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## ON CONTINUOUS SOLUTIONS OF THE MODEL HOMOGENEOUS BELTRAMI EQUATION WITH A POLAR SINGULARITY

This paper consists of two parts. The first part is devoted to the study of the Beltrami model equation with a polar singularity in a circle centered at the origin, with a cut along the positive semi-axis. The coefficients of the equation have a first-order pole at the origin and do not even belong to the class  $L_2(G)$ . For this reason, despite its specific form, this equation is not covered by the analytical apparatus of I.N. Vekua [1] and needs to be independently studied. Using the technique developed by A.B. Tungatarov [2] in combination with the methods of the theory of functions of a complex variable [3] and functional analysis [4], manifolds of continuous solutions of the Beltrami model equation with a polar singularity are obtained. The theory of these equations has numerous applications in mechanics and physics. In the second part of the article, the coefficients of the equation are chosen so that the resulting solutions are continuous in a circle without a cut [5]. These results can be used in the theory of infinitesimal bendings of surfaces of positive curvature with a flat point and in constructing a conjugate isometric coordinate system on a surface of positive curvature with a planar point [6].

**Key words:** Beltrami equation, equation with a polar singularity.

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### Полярлық ерекшелігі бар модельді Бельтрами теңдеуінің үзіліссіз шешімдері

Қарастырып отырған еңбек екі бөлімнен тұрады. Бірінші бөлімі центрі координаталар бас нүктесінде орналасқан оң жақ жарты осьсіз дөңгелекте берілген полярлық ерекшелігі бар модельді Бельтрами теңдеуін зерттеуге арналған. Қарастырылған теңдеудің коэффициенттерінің координаталар бас нүктесінде бірінші ретті полюстері бар және олар  $L_2(G)$  класында жатпайды. Сондықтан И.Н.Векуаның [1] аналитикалық аппаратымен қамтылмаған және жеке зерттеуді қажет етеді. А.Б. Тунгатаров [2] жасаған күрделі әдісті [3] және функционалдық талдаудың [4] теориясы әдістерімен үйлестіре отырып, Белтрами моделді теңдеуінің полярлық ерекшелігі бар үздіксіз шешімдерінің түрлері алынды. Бұл теңдеулер теориясының механика мен физикада көптеген қолданыстары бар. Жұмыстың екінші бөлімінде теңдеудің шешімі қарастырылған дөңгелекте үзіліссіз болатындай етіп коэффициенттер таңдалады [5]. Бұл нәтижелерді тығыздық нүктесі бар қисықтығы оң ақырсыз аз илетін беттер теориясында және тығыздық нүктесі бар қисықтығы оң беттерде изометриялы түйіндес координаттарды құрастыру үшін қолдануға болады [6].

**Түйін сөздер:** Бельтрами теңдеуі, полярлық ерекшелігі бар теңдеу.

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### О непрерывных решениях модельного однородного уравнения Бельтрами с полярной особенностью

Настоящая работа состоит из двух частей. Первая часть посвящена исследованию модельного уравнения Бельтрами с полярной особенностью в круге с центром в начале координат, с разрезом вдоль положительной полуоси. Коэффициенты рассматриваемого уравнения имеют полюс первого порядка в начальной точке координат и не принадлежат даже классу  $L_2(G)$ . По этой причине, несмотря на свой специфический вид это уравнение не охватывается аналитическим аппаратом И.Н. Векуа [1] и нуждается в самостоятельном исследовании. Используя методику разработанной А.Б.Тунгатаровым [2] в сочетании с методами теории функций комплексного переменного [3] и функционального анализа [4] получены многообразия непрерывных решений модельного уравнения Бельтрами с полярной особенностью. Теория этих уравнений имеет многочисленные приложения в механике и физике. Во второй части статьи возникшие произвольные постоянные подобраны так, чтобы построенные решения были непрерывны в круге без разреза [5]. Эти результаты могут быть использованы в теории бесконечно малых изгибов поверхностей положительной кривизны с точкой уплощения и при построении сопряженно изометрической системы координат на поверхности положительной кривизны с точкой уплощения [6].

