IRSTI 27.33.19 https://doi.org/10.26577/JMMCS.2020.v105.i1.04

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ON A COMPARISON THEOREM FOR STOCHASTIC INTEGRO-FUNCTIONAL EQUATIONS OF NEUTRAL TYPE

In this paper, we will discuss a comparison result for solutions to the Cauchy problems for two stochastic differential equations with delay. On this subject number of authors have obtained their comparison results. We deal with the Cauchy problems for two integro-differential equations. Except transient- (or drift-) and diffusion coefficients our equations include also one integro-differential term. Basic difference of our case from the case of all earlier investigated problems is presence of this term. We introduce a concept of solutions to our problems and prove the comparison theorem for them. According to our result under certain assumptions on coefficients of equations under consideration, their solutions depend on the transient-coefficients in a monotone way.
Key words: stochastic differential equation, comparison theorem, Hilbert space.

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Бейтарап типтегі стохастикалық интегралдық-функционалдық теңдеулер үшін салыстыру теоремасы

Бұл мақалада кідіріс әсерлі екі стохастикалық дифференциалдық теңдеулер үшін Коши есебі шешімдерін салыстыру есебі қарастырылған. Осы сияқты есептердің шешімдерін салыстыруға қатысты көптеген авторлар өз нәтижелерін алды. Мақалада бейтарап типті екі стохастикалық интегралдық-дифференциалдық теңдеулер үшін Коши есебі қарастырылған. Қарастырылатын есептің тасымалдау және диффузия коэффициенттерінен басқа бір интегралдық-дифференциалдық бөліктен тұрады. Бұрын зерттелген жұмыстардан негізгі айырмашылығы интегралдық бөліктің бар болуы болып табылады. Алынған нәтижелерге сәйкес, олардың шешімі тасымалдау коэффициентіне монотонды тәуелді болады.
Түйін сөздер: стохастикалық дифференциалдық теңдеу, салыстыру теоремасы, гильберт кеңістігі.

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Об одной теореме сравнения для стохастических интегро-функциональных уравнений нейтрального типа

В данной статье рассматривается задача сравнения решений задач Коши для двух стохастических дифференциальных уравнений с запаздыванием. В этой области множество авторов получили свои результаты, касающиеся сравнения решений подобных задач. В данной работе рассматриваются задачи Коши для двух стохастических интегро-дифференциальных уравнений нейтрального типа. Помимо коэффициента сноса (переноса) и коэффициента диффузии, рассматриваемые уравнения содержат также один интегро-дифференциальный член. Наличие этого интегрального члена является основным отличием этой задачи ото всех ранее исследуемых задач. Для наших задач вводятся понятия решений, для которых доказана теорема сравнения. Согласно полученному результату, при некоторых предположениях на коэффициенты рассматриваемых уравнений, их решения монотонно зависят от коэффициентов переноса
Ключевые слова: стохастическое дифференциальное уравнение, теорема сравнения, гильбертово пространство.

1 Introduction

In the given paper the following Cauchy problems for two neutral stochastic integro-differential equations

$$d\left(u_i(t,x)+\int\limits_{\mathbb{R}^d} b_i(t,x,u_i(\alpha(t),\xi),\xi)d\xi\right)=f_i(t,u_i(\alpha(t),x),x)dt \\ +\sigma(t,x)d\beta(t),\ 0<t\le T,\ x\in\mathbb{R}^d,\ i\in\{1,2\}, \tag{1}$$

$$u_i(t,x)=\phi_i(t,x),\ -r\le t\le 0,\ x\in\mathbb{R}^d,\ r>0,\ i\in\{1,2\}, \tag{1*}$$

are studied, where $T>0$ is fixed, β is one-dimensional Brownian motion, $f_i\colon[0,T]\times\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}$, $i\in\{1,2\}$, $\sigma\colon[0,T]\times\mathbb{R}^d\to\mathbb{R}$ and $b_i\colon[0,T]\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}$, $i\in\{1,2\}$, are some given functions to be specified later, $\phi_i\colon[-r,0]\times\mathbb{R}^d\to\mathbb{R}$, $i\in\{1,2\}$, are initial-datum functions, $\alpha\colon[0,T]\to[-r,+\infty)$ is a delay function. For solutions u_1 and u_2 of these problems a comparison theorem is proved. According to the obtained result, if $f_1\ge f_2$, then $u_1\ge u_2$ with probability one. A comparison problem for solutions to stochastic differential equations in finite-dimensional case has firstly arised in [14]. A comparison theorem for equation of the form $d\xi(t)=f(t,\xi(t))dt+\sigma(t,\xi(t))d\beta(t)$ has been proved in this work by A. V. Skorokhod. According to this theorem, under certain assumptions, a solution of the equation above is monotonously non-decreasing function, depending on drift-coefficient f. A more general presentation of the comparison theorem is given in [11], [13]. Variations of these results have been proposed in [2] – [10]. The aim of the given work was to prove the comparison theorem for solutions of problem (1) –(1*).

2 Literature review

For the first time, the problem of the comparison of solutions of stochastic equations in the finite-dimensional case arose in [14]. It is proved in it that, under certain assumptions, the solution of the equation is a monotone non-decreasing function of the transfer coefficient. A more general form of the comparison theorem is given in [10] [13]. Variations of these results were proposed in [1], [2], [3], [5], [7]–[9], [11]–[25]. Comparison theorems for solutions of stochastic partial differential equations with a multidimensional Wiener process are presented in [3]. In [6], a proof is presented of a comparison theorem for solutions of the Cauchy problem for stochastic differential equations with a multidimensional Wiener process in a Hilbert space. The aim of this paper was to prove a comparison theorem for solutions of problem (1) using ideas from [6] and [14]. This result plays an important role in studying the existence and uniqueness of a solution to this problem under non-Lipschitz conditions on drift coefficients.

The structure of the article is as follows: Section 2 contains the statement of the problem, 3 — preliminary facts and auxiliary results, 4 — proof of the main theorem.

