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THE CONVOLUTION IN ANISOTROPIC BESOV SPACES

We study the boundedness of the convolution operator in Nikol'skii-Besov anisotropic spaces $B^{\boldsymbol{\alpha}\mathbf{q}}_{\mathbf{p}\boldsymbol{\tau}}$. These spaces are constructed on the basis of anisotropic Lorentz spaces $L_{\mathbf{p}\boldsymbol{\tau}}$, where $\mathbf{p}$ и $\boldsymbol{\tau}$ are vector parameters. The properties of anisotropic Nikol'skii-Besov spaces are investigated. The main goal of the paper is to solve the following problem: let f and g be functions from some classes of the Nikol'skii-Besov space scale. It is necessary to determine which space belongs to their convolution $f * g$. We prooved the inequality of different Nikol'skii metrics for trigonometric polynomials with spectrum in binary blocks in anisotropic Lorentz spaces $L_{\mathbf{p}\boldsymbol{\tau}}$. Conditions are obtained in terms of the corresponding vector parameters $\boldsymbol{\alpha}$, $\mathbf{p}$, $\mathbf{q}$, $\boldsymbol{\tau}$, $\mathbf{r}$, $\boldsymbol{\mu}$, $\boldsymbol{\beta}$, $\boldsymbol{\eta}$, $\mathbf{h}$, $\boldsymbol{\nu}$, $\boldsymbol{\gamma}$, $\boldsymbol{\xi}$, which are necessary and sufficient conditions for embeddings

$$B^{\boldsymbol{\beta}\boldsymbol{\eta}}_{\mathbf{r}\boldsymbol{\mu}} * B^{\boldsymbol{\gamma}\boldsymbol{\xi}}_{\mathbf{h}\boldsymbol{\nu}} \hookrightarrow B^{\boldsymbol{\alpha}\mathbf{q}}_{\mathbf{p}\boldsymbol{\tau}}.$$

This statement is an analogue of O'Neil inequality for Lorentz spaces. In particular, the classical O'Neil inequality follows from the proved results. The obtained criterion is generalized by the results of Burenkov and Batyrov, who considered this problem in Besov spaces with scalar parameters.

Key words: Young-O'Neil inequality, anisotropic Besov spaces, convolution operator.

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Анизотропты Бесов кеңістіктеріндегі үйірткі

Берілген жұмыста $B^{\boldsymbol{\alpha}\mathbf{q}}_{\mathbf{p}\boldsymbol{\tau}}$ анизотропты Никольский-Бесов кеңістіктеріндегі үйірткі операторының шенелуі зерттеледі. Бұл кеңістіктер $L_{\mathbf{p}\boldsymbol{\tau}}$ анизотропты Лоренц кеңістіктерінің негізінде құрылған, мұндағы $\mathbf{p}$ және $\boldsymbol{\tau}$ – векторлық параметрлер. Жұмыстың мақсаты келесі есепті шешу болып табылады: айталық, f және g Никольский-Бесов кеңістіктерінің қандай да бір шкаласынан алынған функциялар болсын. Олардың $f * g$ үйірткісі қандай кеңістікке жататынын анықтау керек. $L_{\mathbf{p}\boldsymbol{\tau}}$ анизотропты Лоренц кеңістіктеріндегі екілік бөлшектенуде спектрлі тригонометриялық көпмүшелерге арналған Никольскийдің әр түрлі метрика теңсіздігі дәлелденді. Сәйкес $\boldsymbol{\alpha}$, $\mathbf{p}$, $\mathbf{q}$, $\boldsymbol{\tau}$, $\mathbf{r}$, $\boldsymbol{\mu}$, $\boldsymbol{\beta}$, $\boldsymbol{\eta}$, $\mathbf{h}$, $\boldsymbol{\nu}$, $\boldsymbol{\gamma}$, $\boldsymbol{\xi}$ векторлық параметрлерінің терминдерінде

$$B^{\boldsymbol{\beta}\boldsymbol{\eta}}_{\mathbf{r}\boldsymbol{\mu}} * B^{\boldsymbol{\gamma}\boldsymbol{\xi}}_{\mathbf{h}\boldsymbol{\nu}} \hookrightarrow B^{\boldsymbol{\alpha}\mathbf{q}}_{\mathbf{p}\boldsymbol{\tau}}.$$

енгізуі үшін қажетті және жеткілікті шарттар алынды. Бұл тұжырым Лоренц кеңістіктері үшін О'Нейл теңсіздігінің аналогы болып табылады. Сонымен қатар, дәлелденген нәтижелерден классикалық О'Нейл теңсіздігі шығады. Алынған критерий Бесов кеңістігінде осы есепті скаляр параметрлермен қарастырған Буренков пен Батыровтың нәтижелерін жалпылайды.