**Ключевые слова:** уравнение Бельтрами, уравнение с полярной особенностью.

## 1 Introduction and review of literature

The fundamentals of the theory of generalized analytic functions (representations of the first and second kind, a generalized Cauchy formula, expansions into generalized Taylor, Laurent, and other series, as well as the theory of corresponding boundary value problems), with which the present work is closely related, were constructed by the famous mathematician I.N. Vekua [7] - [9] in the case when the coefficients of elliptic systems are summable to a power of more than two. N.K. Blied [10, 11] extended the theory of generalized analytic functions to cases where the coefficients and free terms belong to the fractional spaces of O.V. Besov. In the work of M.O. Otelbaev and K.N. Ospanov [12], the generalized Cauchy-Riemann system from the space obtained by the completion of infinitely smooth functions is investigated. A.B. Tungatarov [13] found in explicit form the right inverse operator for the Beltrami differential operator  $\bar{\partial} - \mu\partial$  (an analog of the well-known operator  $T$  from [1]) for the case  $\mu(z) = \frac{z}{\bar{z}}$ ,  $0 \leq \beta < 1$ . The solutions of the corresponding Beltrami equation are called  $\beta$ -analytic functions. Ricardo Abreu-Blya, Juan Bori-Reyes, Dixan Peña-Peña and Jean-Maria Villiers [14]-[16] investigated the solvability of analogues of the Riemann boundary value problem and a number of related questions for such functions.

Let  $R > 0$  and  $G = \{z = re^{i\varphi} : 0 \leq r < R, 0 \leq \varphi \leq 2\pi\}$ . In the area  $G$  we consider equation of the form

$$\partial_{\bar{z}}V - \beta e^{2i\varphi} \partial_z V + \frac{a(\varphi)}{2\bar{z}}V + \frac{b(\varphi)}{2\bar{z}}\bar{V} = 0, \quad (1)$$

where  $0 \leq \beta < 1$  is an ellipticity condition,

$$a(\varphi), b(\varphi) \in C[0, 2\pi], \quad a(\varphi + 2\pi) = a(\varphi), \quad b(\varphi + 2\pi) = b(\varphi).$$

The equation (1) if  $\beta = 0$ ,  $a(\varphi) = 0$  will be used in the study of infinitesimal bendings of surfaces of positive curvature with a flattening point, in the neighborhood of which the surface has a special structure [6], [17]. The coefficients of the equation (1) do not even belong to the class  $L_2(G)$ . Therefore, using the known methods of the theory of generalized analytic functions [1], [18], it is not possible to prove the existence of continuous solutions to this

equation. The theory of the equation (1) for  $\beta = 0$ ,  $a(\varphi) = 0$ ,  $b(\varphi) = \lambda \exp(ik\varphi)$ , where  $\lambda$  is an arbitrary complex number and  $k$  is an integer number, is constructed in [19], [20]. A variety of continuous solutions of the equation (1) was obtained in [21], [22]; the boundary value problem for the jump for one particular case of the Beltrami equation was considered in [23].

## 2 Material and methods

### 2.1 Beltrami model homogeneous equation with a polar singularity

We proceed to solve the equation (1).

Equation (1) in the polar coordinate system is written as

$$(1 - \beta) \frac{\partial V}{\partial r} + \frac{i}{r} (1 + \beta) \frac{\partial V}{\partial \varphi} + \frac{1}{r} (a(\varphi)V + b(\varphi)\bar{V}) = 0$$

Finding continuous solutions of the last equation by the method of separation of variables

$$V(r, \varphi) = T(r) \cdot \Psi(\varphi), \quad (2)$$

where  $\overline{T(r)} = T(r)$ ,  $T(r) \in C[0, R] \cap C^1(0, R]$ ,  $\Psi(\varphi) \in C^1[0, 2\pi]$ , we get the following system:

$$(1 - \beta)rT'(r) - \lambda T(r) = 0, \quad (3)$$

$$\Psi'(\varphi) - \frac{i}{1 + \beta} (a(\varphi) + \lambda)\Psi(\varphi) - \frac{i}{1 + \beta} b(\varphi)\overline{\Psi(\varphi)} = 0, \quad (4)$$

where  $\lambda > 0$  is a real parameter.

The solution of the equation (3) is the function  $T(r) = A \cdot r^{\frac{\lambda}{1 - \beta}}$ , where  $A$  is a real number.