3 Material and methods

3.1 Comparison theorem for stochastic differential equations in the finite-dimensional case

We consider the Cauchy problem of the form

$$d\Big(u_i(t,x)+\int\limits_{\mathbb{R}^d} b_i(t,x,u_i(\alpha(t),\xi),\xi)d\xi\Big)=f_i(t,u_i(\alpha(t),x),x)dt+\sigma(t,x)d\beta(t),$$

$$0<t\le T,\ x\in\mathbb{R}^d,\ i\in\{1,2\},$$

$$u_i(t,x)=\phi_i(t,x),\ -r\le t\le 0,\ i\in\{1,2\},$$

where β one-dimensional Brownian motion. We regard the following conditions to be fulfilled:

1. $\alpha\colon[0,T]\to[-r,+\infty)$ belongs to $C^1([0,T])$ with $0<\alpha'<1$.
2. $f_i\colon[0,T]\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}$, $i\in\{1,2\}$, $\sigma\colon[0,T]\times\mathbb{R}\times\mathbb{R}^d\to[0,\infty]$, $b\colon[0,T]\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d\to$ $\to\mathbb{R}$ are measurable functions.
3. The initial-datum functions $\phi(t,x,\omega)\colon[-r,0]\times\mathbb{R}^d\times\Omega\to L_2(\mathbb{R}^d)$, $i\in\{1,2\}$, are $\mathcal{F}_0$-measurable random variables and such that

$$\mathbf{E}\sup_{-r\le t\le 0}\|\phi_i(t,\cdot)\|^2_{L_2(\mathbb{R}^d)}<\infty,\ i\in\{1,2\},\qquad \mathbf{E}\sup_{-r\le t\le 0}\phi^2(t,x)<\infty;.$$

4. b, satisfy the Lipshitz condition in the third argument of the form

$$\begin{aligned}&|b(t,x,u,\xi)-b(t,x,\nu,\xi)|\le l(t,x,\xi)|u-\nu|,\\ &\quad 0\le t\le T,\ \{x,\xi\}\subset\mathbb{R}^d,\ \{u,v\}\subset\mathbb{R},\end{aligned}\tag{2}$$

where the conditions are valid for the function l

$$\sup_{0\le t\le T}\int\limits_{\mathbb{R}^d}\Big(\int\limits_{\mathbb{R}^d}\frac{l^2(t,x,\xi)}{\rho(\xi)}d\xi\Big)\rho(x)dx<\frac{1}{4},\ \sup_{0\le t\le T}\int\limits_{\mathbb{R}^d}\frac{l^2(t,x,\xi)}{\rho(\xi)}d\xi<\infty,x\in\mathbb{R}^d.\tag{3}$$

,

5. There exists a function $b_1 \colon \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty)$, such that

$$\int\limits_{\mathbb{R}^d}\Big(\int\limits_{\mathbb{R}^d} b_1(x,\xi)d\xi\Big)^2\rho(x)dx<\infty, \qquad\qquad \int\limits_{\mathbb{R}^d} b_1(x,\xi)d\xi<\infty, x\in\mathbb{R}^d;$$

such that

$$\sup_{0\le t\le T}|b(t,x,0,\xi)|\le b_1(x,\xi),\ x\in\mathbb{R}^d,\xi\in\mathbb{R}^d. \tag{4}$$

6. The function $f_i,\sigma,\ i\in 1,2$, satisfy the conditions of linear growth and Lipschitz in the second argument, that is, there are $L>0$ such that

$$|f_i(t,u,x)|+|\sigma(t,u,x)|\le L(1+|u|),\ 0\le t\le T, u\in\mathbb{R},\ x\in\mathbb{R}^d, \tag{5}$$

$$|f_i(t,u,x)-f_i(t,\nu,x)|+|\sigma(t,u,x)-|\sigma(t,\nu,x)|\le L|u-\nu|,\ 0\le t\le T, \{u,\nu\}\in\mathbb{R},\ x\in\mathbb{R}^d,\ i\in 1,2. \tag{6}$$

Let $u\equiv u_i,\ \phi\equiv\phi_i,\ b\equiv b_i,\ f\equiv f_i,\ i\in\{1,2\}$.

Definition. A continuous random process $u_i(t,\cdot,\omega):[-r,T]\times\Omega\to\mathbb{R}, i\in\{1,2\}$ is called a **solution** to (1) – (1*) provided

1. It is $\mathcal{F}_t$-measurable for almost all $-r\le t\le T$.
2. It satisfies the following integral equation

$$u_i(t,\cdot)=\phi(0,\cdot)+\int\limits_{\mathbb{R}^d} b(0,\cdot,\phi(-r,\xi),\xi)d\xi-\int\limits_{\mathbb{R}^d} b(t,\cdot,u_i(\alpha(t),\xi),\xi)d\xi$$
$$+\int\limits_0^t f_i(s,u_i(\alpha(s),\cdot),\cdot)ds+\int\limits_0^t \sigma(s,u_i(\alpha(s),\cdot),\cdot)d\beta(s),\ 0\le t\le T,\ i\in\{1,2\},$$

$$u_i(t,\cdot)=\phi(t,),\ -r\le t\le 0,\ i\in\{1,2\}. \tag{7}$$

3. It satisfies the condition

$$\mathbf{E}\int\limits_0^T\|u_i(t,\,\cdot)\|^2_{L_2(\mathbb{R}^d)}dt<\infty,\quad \mathbf{E}\int\limits_0^T {u_i}^2(t,\cdot)dt<\infty, i\in\{1,2\}.$$

The following theorems are true.

Theorem 1. Denote by $u=u_i, f=f_i, i\in 1,2$. Assume that the conditions (1)–(6) are satisfied. Then (7) has a solution continuous with probability one, unique in the sense that if

$u(t,\cdot)$, $\nu(t,\cdot)$, $0<t<T$ are two continuous solutions to (7), then $\mathbf{P}\{\sup\limits_{0\le t\le T}|u(t,\cdot)-\nu(t,\cdot)|>0\}=0$.

