alpha = \frac{q}{p}$, we obtain that

$$I^q(f) \leq \left(\sum_{m=1}^{\infty} \sum_{i=j_{m+1}-1}^{j_{m+2}-1} |f_i \cdot u_i|^p \right)^{\frac{q}{p}} \left[\sup_{m \geq 1} \left(\sum_{n=j_1}^{j_{m+1}-1} \omega_n^\theta \left(\sum_{s=n}^{j_{m+1}-1} \varphi_s^q \right)^{\frac{\theta}{q}} \right)^{\frac{1}{\theta}} \left(\sum_{i=j_{m+1}-1}^{\infty} u_i^{-p'} \right)^{\frac{1}{p'}} \right]^q \leq$$

$$\leq \left(\sum_{i=1}^{\infty} |f_i u_i|^p \right)^{\frac{q}{p}} \left(\sup_{r \geq 1} \left(\sum_{n=1}^{r} \omega_n^\theta \left(\sum_{s=n}^{r} \varphi_s^q \right)^{\frac{\theta}{q}} \right)^{\frac{1}{\theta}} \left(\sum_{i=r}^{\infty} u_i^{-p'} \right)^{\frac{1}{p'}} \right)^q = \left(A_1 \|f\|_{p,u} \right)^q,$$

so that

$$I(f) \ll A_1 \|f\|_{p,u}, \quad \text{when } 1 < p \leq q < \theta. \tag{20}$$

(ii) Let $0 < p \leq 1$. We start with the inequality (19):

$$I^q(f) \ll \sum_{m=1}^{\infty} \left(\sum_{i=j_{m+1}-1}^{j_{m+2}-1} f_i \cdot u_i \cdot u_i^{-1} \right)^{p \cdot \frac{q}{p}} \left(\sum_{n=1}^{j_{m+1}-1} \omega_n^\theta \left(\sum_{s=n}^{j_{m+1}-1} \varphi_s^q \right)^{\frac{\theta}{q}} \right)^{\frac{q}{\theta}}.$$

By applying (2) with $0 < p \le 1$, we obtain that

$$I^q(f) \le \sum_{m=1}^{\infty} \left(\sum_{i=j_{m+1}-1}^{j_{m+2}-1} |f_i \cdot u_i|^p \right)^{\frac{q}{p}} \left[\sup_{j_{m+1}-1 \le k} u_k^{-1} \left(\sum_{n=1}^{j_{m+1}-1} \omega_n^\theta \left(\sum_{s=n}^{j_{m+1}-1} \varphi_s^q \right)^{\frac{\theta}{q}} \right)^{\frac{1}{\theta}} \right]^q.$$

By using (3), we get that

$$I^q(f) \le \left(\sum_{m=1}^{\infty} \sum_{i=j_{m+1}-1}^{j_{m+2}-1} |f_i \cdot u_i|^p \right)^{\frac{q}{p}} \left[\sup_{m \ge 1} \left(\sum_{n=j_1}^{j_{m+1}-1} \omega_n^\theta \left(\sum_{s=n}^{j_{m+1}-1} \varphi_s^q \right)^{\frac{\theta}{q}} \right)^{\frac{1}{\theta}} \sup_{j_{m+1}-1 \le k} u_k^{-1} \right]^q \le$$

$$\le \left(\sum_{i=1}^{\infty} |f_i \cdot u_i|^p \right)^{\frac{q}{p}} \left[\sup_{r \ge 1} \left(\sum_{n=1}^{r} \omega_n^\theta \left(\sum_{s=n}^{r} \varphi_s^q \right)^{\frac{\theta}{q}} \right)^{\frac{1}{\theta}} \sup_{r \le k} u_k^{-1} \right]^q = \left(A_2 \|f\|_{p,u} \right)^q,$$

so that

$$I(f) \ll A_2 \|f\|_{p,u}, \quad if \quad p \le q < \theta, \tag{21}$$

where $p \in (0,1]$.

The estimates (17), (20) and (18), (21) were obtained under the condition (12). Let $M = \{f \in l_{p,u} : \exists N = N(f) > 0 \text{ and } f_i = 0, \ i \ge N\}$. Since f from M satisfy the condition (12) and the set M is everywhere dense in $l_{p,u}$, then the estimates (17), (20) and (18), (21) are satisfied for all $f \in l_{p,u}$. Therefore from the inequalities (8), (17) and (20), we get $C \approx A_1$ and from the inequalities (8) and (18), (21), we get $C \approx A_2$, where C is the best constant in (1). The proof of Theorem is complete.

5 Conclusion

In conclusion, we have established necessary and sufficient conditions on functions u and ω are ensuring boundedness of a discrete Hardy-type operator from a weighted sequence space $l_{p,u}$ to a weighted sequence space for a wide range of the numerical parameters p, u and θ.

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