We seek the solution of the equation (4) in the form

$$\Psi(\varphi) = P_\lambda(\varphi) \exp\left(\frac{i}{1 + \beta} \left(\lambda \cdot \varphi + \int_0^\varphi a(\gamma) d\gamma\right)\right)$$

Substituting the function  $\Psi(\varphi)$  into (4), we obtain the equation for  $P_\lambda(\varphi)$ :

$$P'_\lambda(\varphi) - A_\lambda(\varphi)\overline{P_\lambda(\varphi)} = 0, \quad (5)$$

where

$$A_\lambda(\varphi) = \frac{i}{1 + \beta} b(\varphi) \exp\left(-\frac{2i}{1 + \beta} (\lambda\varphi + \operatorname{Re}B(\varphi))\right), \quad B(\varphi) = \int_0^\varphi a(\gamma) d\gamma.$$

Thus, the solution of the equation (1) has the following form

$$V_\lambda(r, \varphi) = r^{\frac{\lambda}{1-\beta}} P_\lambda(\varphi) \exp\left(\frac{i}{1+\beta}(\lambda\varphi + B(\varphi))\right), \quad (6)$$

where  $P_\lambda(\varphi)$  is the solution of the equation (5).

Now let's start solving the equation (5).

Integrating the equation (5) we obtain

$$P_\lambda(\varphi) = \int_0^\varphi A_\lambda(\gamma) \overline{P_\lambda(\gamma)} d\gamma + c_\lambda, \quad (7)$$

where  $c_\lambda$  is an arbitrary complex number.

Hence

$$\int_0^\varphi A_\lambda(\gamma) \overline{P_\lambda(\gamma)} d\gamma = \int_0^\varphi A_\lambda(\gamma) \int_0^\gamma \overline{A_\lambda(\gamma_1)} P_\lambda(\gamma_1) d\gamma_1 d\gamma + \overline{c_\lambda} \int_0^\varphi A_\lambda(\gamma) d\gamma \quad (8)$$

Using (8) and (7) we obtain the following equation

$$P_\lambda(\varphi) = (B_\lambda P_\lambda)(\varphi) + \overline{c_\lambda} I_{\lambda,1}(\varphi) + c_\lambda, \quad (9)$$

where

$$(B_\lambda f)(\varphi) = \int_0^\varphi A_\lambda(\gamma) \int_0^\gamma \overline{A_\lambda(\gamma_1)} f(\gamma_1) d\gamma_1 d\gamma, \quad I_{\lambda,1}(\varphi) = \int_0^\varphi A_\lambda(\gamma) d\gamma.$$

Further we use operators of the form

$$(B_\lambda^2 f)(\varphi) = (B_\lambda(B_\lambda f)(\varphi))(\varphi), \quad (B_\lambda^k f)(\varphi) = (B_\lambda(B_\lambda^{k-1} f)(\varphi))(\varphi),$$

$$I_{\lambda,k}(\varphi) = \int_0^\varphi A_\lambda(\gamma) \overline{I_{\lambda,k-1}(\gamma)} d\gamma, \quad (k = \overline{2, \infty}).$$

The following easily verifiable relations hold for these operators

$$(B_\lambda I_{\lambda,k})(\varphi)(\varphi) = I_{\lambda,k+2}(\varphi), \quad (B_\lambda c_\lambda)(\varphi) = c_\lambda I_{\lambda,2}(\varphi), \quad (10)$$

$$|B_\lambda^n P_\lambda(\varphi)| \leq \frac{(|A_\lambda|_0 \cdot \varphi)^{2n}}{(2n)!} |P_\lambda(\varphi)|_0, \quad |I_{\lambda,k}(\varphi)| \leq \frac{(|A_\lambda|_0 \cdot \varphi)^k}{k!}, \quad (11)$$

where  $|f|_0 = \|f\|_{C[0,2\pi]}$ .