Theorem 2. Suppose that conditions (1)–(6) are satisfied. Suppose further that $u_i(t,x), 0<t<T, i\in\{1,2\}$ are continuous (with probability one) solutions to problem (7). That, if $f_1(t,u,\cdot)\le f_2(t,u,\cdot)$ for all $0\le t\le T$ the condition $u_1(t,x)\le u_2(t,x)$ are satisfied.

Proof of the theorem 1. In order to prove existence and uniqueness of solution to (7) we use the method of successive approximations. The idea of the proof is to construct a sequence of approximations, which converges to the solution u. From now on x is supposed to be fixed. Let

$$u^{(0)}(t,\cdot\,)=\phi(0,\cdot\,),\ 0<t\le T, \tag{8}$$
$$u^{(0)}(t,\cdot\,)=\phi(t,\cdot\,),\ -r\le t\le 0, \tag{8*}$$

and for $n\in\{1,2,\dots\}$ define $u^{(n)}$ as

$$u^{(n)}(t,\cdot\,)=\phi(0,\cdot\,)+\int\limits_{\mathbb{R}^d}b(0,\cdot\,,\phi(-r,\xi),\xi)d\xi-\int\limits_{\mathbb{R}^d}b(t,\cdot\,,u^{(n-1)}(\alpha(t),\xi),\xi)d\xi$$
$$+\int\limits_0^t f(s,u^{(n-1)}(\alpha(s),\cdot\,),\cdot\,)ds+\int\limits_0^t\sigma(s,\cdot)d\beta(s),\ 0<t\le T, \tag{9}$$
$$u^{(n)}(t,\cdot\,)=\phi(t,\cdot\,),\ -r\le t\le 0. \tag{9*}$$

1.1 Firstly let us choose a small $0\le T_1\le T$ and prove that $\sup\limits_{0\le t\le T_1}\mathbf{E}\|u^{(n)}(t,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}$ has a bound, independent of n. We obtain

$$\sup_{0\le t\le T_1}\mathbf{E}\|u^{(n)}(t,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}\le 8\mathbf{E}\|\phi(0,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}+8\mathbf{E}\left\|\int\limits_{\mathbb{R}^d}|b(0,\cdot\,,\phi(-r,\xi),\xi)|d\xi\right\|^2_{L_2(\mathbb{R}^d)}$$
$$+2\sup_{0\le t\le T_1}\mathbf{E}\left\|\int\limits_{\mathbb{R}^d}|b(t,\cdot\,,u^{(n-1)}(\alpha(t),\xi),\xi)|d\xi\right\|^2_{L_2(\mathbb{R}^d)}$$
$$+8\sup_{0\le t\le T_1}\mathbf{E}\left\|\int\limits_0^t|f(s,u^{(n-1)}(\alpha(s),\cdot\,),\cdot\,)|ds\right\|^2_{L_2(\mathbb{R}^d)}+8\sup_{0\le t\le T_1}\mathbf{E}\left\|\int\limits_0^t\sigma(s,\cdot)d\beta(s)\right\|^2_{L_2(\mathbb{R}^d)}$$
$$=8\mathbf{E}\|\phi(0,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}+\sum_{j=1}^4 S_j,\ 0<t\le T. \tag{10}$$

From (2) and (4) we have

$$|b(t,\cdot\,,u,\xi)|\le|b(t,\cdot\,,u,\xi)-b(t,\cdot\,,0,\xi)|+|b(t,\cdot\,,0,\xi)|\le l(t,\cdot\,,\xi)|u|+\chi(\cdot\,,\xi),$$
$$0\le t\le T,\ u\in\mathbb{R},\ \xi\in\mathbb{R}^d.$$

,

Then we obtain

$$S_1 = 8\mathbf{E}\int\limits_{\mathbb{R}^d}\left(\int\limits_{\mathbb{R}^d}|b(0,x,\phi(-r,\xi),\xi)|d\xi\right)^2 dx \le 16\mathbf{E}\int\limits_{\mathbb{R}^d}\left(\int\limits_{\mathbb{R}^d} l(0,x,\xi)\phi(-r,\xi)d\xi\right)^2 dx$$
$$+ 16\int\limits_{\mathbb{R}^d}\left(\int\limits_{\mathbb{R}^d}\chi(x,\xi)d\xi\right)^2 dx \le 16\left(\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} l^2(0,x,\xi)d\xi dx\right)\mathbf{E}\|\phi(-r,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}$$
$$+ 16\int\limits_{\mathbb{R}^d}\left(\int\limits_{\mathbb{R}^d}\chi(x,\xi)d\xi\right)^2 dx,$$

$$S_2 = 2\sup_{0\le t\le T_1}\mathbf{E}\int\limits_{\mathbb{R}^d}\left(\int\limits_{\mathbb{R}^d}|b(t,x,u^{(n-1)}(\alpha(t),\xi),\xi)|d\xi\right)^2 dx \le 4\left(\sup_{0\le t\le T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx\right)$$
$$\times \sup_{0\le t\le T_1}\mathbf{E}\|u^{(n-1)}(\alpha(t),\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)} + 4\int\limits_{\mathbb{R}^d}\left(\int\limits_{\mathbb{R}^d}\chi(x,\xi)d\xi\right)^2 dx. \tag{11}$$