Түйін сөздер: Юнг-О'Нейл теңсіздігі, анизотропты Бесов кеңістіктері, үйірткі операторы.

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Свертка в анизотропных пространствах Бесова

В работе исследуется ограниченность оператора свертки в анизотропных пространствах Никольского-Бесова $B^{\boldsymbol{\alpha}\mathbf{q}}_{\mathbf{p}\boldsymbol{\tau}}$. Данные пространства построены на основе анизотропных пространств Лоренца $L_{\mathbf{p}\boldsymbol{\tau}}$, где $\mathbf{p}$ и $\boldsymbol{\tau}$ векторные параметры. Исследованы свойства анизотропных пространств Никольского-Бесова. Целью работы является решение следующей задачи: пусть f и g функции из некоторых классов шкалы пространств Никольского-Бесова. Нужно определить, какому пространству принадлежит их свертка $f * g$. Доказано неравенство разных метрик Никольского для тригонометрических полиномов со спектром в двоичных пачках в анизотропных пространствах Лоренца $L_{\mathbf{p}\boldsymbol{\tau}}$. Получены условия в терминах соответсвующих векторных параметров $\boldsymbol{\alpha}$, $\mathbf{p}$, $\mathbf{q}$, $\boldsymbol{\tau}$, $\mathbf{r}$, $\boldsymbol{\mu}$, $\boldsymbol{\beta}$, $\boldsymbol{\eta}$, $\mathbf{h}$, $\boldsymbol{\nu}$, $\boldsymbol{\gamma}$, $\boldsymbol{\xi}$, являющихся необходимыми и достаточными условиями для вложений

$$B^{\boldsymbol{\beta}\boldsymbol{\eta}}_{\mathbf{r}\boldsymbol{\mu}} * B^{\boldsymbol{\gamma}\boldsymbol{\xi}}_{\mathbf{h}\boldsymbol{\nu}} \hookrightarrow B^{\boldsymbol{\alpha}\mathbf{q}}_{\mathbf{p}\boldsymbol{\tau}}.$$

Данное утверждение является аналогом неравенства О'Нейла для пространств Лоренца. В частности, из доказанных результатов следует классическое неравенство О'Нейла. Полученный критерий обобщают результаты Буренкова и Батырова, которые рассмотрели данную задачу в пространствах Бесова со скалярными параметрами.

Ключевые слова: неравенство Юнга-О'Нейла, анизотропные пространства Бесова, оператор свертки.

1 Introduction and review of literature

Let I be either a n-dimensional torus $\mathbb{T}^n = [0,1)^n$, or a Euclidean space $\mathbb{R}^n$. Let $f(x)$ and $g(x)$ be determined and measurable functions on I with respect to the n-dimensional Lebesgue measure such that for almost all $x \in I$ there exists an integral

$$\int\limits_I f(x-y)g(y)dy.$$

In this case, it is said that the convolution of these functions is defined

$$(f*g)(x) = \int\limits_I f(x-y)g(y)dy. \tag{1}$$

The classical Young's inequality [1, 199] has the form: suppose

$$1 \le p, r, q \le \infty, \quad \frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{r}. \tag{2}$$

If $f \in L_p(I)$, $g \in L_r(I)$, then almost everywhere in I there exists a convolution $f*g$, belonging to the space $L_q(I)$ and the following inequality holds

$$\|f*g\|_{L_q(I)} \le \|f\|_{L_p(I)}\|g\|_{L_r(I)}. \tag{3}$$

We write this statement in the form of a relation

$$L_p(I) * L_r(I) \hookrightarrow L_q(I).$$

These inequalities play an important role in harmonic analysis and in the theory of partial differential equations [1–3].

If

$$1 < p, r, q < \infty, \quad \frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{r}, \tag{4}$$

then for $g_0(x) = \frac{1}{|x|^{\frac{n}{r}}}$ the inequality holds

$$\|f * g_0\|_{L_q(I)} \leq C\|f\|_{L_p(I)}.$$

This inequality is called the Hardy-Littlewood-Sobolev inequality. It does not follow from Young's inequality, since $\|g_0\|_{L_r(I)} = \infty$. Generalization of inequality (3) obtained by O'Neil [4] (see also [5, 6]).