Applying the operator  $(B_\lambda f)(\varphi)$  to both sides of equality (9) and using (10), we obtain

$$(B_\lambda P_\lambda(\varphi))(\varphi) = (B_\lambda^2 P_\lambda)(\varphi) + \overline{c_\lambda} I_{\lambda,3}(\varphi) + c_\lambda I_{\lambda,2}(\varphi) \quad (12)$$

From (9) and (12) the following equation follows

$$P_\lambda(\varphi) = (B_\lambda^2 P_\lambda)(\varphi) + \overline{c_\lambda}(I_{\lambda,1}(\varphi) + I_{\lambda,3}(\varphi)) + c_\lambda(1 + I_{\lambda,2}(\varphi)) \quad (13)$$

Applying the operator  $(B_\lambda f)(\varphi)$  to both sides of equality (13) we obtain

$$(B_\lambda P_\lambda)(\varphi) = (B_\lambda^3 P_\lambda)(\varphi) + \overline{c_\lambda}(I_{\lambda,3}(\varphi) + I_{\lambda,5}(\varphi)) + c_\lambda(I_{\lambda,2}(\varphi) + I_{\lambda,4}(\varphi)) \quad (14)$$

From (9) and (14) we get

$$P_\lambda(\varphi) = (B_\lambda^3 P_\lambda)(\varphi) + \overline{c_\lambda}(I_{\lambda,1}(\varphi) + I_{\lambda,3}(\varphi) + I_{\lambda,5}(\varphi)) + c_\lambda(1 + I_{\lambda,2}(\varphi) + I_{\lambda,4}(\varphi))$$

Repeating this process, we have

$$P_\lambda(\varphi) = (B_\lambda^n P_\lambda)(\varphi) + \overline{c_\lambda} \sum_{k=1}^n I_{\lambda,2k-1}(\varphi) + c_\lambda \left(1 + \sum_{k=1}^{n-1} I_{\lambda,2k}(\varphi)\right) \quad (15)$$

Passing to the limit in (15) as  $n \rightarrow \infty$ , and by (11) we obtain

$$P_\lambda(\varphi) = \overline{c_\lambda} P_{\lambda,1}(\varphi) + c_\lambda P_{\lambda,2}(\varphi), \quad (16)$$

where  $P_{\lambda,1}(\varphi) = \sum_{k=1}^{\infty} I_{\lambda,2k-1}(\varphi)$ ,  $P_{\lambda,2}(\varphi) = 1 + \sum_{k=1}^{\infty} I_{\lambda,2k}(\varphi)$ .

Using inequalities (11)  $P_{\lambda,1}(\varphi)$ ,  $P_{\lambda,2}(\varphi)$  we obtain the following expressions:

$$P'_{\lambda,1}(\varphi) - A_\lambda(\varphi) \overline{P_{\lambda,2}(\varphi)} = 0, \quad P'_{\lambda,2}(\varphi) - A_\lambda(\varphi) \overline{P_{\lambda,1}(\varphi)} = 0, \quad (17)$$

$$|P_{\lambda,1}(\varphi)| \leq \text{sh}(|A_\lambda|_0 \varphi), \quad |P_{\lambda,2}(\varphi)| \leq \text{ch}(|A_\lambda|_0 \varphi), \quad (18)$$

$$P_{\lambda,2}(\varphi) - 1 = \int_0^\varphi A_\lambda(\gamma) \overline{P_{\lambda,1}(\gamma)} d\gamma, \quad P_{\lambda,1}(\varphi) = \int_0^\varphi A_\lambda(\gamma) \overline{P_{\lambda,2}(\gamma)} d\gamma \quad (19)$$

From the second equation (19) and (17) it follows

$$P_{\lambda,2}(\varphi) - 1 = \overline{P_{\lambda,1}(\varphi)} \sum_{k=1}^n I_{\lambda,2k-1}(\varphi) -$$

$$-P_{\lambda,2}(\varphi) \sum_{k=1}^n \overline{I_{\lambda,2k}(\varphi)} + \int_0^\varphi A_\lambda(\gamma) I_{\lambda,2k-1}(\gamma) \overline{P_{\lambda,1}(\gamma)} d\gamma$$

where  $n \geq 1$  is a whole number.