According to properties of α, there exists a point $0 \le t^* \le T_1$, $\alpha(t^*) = 0$. Then

$$\sup_{0\le t\le T_1}\mathbf{E}\|u^{(n-1)}(\alpha(t),\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)} \le \sup_{0\le t\le t^*}\mathbf{E}\|u^{(n-1)}(\alpha(t),\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}$$
$$+ \sup_{t^*\le t\le \alpha(T_1)}\mathbf{E}\|u^{(n-1)}(\alpha(t),\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)} \le \sup_{-r\le t\le 0}\mathbf{E}\|\phi(t,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)} + \sup_{0\le t\le T_1}\mathbf{E}\|u^{(n-1)}(t,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}$$

and we get from (11)

$$S_2 \le 4\left(\sup_{0\le t\le T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx\right)\left(\sup_{-r\le t\le 0}\mathbf{E}\|\phi(t,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)} + \sup_{0\le t\le T_1}\mathbf{E}\|u^{(n-1)}(t,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}\right)$$
$$+ 4\int\limits_{\mathbb{R}^d}\left(\int\limits_{\mathbb{R}^d}\chi(x,\xi)d\xi\right)^2 dx.$$

If t^* does not exist, then $\alpha(t) < 0$ for all t and further conclusions are obvious, because

$$\sup_{0\le t\le T_1}\mathbf{E}\|u^{(n-1)}(\alpha(t),\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)} = \sup_{-r\le t\le 0}\mathbf{E}\|\phi(t,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}.$$

In order to estimate S_3, we take (5) into account and obtain

$$S_3 = 8\sup_{0\le t\le T_1}\mathbf{E}\int\limits_{\mathbb{R}^d}\left(\int\limits_0^t|f(s,u^{(n-1)}(\alpha(s),x),x)|ds\right)^2 dx \le 16T_1\sup_{0\le t\le T_1}\mathbf{E}\int\limits_0^t\int\limits_{\mathbb{R}^d}\Big(\eta^2(s,x)$$

$$+L^2\Big(u^{(n-1)}(\alpha(s),x)\Big)^2\Bigg)dxds \le 16T_1\Bigg(T_1\sup_{0\le t\le T_1}\int\limits_{\mathbb{R}^d}\eta^2(t,x)dx+L^2\int\limits_{-r}^{\alpha(T_1)}\mathbf{E}\|u^{(n-1)}(s,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}ds\Bigg)$$

$$\le 16T_1^2\sup_{0\le t\le T_1}\int\limits_{\mathbb{R}^d}\eta^2(t,x)dx+16L^2T_1\Bigg(r\sup_{-r\le s\le 0}\mathbf{E}\|\phi(s,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}+\int\limits_0^{T_1}\mathbf{E}\|u^{(n-1)}(s,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}ds\Bigg).$$

For S_4 we conclude

$$S_4=8\sup_{0\le t\le T_1}\int\limits_{\mathbb{R}^d}\int\limits_0^t\Bigg(\sigma^2(s,x)ds\Bigg)dx\le 8\int\limits_0^{T_1}\|\sigma(s,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}ds.$$

Let denote

$$S(T_1)=8\mathbf{E}\|\phi(0,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}+16\Bigg(\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}l^2(0,x,\xi)d\xi dx\Bigg)\mathbf{E}\|\phi(-r,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}$$

$$+20\int\limits_{\mathbb{R}^d}\Bigg(\int\limits_{\mathbb{R}^d}\chi(x,\xi)d\xi\Bigg)^2dx+4\Bigg(\sup_{0\le t\le T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}l^2(t,x,\xi)d\xi dx\Bigg)\sup_{-r\le t\le 0}\mathbf{E}\|\phi(t,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}$$

$$+16T_1^2\sup_{0\le t\le T_1}\int\limits_{\mathbb{R}^d}\eta^2(t,x)dx+16rL^2T_1\sup_{-r\le t\le 0}\mathbf{E}\|\phi(t,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}+8\int\limits_0^{T_1}\|\sigma(t,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}dt<\infty.$$

Then from (10) we obtain

$$\sup_{0\le t\le T_1}\mathbf{E}\|u^{(n)}(t,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}\le S(T_1)+4\Bigg(\sup_{0\le t\le T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}l^2(t,x,\xi)d\xi dx\Bigg)$$
$$\times\sup_{0\le t\le T_1}\mathbf{E}\|u^{(n-1)}(t,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}+16L^2T_1\int\limits_0^{T_1}\mathbf{E}\|u^{(n-1)}(t,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}dt. \tag{12}$$

If $n=1$, then from (12) we have

$$\sup_{0\le t\le T_1}\mathbf{E}\|u^{(1)}(t,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}\le S(T_1)+4\Bigg(\sup_{0\le t\le T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}l^2(t,x,\xi)d\xi dx\Bigg)\mathbf{E}\|\phi(0,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}$$
$$+16L^2T_1\int\limits_0^{T_1}\mathbf{E}\|\phi(0,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}dt.$$

For an arbitrary $n\in\{2,3,\,\dots\}$ we obtain

$$\sup_{0\le t\le T_1}\mathbf{E}\|u^{(n)}(t,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}\le S(T_1)\Bigg[1+4\sup_{0\le t\le T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}l^2(t,x,\xi)d\xi dx+\dots$$

,

$$+\left(4\sup_{0\le t\le T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx\right)^{n-1}\Bigg] + 16L^2T_1\int\limits_0^{T_1} S(T_1)\Bigg[1+4\sup_{0\le t\le T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx$$

$$+\ldots+\left(4\sup_{0\le t\le T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx\right)^{n-2}\Bigg]ds + 16L^2T_1\int\limits_0^{T_1}\big(16L^2T_1\big(T_1-s\big)\big)S(T_1)$$

$$\times\Bigg[1+\ldots+\left(4\sup_{0\le t\le T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx\right)^{n-3}\Bigg]ds$$

$$+\ldots+16L^2T_1\int\limits_0^{T_1}\frac{\big(16L^2T_1\big(T_1-s\big)\big)^{n-3}}{\big(n-3\big)!}S(T_1)\Bigg[1+4\sup_{0\le t\le T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx\Bigg]ds$$

$$+\left(4\sup_{0\le t\le T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx\right)^{n-1}\Bigg[\left(4\sup_{0\le t\le T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx\right)\mathbf{E}\|\phi(0,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}$$