If (4) is true and $0 < s_1, s_2, s \leq \infty, \frac{1}{s} = \frac{1}{s_1} + \frac{1}{s_2}$, then

$$L_{ps_1} * L_{rs_2} \hookrightarrow L_{qs} \tag{5}$$

and in particular

$$L_p * L_{r\infty} \hookrightarrow L_q, \tag{6}$$

where L_{ps} is Lorentz space.

Note that in relation (5), condition (4) is essential. The limiting cases of the O'Neil inequality with condition (2) were considered in [7].

The O'Neil inequality for anisotropic Lorentz spaces was studied in [8–10]. In the case of $n \geq 2$ these results are extended the inequality (6). In the one-dimensional case, the O'Neil inequality was extended in [11, 12].

There are generalizations of the Young and O'Neil inequalities for various functional spaces: weighted L_p spaces, classical and Lorentz weighted spaces, Hardy spaces, Wiener spaces, Orlicz spaces; see [5, 6, 8, 13–18], and references therein.

Convolution operators were studied in spaces of smooth functions in [19–24].

V.I. Burenkov and B.E. Batyrov in [21] received the following statement: Let $-\infty < l_1, l_2, l_3 < \infty$, $0 < p_1, p_2, p_3 \leq \infty$, $0 < \theta_1, \theta_2, \theta_3 \leq \infty$. In order for there to exist a number $c_3 > 0$ such that for any $f_1 \in B^{l_1}_{p_1\theta_1}(\mathbb{R}^n)$, $f_2 \in B^{l_2}_{p_2\theta_2}(\mathbb{R}^n)$ such that Ff_1 and Ff_2 are regular generalized functions and their (pointwise) product $Ff_1 \cdot Ff_2 \in S(\mathbb{R}^n)$, the inequality

$$\|f_1 * f_2\|_{B^{l_3}_{p_3\theta_3}(\mathbb{R}^n)} \leq c_3 \|f_1\|_{B^{l_1}_{p_1\theta_1}(\mathbb{R}^n)} \|f_2\|_{B^{l_2}_{p_2\theta_2}(\mathbb{R}^n)} \tag{7}$$

it is necessary and sufficient that the following conditions be fulfilled:

1) $p_3 \geq p_1$, $p_3 \geq p_2$;

2) $\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3} - 1 \geq 0$;

and one of the conditions

3a) $l_3 < l_1 + l_2 - n\left(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3} - 1\right)$

or

3b) $l_3 = l_1 + l_2 - n\left(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3} - 1\right)$ и $\frac{1}{\theta_3} \leq \frac{1}{\theta_1} + \frac{1}{\theta_2}$,
where Ff is the Fourier transform of the function f:

$$(Ff)(x) = (2\pi)^{-\frac{n}{2}} \int\limits_{\mathbb{R}^n} e^{-ix\xi} f(\xi) d\xi.$$

For $p_2 = p_3$, $\theta_2 = \theta_3$, $0 < l_2 < l_3 < \infty$ inequality (7) and some of its generalizations follow from the results obtained in the works of K.K. Golovkin and V.A. Solonnikov [19, 20] and [23].

In [24] we investigated the boundedness of the norm of the convolution operator in Sobolev spaces, with the dominant mixed derivative and anisotropic Nikol'skii-Besov spaces. For Sobolev spaces with the dominant mixed derivative, an analogue of Young's inequality is obtained, namely, relations of the form

$$W^{\boldsymbol{\gamma}}_{\mathbf{p}} * W^{\boldsymbol{\beta}}_{\mathbf{r}} \hookrightarrow W^{\boldsymbol{\alpha}}_{\mathbf{q}} \tag{8}$$

are proved when the corresponding conditions on the parameters are satisfied. Using relation (8) and the Nursultanov interpolation theorem for anisotropic spaces, an analogue of the O'Neil theorem was obtained for the Nikol'skii-Besov space scale $B^{\boldsymbol{\alpha}}_{\mathbf{pq}}$, where $\boldsymbol{\alpha}$, $\mathbf{p}$, $\mathbf{q}$ are vector parameters. Relations of the form $B^{\boldsymbol{\gamma}}_{\mathbf{ps_1}} * B^{\boldsymbol{\beta}}_{\mathbf{rs_2}} \hookrightarrow B^{\boldsymbol{\alpha}}_{\mathbf{qs}}$ are obtained, with the corresponding ratios of vector parameters.