Passing to the limit in (15) as  $n \rightarrow \infty$  and using the inequalities (11), (18), we get the identity

$$|P_{\lambda,2}(\varphi)|^2 - |P_{\lambda,1}(\varphi)|^2 = 1 \quad (20)$$

It follows from (6) and (16) that any solutions of equation (1) can be found by the formula

$$V_\lambda(r, \varphi) = r^{\frac{\lambda}{1-\beta}} (\overline{c_\lambda} P_{\lambda,1}(\varphi) + c_\lambda P_{\lambda,2}(\varphi)) \times \exp\left(\frac{i}{1+\beta}(\lambda\varphi + B(\varphi))\right) \quad (21)$$

Using the formula (17), we find

$$\partial_{\bar{z}} V = r^{\frac{\lambda}{1-\beta}-1} \exp\left(i\varphi + i\frac{\lambda\varphi + B(\varphi)}{1+\beta}\right) \left(\left(1 + \frac{i}{1+\beta}(\lambda\varphi + a(\varphi))\right) \times\right.$$

$$\left. \times (\overline{c_k} P_{\lambda,1}(\varphi) + c_k P_{\lambda,2}(\varphi))\right) + iA_\lambda(\varphi) \left(c_k \overline{P_{\lambda,1}(\varphi)} + \overline{c_k} P_{\lambda,2}(\varphi)\right)$$

Obviously, the following relation holds  $\partial_{\bar{z}} V \in L_p$ ,  $1 < p < \frac{2(1-\beta)}{1-\lambda-\beta}$ ,  $0 < \lambda + \beta < 1$ .

Similarly, the following relation holds  $\partial_z V \in L_p$ ,  $1 < p < \frac{2(1-\beta)}{1-\lambda-\beta}$ ,  $0 < \lambda + \beta < 1$ .

We use the theorem of I.N. Vekua from [1] : if  $\partial_{\bar{z}} V \in L_p$ ,  $p >$ , then there is  $\partial_z V$  and also belongs to  $L_p$ .

Therefore, by definition we get  $V \in W_p^1(G)$ ,  $1 < p < \frac{2(1-\beta)}{1-\lambda-\beta}$ ,  $0 < \lambda + \beta < 1$ . So the function  $V_\lambda(r, \varphi)$  will be a solution of the equation (1) from the class  $C(G_0) \cap W_p^1(G)$ ,  $1 < p < \frac{2(1-\beta)}{1-\lambda-\beta}$ ,  $0 < \lambda + \beta < 1$  where  $W_p^1(G)$  is the space of S.L. Sobolev from [1] and  $G_0$  is a region of  $G$  with a cut along the semiaxis  $\{z = re^{i\varphi} : r > 0, \varphi = 0\}$ .

Thus, we have obtained the lemma.

**Лемма 1** Equation (1) has the solutions from the class  $C(G_0) \cap W_p^1(G)$ , where  $1 < p < \frac{2(1-\beta)}{1-\lambda-\beta}$ ,  $0 < \lambda + \beta < 1$ . Any such solutions can be found by formula (21).

## 2.2 Construction of continuous solutions of the Beltrami model equation with a polar singularity

Now we proceed to the construction of continuous solutions of the equation (1) in  $G$ . It follows from (21) that in general case we have that  $V_\lambda(r, 0) \neq V_\lambda(r, 2\pi)$ . Therefore, we choose the numbers  $c_\lambda$  and  $\lambda$  from (21) so that the following equality holds

$$V_\lambda(r, 0) = V_\lambda(r, 2\pi). \quad (22)$$

For this purpose, substituting (21) in (22), we obtain

$$\Delta_1(\lambda)\bar{c}_\lambda + \Delta_2(\lambda)c_\lambda = 0, \quad (23)$$

where  $\Delta_1(\lambda) = P_{\lambda,1}(2\pi)$ ,

$$\Delta_2(\lambda) = P_{\lambda,2}(2\pi) - \exp\left(\frac{-2\pi i}{1+\beta}(\lambda+d)\right), d = \frac{B(2\pi)}{2\pi}.$$

Expanding on the real and imaginary parts of a complex number  $c_\lambda$  and expressions  $\Delta_1(\lambda)$ ,  $\Delta_2(\lambda)$  and then substituting them in (23), we obtain the following system of homogeneous equations with respect to unknowns  $c'_\lambda$  and  $c''_\lambda$ , where  $c'_\lambda$  and  $c''_\lambda$  are real and imaginary parts of the complex number  $c_\lambda$  respectively.