$$+16L^2T_1\int\limits_0^{T_1}\mathbf{E}\|\phi(0,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}ds\Bigg] + 16L^2T_1\left(4\sup_{0\le t\le T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx\right)^{n-2}$$

$$\times\int\limits_0^{T_1}\Bigg[\left(4\sup_{0\le t\le T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx\right)\mathbf{E}\|\phi(0,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)} + 16L^2T_1\int\limits_0^{T_1}\mathbf{E}\|\phi(0,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}ds\Bigg]d\tau$$

$$\big(16L^2T_1\big)^2\left(4\sup_{0\le t\le T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx\right)^{n-3}\int\limits_0^{T_1}(T_1-\tau)$$

$$\times\Bigg[\left(4\sup_{0\le t\le T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx\right)\mathbf{E}\|\phi(0,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)} + 16L^2T_1\int\limits_0^{T_1}\mathbf{E}\|\phi(0,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}ds\Bigg]d\tau$$

$$+\ldots+\big(16L^2T_1\big)^{n-3}\left(4\sup_{0\le t\le T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx\right)^2\int\limits_0^{T_1}\frac{(T_1-\tau)^{n-4}}{(n-4)!}$$

$$\times\Bigg[\left(4\sup_{0\le t\le T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx\right)\mathbf{E}\|\phi(0,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)} + 16L^2T_1\int\limits_0^{T_1}\mathbf{E}\|\phi(0,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}ds\Bigg]d\tau$$

$$+\big(16L^2T_1\big)^{n-2}\left(4\sup_{0\le t\le T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx\right)\int\limits_0^{T_1}\frac{(T_1-\tau)^{n-3}}{(n-3)!}$$

$$\times\Bigg[\left(4\sup_{0\le t\le T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx\right)\mathbf{E}\|\phi(0,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)} + 16L^2T_1\int\limits_0^{T_1}\mathbf{E}\|\phi(0,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}ds\Bigg]d\tau$$

$$+\left(4\sup_{0\le t\le T}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx\right)^{n-2}16L^2T_1\int_0^{T_1}C(T_1)ds+\left(4\sup_{0\le t\le T}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx\right)^{n-3}$$

$$\times 16L^2T_1\int_0^{T_1}\big(16L^2T_1(T_1-s)\big)C(T_1)ds+\ldots+\left(4\sup_{0\le t\le T}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx\right)^{2}16L^2T_1$$

$$\times\int_0^{T_1}\frac{\big(16L^2T_1(T_1-s)\big)^{n-4}}{(n-4)!}C(T_1)ds+\left(4\sup_{0\le t\le T}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx\right)16L^2T_1$$

$$\times\int_0^{T_1}\frac{\big(16L^2T_1(T_1-s)\big)^{n-3}}{(n-3)!}C(T_1)ds+16L^2T_1\int_0^{T_1}\frac{\big(16L^2T_1(T_1-s)\big)^{n-2}}{(n-2)!}C(T_1)ds$$

$$+\left(4\sup_{0\le t\le T}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx\right)^{n-2}16L^2T_1\int_0^{T_1}\big(16L^2T_1(T_1-s)\big)\mathbf{E}\|\phi(0,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}ds$$

$$+\left(4\sup_{0\le t\le T}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx\right)^{n-3}16L^2T_1\int_0^{T_1}\frac{\big(16L^2T_1(T_1-s)\big)^{2}}{2}\mathbf{E}\|\phi(0,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}ds$$

$$+\ldots+\left(4\sup_{0\le t\le T}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx\right)^{2}16L^2T_1\int_0^{T_1}\frac{\big(16L^2T_1(T_1-s)\big)^{n-3}}{(n-3)!}\mathbf{E}\|\phi(0,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}ds$$

$$+\left(4\sup_{0\le t\le T}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx\right)16L^2T_1\int_0^{T_1}\frac{\big(16L^2T_1(T_1-s)\big)^{n-2}}{(n-2)!}\mathbf{E}\|\phi(0,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}ds$$

$$+16L^2T_1\int_0^{T_1}\frac{\big(16L^2T_1(T_1-s)\big)^{n-1}}{(n-1)!}\mathbf{E}\|\phi(0,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}ds, \tag{13}$$

where $C(T_1)=S(T_1)+\left(4\sup_{0\le t\le T}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx\right)\mathbf{E}\|\phi(0,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}$. It is easy to see that if T_1 is small enough and assumption (3) is true, then the the right-hand of (13) is not more than

$$\frac{S(T_1)}{1-4\sup\limits_{0\le t\le T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx}+\frac{16L^2T_1\cdot S(T_1)\cdot\int\limits_0^{T_1}\exp\{16L^2T_1(T_1-s)\}ds}{1-4\sup\limits_{0\le t\le T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx}$$

$$+\frac{C(T_1)\cdot\exp\{16L^2T_1^2\}}{1-4\sup\limits_{0\le t\le T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx}+\frac{16L^2T_1\cdot\int\limits_0^{T_1}\exp\{16L^2T_1(T_1-s)\}ds}{1-4\sup\limits_{0\le t\le T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx}\mathbf{E}\|\phi(0,\cdot)\|^2_{L_2(\mathbb{R}^d)}$$

$$=\frac{\big(S(T_1)+C(T_1)\big)\cdot\exp\{16L^2T_1^2\}+\big(\exp\{16L^2T_1^2\}-1\big)\mathbf{E}\|\phi(0,\cdot)\|^2_{L_2(\mathbb{R}^d)}}{1-4\sup\limits_{0\le t\le T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx}>0.$$

Thus there exists $c(T_1)>0$ such that for an arbitrary $n\in\{1,2,\ldots\}$

$$\sup_{0\le t\le T_1}\mathbf{E}\|u^{(n)}(t,\cdot)\|^2_{L_2(\mathbb{R}^d)}\le c(T_1). \tag{14}$$

1.2 Second let us prove that $\big(u^{(n)}(t,\cdot), n\in\{1,2,\ldots\}\big)$, $0<t\le T_1$, is convergent. In order to do it we estimate $\sup\limits_{0\le t\le T_1}\mathbf{E}\|u^{(n+1)}(t,\cdot)-u^{(n)}(t,\cdot)\|^2_{L_2(\mathbb{R}^d)}$, $n\in\{0,1,\ldots\}$.