The theorems obtained in [24] complement the results of Batyrov and Burenkov, where similar problems were considered in isotropic Nikol'skii-Besov spaces, that is, in spaces where the parameters are scalars.

2 Material and methods

Let $\boldsymbol{\alpha} \in \mathbb{R}^2$, $0 < \mathbf{q} = (q_1, q_2), \boldsymbol{\tau} = (\tau_1, \tau_2) \leq \infty$, $1 \leq \mathbf{p} = (p_1, p_2) < \infty$, $\mathbb{T}^2 = [0, 1)^2$.

We denote the space $B^{\boldsymbol{\alpha}\mathbf{q}}_{\mathbf{p}\boldsymbol{\tau}}(\mathbb{T}^2)$ as the set of all trigonometric series $f = \sum\limits_{\mathbf{m}\in\mathbb{Z}^2} a_{\mathbf{m}}(f) e^{2\pi i(\mathbf{m},\mathbf{x})}$ (generally speaking, divergent) for which

$$\|f\|_{B^{\boldsymbol{\alpha}\mathbf{q}}_{\mathbf{p}\boldsymbol{\tau}}(\mathbb{T}^2)} = \left(\sum_{k_2=0}^{\infty}\left(\sum_{k_1=0}^{\infty}\left(2^{\alpha_1 k_1 + \alpha_2 k_2}\|\Delta_{\mathbf{k}} f)\|_{L_{\mathbf{p},\boldsymbol{\tau}}(\mathbb{T}^2)}\right)^{q_1}\right)^{q_2/q_1}\right)^{1/q_2}$$

is finite, are called as Besov type spaces $B^{\boldsymbol{\alpha}\mathbf{q}}_{\mathbf{p}\boldsymbol{\tau}}(\mathbb{T}^2)$, where

$$\Delta_{\mathbf{k}} f(x_1, x_2) = \sum_{2^{k_2-1} \leq |m_2| < 2^{k_2}} \sum_{2^{k_1-1} \leq |m_1| < 2^{k_1}} a_{m_1,m_2}(f) e^{2\pi i(m_1 x_1 + m_2 x_2)},$$

$k \in \mathbb{N}$. In the isotropic case, these spaces were investigated in [25], where the interpolation properties were studied.

We define the concept of convolution for the elements of this spaces.

Let $f = \sum\limits_{k_2=-\infty}^{\infty} \sum\limits_{k_1=-\infty}^{\infty} a_{k_1,k_2} e^{2\pi i(k_1 x_1 + k_2 x_2)}$ and $g = \sum\limits_{k_2=-\infty}^{\infty} \sum\limits_{k_1=-\infty}^{\infty} b_{k_1,k_2} e^{2\pi i(k_1 x_1 + k_2 x_2)}$ be trigonometric series. By convolution of these series we mean the series

$$(f * g)(y_1, y_2) = \sum_{k_2=-\infty}^{\infty} \sum_{k_1=-\infty}^{\infty} a_{k_1,k_2}(f) b_{k_1,k_2}(g) e^{2\pi i(k_1 y_1 + k_2 y_2)}. \tag{9}$$

Note that for the "good" functions f and g, the convolution defined by equality (9) coincides with the classical definition (1). If the functions f and g from the corresponding spaces in (3), then $f(x)\overset{L_{\mathbf{p}}}{=}\sum\limits_{k\in\mathbb{Z}^n}\hat{f}(k)e^{2\pi i(k,x)}$ and $g(x)\overset{L_{\mathbf{r}}}{=}\sum\limits_{k\in\mathbb{Z}^n}\hat{g}(k)e^{2\pi i(k,x)}$ and

$$(f*g)(x)=\int\limits_{\mathbb{T}^n}f(x-y)g(y)dy\overset{L_{\mathbf{q}}}{=}\sum_{k\in\mathbb{Z}^n}\hat{f}(k)\hat{g}(k)e^{2\pi i(k,x)}.$$

Here, equalities are understood in the sense of the corresponding metrics.