$$\begin{aligned} & c'_\lambda \left( \operatorname{Re}P_{\lambda,1}(2\pi) + \operatorname{Re} \left( P_{\lambda,1}(2\pi) - \exp\left(\frac{-2\pi i}{1+\beta}(\lambda+d)\right) \right) \right) + \\ & + c''_\lambda \left( \operatorname{Im}P_{\lambda,1}(2\pi) - \operatorname{Im} \left( P_{\lambda,2}(2\pi) - \exp\left(-\frac{2\pi i}{1+\beta}(\lambda+d)\right) \right) \right) = 0, \\ & c'_\lambda \left( \operatorname{Im}P_{\lambda,1}(2\pi) + \operatorname{Im} \left( P_{\lambda,2}(2\pi) - \exp\left(-\frac{2\pi i}{1+\beta}(\lambda+d)\right) \right) \right) - \\ & - c''_\lambda \left( \operatorname{Re}P_{\lambda,1}(2\pi) - \operatorname{Re} \left( P_{\lambda,2}(2\pi) - \exp\left(-\frac{2\pi i}{1+\beta}(\lambda+d)\right) \right) \right) = 0. \end{aligned}$$

This system is linear and homogeneous with respect to  $c'_\lambda$  and  $c''_\lambda$ . Therefore, it has a nonzero solution if  $|\Delta_2(\lambda)|^2 - |\Delta_1(\lambda)|^2 = 0$ . Thus, equation (23) has a nonzero solution only when  $|\Delta_2(\lambda)| = |\Delta_1(\lambda)|$ . This equality is equivalent to the equation for  $\lambda$ :

$$\operatorname{Re}P_{\lambda,2}(2\pi) \cos\left(\frac{2\pi}{1+\beta}(\lambda+d)\right) - \operatorname{Im}P_{\lambda,2}(2\pi) \sin\left(\frac{2\pi}{1+\beta}(\lambda+d)\right) = 1 \quad (24)$$

Since, by virtue of (20)  $|P_{\lambda,2}(2\pi)| \geq 1$ , then for each integer  $k \geq 0$  there exists a solution of the equation (24)  $\lambda = \lambda_k$  belonging to the segment  $[k, k+1]$ . Let  $k$  be an integer,  $k \leq \lambda_k \leq k+1$  and  $\lambda_k$  be a solution to equation (24). Then  $|\Delta_1(\lambda_k)| = |\Delta_2(\lambda_k)|$  and the equation  $\Delta_1(\lambda_k)\bar{c}_k + \Delta_2(\lambda_k)c_k = 0$  has a nonzero solution  $c_k$ , which is found by the formula

$$c_k = \begin{cases} \alpha_k, & \text{if } \Delta_1(\lambda_k) = 0, \\ \Delta_1(\lambda_k)\alpha_k - \overline{\Delta_2(\lambda_k)\alpha_k}, & \text{if } \Delta_1(\lambda_k) \neq 0 \end{cases} \quad (25)$$

Here  $\alpha_k$  is an arbitrary complex number. Therefore, for each integer  $k$  the function

$$V_{\lambda_k}(r, \varphi) = r^{\frac{\lambda_k}{1-\beta}} (\overline{c_k} P_{\lambda_k,1}(\varphi) + c_k P_{\lambda_k,2}(\varphi)) \times \exp\left(\frac{i}{1+\beta}(\lambda_k \varphi + B(\varphi))\right) \quad (26)$$

will be a solution of the equation (1) from the class  $C(G_0) \cap W_p^1(G)$ ,  $1 < p < \frac{2(1-\beta)}{1-\lambda-\beta}$  and satisfies the condition (22). Thus, we have obtained the theorem.

**Теорема 1** *If  $\lambda_k + \beta \geq 1$ , then for any integer  $k \geq 0$  the equation (1) has solutions from the class  $C^1(\overline{G})$ . These solutions can be found by formulas (26) and (25), where  $\lambda_k$  is the solution of the equation (24) from the interval  $[k, k+1]$ .*

**Теорема 2** *If  $\lambda_0 + \beta < 1$ , then the equation (1) always has solutions from the class  $C(\overline{G}) \cap W_p^1(G)$ , where  $1 < p < \frac{2(1-\beta)}{1-\lambda_0-\beta}$ . These solutions can be found by formulas (26) and (25), where  $\lambda_0$  is the solution of the equation (24) from the interval  $(0; 1)$ .*

### 3 Conclusion

In conclusion, we note that in the article we construct varieties of continuous solutions of the Beltrami equation with a polar singularity in a circle centered at the origin, with a cut along the positive semiaxis. In the second part of the article, the arising arbitrary constants are chosen so that the constructed solutions are continuous in a circle without a cut.

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