If $n=0$, then we obtain, taking into account estimate (14),

$$\sup_{0\le t\le T_1}\mathbf{E}\|u^{(1)}(t,\cdot)-u^{(0)}(t,\cdot)\|^2_{L_2(\mathbb{R}^d)}\le 2\sup_{0\le t\le T_1}\mathbf{E}\|u^{(1)}(t,\cdot)\|^2_{L_2(\mathbb{R}^d)}$$
$$+2\mathbf{E}\|\phi(0,\cdot)\|^2_{L_2(\mathbb{R}^d)}<\infty.|$$

If $n\in\{1,2,\ldots\}$, then we obtain, taking into account estimates from **1.1**,

$$\sup_{0\le t\le T_1}\mathbf{E}\|u^{(n+1)}(t,\cdot)-u^{(n)}(t,\cdot)\|^2_{L_2(\mathbb{R}^d)}\le 2\left(\sup_{0\le t\le T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx\right)$$
$$\times\sup_{0\le t\le T_1}\mathbf{E}\|u^{(n-1)}(t,\cdot)-u^{(n)}(t,\cdot)\|^2_{L_2(\mathbb{R}^d)}+2L^2T_1\int\limits_0^{T_1}\mathbf{E}\|u^{(n-1)}(s,\cdot)-u^{(n)}(s,\cdot)\|^2_{L_2(\mathbb{R}^d)}ds$$
$$\le\left(2\sup_{0\le t\le T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx+2L^2T_1^2\right)\sup_{0\le t\le T_1}\mathbf{E}\|u^{(n-1)}(t,\cdot)-u^{(n)}(t,\cdot)\|^2_{L_2(\mathbb{R}^d)}\le\ldots$$
$$\le\left(2\sup_{0\le t\le T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx+2L^2T_1^2\right)^n\sup_{0\le t\le T_1}\mathbf{E}\|u^{(0)}(t,\cdot)-u^{(1)}(t,\cdot)\|^2_{L_2(\mathbb{R}^d)}.$$

Due to assumption (3) and choose of small T_1, $\sup\limits_{0\le t\le T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx+L^2T_1^2<\frac{1}{2}$, therefore $\left(2\sup\limits_{0\le t\le T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx+2L^2T_1^2\right)^n<1$ and we conclude

$$\lim_{m,n\to\infty}\sup_{0\le t\le T_1}\sqrt{\mathbf{E}\|u^{(n)}(t,\,\cdot\,)-u^{(m)}(t,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}}$$
$$=\lim_{m,n\to\infty}\sup_{0\le t\le T_1}\sqrt{\mathbf{E}\left\|\sum_{i=m-1}^{n-1}\left(u^{(i+1)}(t,\,\cdot\,)-u^{(i)}(t,\,\cdot\,)\right)\right\|^2_{L_2(\mathbb{R}^d)}}$$
$$\le\lim_{m,n\to\infty}\sum_{i=m-1}^{n-1}\sqrt{\sup_{0\le t\le T_1}\mathbf{E}\|u^{(i+1)}(t,\,\cdot\,)-u^{(i)}(t,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}}$$
$$\le\sqrt{\sup_{0\le t\le T_1}\mathbf{E}\|u^{(1)}(t,\,\cdot\,)-u^{(0)}(t,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}}$$
$$\times\lim_{m,n\to\infty}\sum_{i=m-1}^{n-1}\sqrt{\left(2\sup_{0\le t\le T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}l^2(t,x,\xi)d\xi dx+2L^2T_1^2\right)^i}=0.$$

Thus, $\left(u^{(n)}(t,\,\cdot\,),n\in\{1,2,\dots\}\right)$, $0<t\le T_1$, is a Cauchy sequence. Consequently, there is a limiting function $u(t,\,\cdot\,)\in L_2(\mathbb{R}^d)$, $0<t\le T_1$, such that

$$\lim_{n\to\infty}\sup_{0\le t\le T_1}\mathbf{E}\|u^{(n)}(t,\,\cdot\,)-u(t,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}=0. \tag{15}$$

From (14), it follows from Fatou's Lemma that

$$\sup_{0\le t\le T_1}\mathbf{E}\|u(t,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}\le c(T_1).$$

The function u is $\mathcal{F}_t$-measurable as a limit of $\mathcal{F}_t$-measurable functions.

1.3 Next we show that $u(t,\,\cdot\,)$, $0<t\le T_1$, solves the equation (7). To this end, we need to pass to the limit in the identity (9). Taking into account (15), we have

$$\lim_{n\to\infty}\sup_{0\le t\le T_1}\mathbf{E}\left\|\int\limits_{\mathbb{R}^d}\left(b(t,\,\cdot\,,u^{(n-1)}(\alpha(t),\xi),\xi)-b(t,\,\cdot\,,u(\alpha(t),\xi),\xi)\right)d\xi\right\|^2_{L_2(\mathbb{R}^d)}$$
$$\le\left(\sup_{0\le t\le T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}l^2(t,x,\xi)d\xi dx\right)\lim_{n\to\infty}\sup_{0\le t\le T_1}\mathbf{E}\|u^{(n-1)}(t,\,\cdot\,)-u(t,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}=0,$$

$$\lim_{n\to\infty}\sup_{0\le t\le T_1}\mathbf{E}\left\|\int\limits_0^t\left(f(s,u^{(n-1)}(\alpha(s),\,\cdot\,),\,\cdot\,)-f(s,u(\alpha(s),\,\cdot\,),\,\cdot\,)\right)ds\right\|^2_{L_2(\mathbb{R}^d)}$$
$$\le L^2T_1\lim_{n\to\infty}\int\limits_{-r}^{\alpha(T_1)}\mathbf{E}\|u^{(n-1)}(s,\,\cdot\,)-u(s,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}ds$$
$$\le L^2T_1^2\lim_{n\to\infty}\sup_{0\le t\le T_1}\mathbf{E}\|u^{(n-1)}(t,\,\cdot\,)-u(t,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}=0.$$

,

Therefore, passing to the limit in (9), we have

$$u(t,\cdot)=\phi(0,\,\cdot\,)+\int\limits_{\mathbb{R}^d}b(0,\,\cdot\,,\phi(-r,\xi),\xi)d\xi-\int\limits_{\mathbb{R}^d}b(t,\,\cdot\,,u(\alpha(t),\xi),\xi)d\xi$$
$$+\int\limits_0^t f(s,u(\alpha(s),\,\cdot\,),\,\cdot\,)ds+\int\limits_0^t\sigma(s,\cdot)d\beta(s),\;0<t\le T_1,$$

— the solution to (7) on $[0,T_1]$. This procedure can be repeated in order to extend the solution to the entire interval $[0,T]$ in finitely many steps, thereby completing the proof.

Proof of the theorem 2. Let prove the desired result under the hypothesis *M1*. From now on, suppose that x is fixed.

2.1 Let u_2 solve the problem

$$d\Bigg(u_2(t,\,\cdot\,)+\int\limits_{\mathbb{R}^d}b_2(t,\,\cdot\,,u_2(\alpha(t),\xi),\xi)d\xi\Bigg)=f_2(t,u_2(\alpha(t),\,\cdot\,),\,\cdot\,)dt+\sigma(t,\,\cdot\,)d\beta(t),\;0<t\le T,$$
$$u_2(t,\,\cdot\,)=\phi_2(t,\,\cdot\,),\;\;-r\le t\le 0,$$

i.e. satisfy the following identities

$$\Bigg(u_2(t,\,\cdot\,)+\int\limits_{\mathbb{R}^d}b_2(t,\,\cdot\,,u_2(\alpha(t),\xi),\xi)d\xi\Bigg)-\Bigg(u_2(0,\,\cdot\,)+\int\limits_{\mathbb{R}^d}b_2(0,\,\cdot\,,u_2(\alpha(0),\xi),\xi)d\xi\Bigg)$$
$$=\int\limits_0^t f_2(s,u_2(\alpha(s),\,\cdot\,),\,\cdot\,)ds+\int\limits_0^t\sigma(s,\,\cdot\,)d\beta(s),\;0<t\le T,\tag{16}$$

$$u_2(t,\,\cdot\,)=\phi_2(t,\,\cdot\,),\;\;-r\le t\le 0.\tag{16*}$$

Let u_3 solve the problem

$$d\Bigg(u_3(t,\,\cdot\,)+\int\limits_{\mathbb{R}^d}b_1(t,\,\cdot\,,u_2(\alpha(t),\xi),\xi)d\xi\Bigg)=f_1(t,u_2(\alpha(t),\,\cdot\,),\,\cdot\,)dt+\sigma(t,\,\cdot\,)d\beta(t),\;0<t\le T,$$
$$u_3(t,\,\cdot\,)=\phi_1(t,\,\cdot\,),\;\;-r\le t\le 0,$$

i.e. satisfy the following identities

$$\Bigg(u_3(t,\,\cdot\,)+\int\limits_{\mathbb{R}^d}b_1(t,\,\cdot\,,u_2(\alpha(t),\xi),\xi)d\xi\Bigg)-\Bigg(u_3(0,\,\cdot\,)+\int\limits_{\mathbb{R}^d}b_1(0,\,\cdot\,,u_2(\alpha(0),\xi),\xi)d\xi\Bigg)$$
$$=\int\limits_0^t f_1(s,u_2(\alpha(s),\,\cdot\,),\,\cdot\,)ds+\int\limits_0^t\sigma(s,\,\cdot\,)d\beta(s),\;0<t\le T,\tag{17}$$

$$u_3(t,\cdot) = \phi_1(t,\cdot),\ -r \le t \le 0. \tag{ 17*}$$

Subtracting (17) – (17*) from (16) – (16*), we obtain

$$\big(u_2(t,\cdot) - u_3(t,\cdot)\big) + \underbrace{\int\limits_{\mathbb{R}^d} \big(b_2(t,\cdot,u_2(\alpha(t),\xi),\xi)d\xi - b_1(t,\cdot,u_2(\alpha(t),\xi),\xi)\big)d\xi}_{\geqslant 0}$$

$$+ \underbrace{\big(u_3(0,\cdot) - u_2(0,\cdot)\big)}_{\geqslant 0} + \underbrace{\int\limits_{\mathbb{R}^d} \big(b_1(0,\cdot,u_2(\alpha(0),\xi),\xi)d\xi - b_2(0,\cdot,u_2(\alpha(0),\xi),\xi)\big)d\xi}_{=0}$$

$$= \underbrace{\int\limits_0^t \big(f_2(s,u_2(\alpha(s),\cdot),\cdot) - f_1(s,u_2(\alpha(s),\cdot),\cdot)\big)ds}_{\leq 0},\ 0 < t \le T,$$

$$u_2(t,\cdot) - u_3(t,\cdot) = \underbrace{\phi_2(t,\cdot) - \phi_1(t,\cdot)}_{\leq 0},\ -r \le t \le 0,$$

therefore $u_2 \le u_3$ with probability one.

Now let consider u_4 — a solution to

$$d\left(u_4(t,\cdot) + \int\limits_{\mathbb{R}^d} b_1(t,\cdot,u_3(\alpha(t),\xi),\xi)d\xi\right) = f_1(t,u_3(\alpha(t),\cdot),\cdot)dt + \sigma(t,\cdot)d\beta(t),\ 0 < t \le T,$$

$$u_3(t,\cdot) = \phi_1(t,\cdot),\ -r \le t \le 0,$$

i.e. is defined from

$$\left(u_4(t,\cdot) + \int\limits_{\mathbb{R}^d} b_1(t,\cdot,u_3(\alpha(t),\xi),\xi)d\xi\right) - \left(u_4(0,\cdot) + \int\limits_{\mathbb{R}^d} b_1(0,\cdot,u_3(\alpha(0),\xi),\xi)d\xi\right)$$

$$= \int\limits_0^t f_1(s,u_3(\alpha(s),\cdot),\cdot)ds + \int\limits_0^t \sigma(s,\cdot)d\beta(s),\ 0 < t \le T, \tag{18}$$

$$u_4(t,\cdot) = \phi_1(t,\cdot),\ -r \le t \le 0. \tag{ 18*}$$

,

Subtracting (18) – (18*) from (17) – (17*), we conclude

$$\big(u_3(t,\cdot)-u_4(t,\cdot)\big)+\underbrace{\int\limits_{\mathbb{R}^d}\big(b_1(t,\cdot,u_2(\alpha(t),\xi),\xi)d\xi-b_1(t,\cdot,u_3(\alpha(t),\xi),\xi)\big)d\xi}_{\geqslant 0}$$

$$+\underbrace{\big(u_4(0,\cdot)-u_3(0,\cdot)\big)}_{=0}+\underbrace{\int\limits_{\mathbb{R}^d}\big(b_1(0,\cdot,u_3(\alpha(0),\xi),\xi)d\xi-b_1(0,\cdot,u_2(\alpha(0),\xi),\xi)\big)d\xi}_{=0}$$

$$=\underbrace{\int\limits_0^t\big(f_1(s,u_2(\alpha(s),\cdot),\cdot)-f_1(s,u_3(\alpha(s),\cdot),\cdot)\big)ds}_{\leq 0},\ 0<t\leq T,$$

$$u_3(t,\cdot)-u_4(t,\cdot)=\phi_1(t,\cdot)-\phi_1(t,\cdot)=0,\ \ -r\leq t\leq 0,$$

therefore $u_3\leq u_4$ with probability one. Continuing in a similar way, one obtain a sequence $\big(u_n,n\in\{2,3,\ldots\}\big)$, fulfilling

$$u_2\leq u_3\leq u_4\leq\ldots\leq u_n\leq\ldots,$$

where u_n, $n\in\{5,6,\ldots\}$, is defined as

$$\left(u_n(t,\cdot)+\int\limits_{\mathbb{R}^d}b_1(t,\cdot,u_{n-1}(\alpha(t),\xi),\xi)d\xi\right)-\left(u_n(0,\cdot)+\int\limits_{\mathbb{R}^d}b_1(0,\cdot,u_{n-1}(\alpha(0),\xi),\xi)d\xi\right)$$

$$=\int\limits_0^t f_1(s,u_{n-1}(\alpha(s),\cdot),\cdot)ds+\int\limits_0^t\sigma(s,\cdot)d\beta(s),\ 0<t\leq T, \tag{19}$$

$$u_n(t,\cdot)=\phi_1(t,\cdot),\ \ -r\leq t\leq 0. \tag{19*}$$

2.2 Hereafter we argue in a similar way as in the proof of theorem 1. We establish that $\big(u_n,n\in\{2,3,\ldots\}\big)$ is convergent. In order to do it, we prove that

$$\lim_{n\to\infty}\sup_{0\leq t\leq T}\mathbf{E}\|u_n(t,\cdot)-u_1(t,\cdot)\|^2_{L_2(\mathbb{R}^d)}=0,$$

where u_1 is defined from

$$\left(u_1(t,\cdot)+\int\limits_{\mathbb{R}^d}b_1(t,\cdot,u_1(\alpha(t),\xi),\xi)d\xi\right)-\left(u_1(0,\cdot)+\int\limits_{\mathbb{R}^d}b_1(0,\cdot,u_1(\alpha(0),\xi),\xi)d\xi\right)$$

$$=\int\limits_0^t f_1(s,u_1(\alpha(s),\cdot),\cdot)ds+\int\limits_0^t\sigma(s,\cdot)d\beta(s),\ 0<t\leq T, \tag{20}$$

$$u_1(t,\cdot)=\phi_1(t,\cdot),\ \ -r\leq t\leq 0. \tag{20*}$$

It follows from the proof of theorem 1 that there exists a constant $c(T)>0$ such that $\sup\limits_{0\leq t\leq T}\mathbf{E}\|u_2(t,\cdot)\|^2_{L_2(\mathbb{R}^d)}\leq c(T)$ and $\sup\limits_{0\leq t\leq T}\mathbf{E}\|u_n(t,\cdot)\|^2_{L_2(\mathbb{R}^d)}\leq c(T)$ for $n\in\{3,4,\ldots\}$. The rest of the proof is similar to the case of theorem 1.

4 Results and discussion

As mentioned in the introduction, comparison theorems play an important role in the study of solutions with non-Lipschitz coefficients, in the study of the behavior of solutions at infinity, for optimal control of stochastic systems ([1],[2],[4],[5], [7]–[9], [11]–[25]). However, for equations with a delay of the neutral type, such studies have not been carried out before. This is due to the fact that lag is among the stochastic derivative, and therefore it is impossible to apply the classical Ito formula of differentiated functioning. Namely, on the application of Ito's formula, the proof of the classical comparison theorems is constructed.

5 Conclusion

Thus, the paper considers the existence, uniqueness and comparison theorems for stochastic functional-differential equations of neutral type with variable delay. When obtaining these results, the methods of stochastic and functional analysis were used, namely, the principle of compressed mappings, monotonicity methods, coupling method and others. Using these methods, we obtain local and global theorems on the existence and uniqueness of initial problems for stochastic functional differential equations with variable delay of a neutral type, as well as theorems for comparing two solutions. In the future, this method will allow us to obtain similar results for equations with unbounded operators, in particular for stochastic functional differential equations of the neutral type of partial differential equations.

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