Lemma 1

$$\Delta_{\mathbf{k}}(f*g)(y_1,y_2)=\int\limits_0^1\int\limits_0^1\Delta_{\mathbf{k}}f(x_1,x_2)\Delta_{\mathbf{k}}g(y_1-x_1,y_2-x_2)dx_1dx_2.$$

Lemma 2 ([10]) *Let* $1<\mathbf{q},\mathbf{p},\mathbf{r}<\infty$, $1\le\mathbf{h},\boldsymbol{\xi},\boldsymbol{\eta}<\infty$, *and* $1+\dfrac{1}{\mathbf{q}}=\dfrac{1}{\mathbf{p}}+\dfrac{1}{\mathbf{r}}$, $\dfrac{1}{\mathbf{h}}=\dfrac{1}{\boldsymbol{\xi}}+\dfrac{1}{\boldsymbol{\eta}}$. *Suppose that* f *and* K *are respectively measurable on* $[0,1]^2$ *and* $[-1,1]^2$ *functions such that* $f\in L_{\mathbf{p}\boldsymbol{\xi}}\big([0,1]^2\big)$ *and* $K\in L_{\mathbf{r}\boldsymbol{\eta}}\big([0,1]^2\big)$. *Then* $f*K\in L_{\mathbf{qh}}\big([0,1]^2\big)$ *and*

$$\|f*K\|_{L_{\mathbf{qh}}}\le 4(q_1'q_2')^2\|f\|_{L_{\mathbf{p}\boldsymbol{\xi}}}\|K\|_{L_{\mathbf{r}\boldsymbol{\eta}}}. \tag{10}$$

Lemma 3 *Let* $0<\mathbf{h},\boldsymbol{\xi},\boldsymbol{\eta}\le\infty$, $1<\mathbf{p},\mathbf{r},\mathbf{q}<\infty$

1. *If* $\dfrac{1}{q_1}+1=\dfrac{1}{p_1}+\dfrac{1}{r_1}$ *and* $\dfrac{1}{h_1}=\dfrac{1}{\xi_1}+\dfrac{1}{\eta_1}$, *then the following inequality*

$$\|f*_1g\|_{L_{(q_1,q_2),(h_1,h_2)}}\le C\|f\|_{L_{(p_1,q_2),(\xi_1,h_2)}}\|g\|_{L_{(r_1,q_2),(\eta_1,h_2)}}$$

holds for the transformation

$$(f*_1g)(x_1,x_2)=\int\limits_{-\infty}^{\infty}f(y_1,x_2)g(x_1-y_1,x_2)dy_1.$$

2. *If* $\dfrac{1}{q_2}+1=\dfrac{1}{p_2}+\dfrac{1}{r_2}$ *and* $\dfrac{1}{h_2}=\dfrac{1}{\xi_1}+\dfrac{1}{\eta_1}$, *then the following inequality*

$$\|f*_2g\|_{L_{(q_1,q_2),(h_1,h_2)}}\le C\|f\|_{L_{(q_1,p_2),(h_1,\xi_2)}}\|g\|_{L_{(q_1,r_2),(h_1,\eta_2)}}$$

holds for the transformation

$$(f*_2g)(x_1,x_2)=\int\limits_{-\infty}^{\infty}f(x_1,y_2)g(x_1,x_2-y_2)dy_2.$$

Proof. The proof of the lemma is similar to the proof of Theorem 3.1 from [10].

We note here that the application of the classical O'Neil inequality in one variable does not give the statement we need.

Lemma 4 *Let*

$$T_{\mathbf{k}}(x_1,x_2)=\sum_{2^{k_2-1}\le|m_2|<2^{k_2}}\ \sum_{2^{k_1-1}\le|m_1|<2^{k_1}} a_{m_1,m_2}e^{2\pi i(m_1x_1+m_2x_2)}.$$

Let $1\le\mathbf{p},\mathbf{q}<\infty$, $0<\boldsymbol{\tau}\le\infty$,

$$\theta_i=\frac{1}{p_i}-\frac{1}{q_i}\ge 0,\quad \frac{1}{t_i}=\frac{1-\operatorname{sgn}\theta_i}{\tau_i},\quad i=1,2,$$

then

$$\|T_{\mathbf{k}}\|_{L_{\mathbf{q}\boldsymbol{\tau}}}\le C2^{k_1\theta_1+k_2\theta_2}\|T_{\mathbf{k}}\|_{L_{\mathbf{pt}}}.$$

Proof. Let $\theta_i>0$, $i=1,2$. Note that for $T_{\mathbf{k}}$ there is a representation

$$T_{\mathbf{k}}(x)=T_{\mathbf{k}}*D_{\mathbf{k}},$$

where $D_{\mathbf{k}}(x_1,x_2)=\sum_{2^{k_2-1}\le|m_2|<2^{k_2}}\ \sum_{2^{k_1-1}\le|m_1|<2^{k_1}} e^{2\pi i(m_1x_1+m_2x_2)}$.

Using Lemma 2, we have

$$\|T_{\mathbf{k}}\|_{L_{\mathbf{q}\boldsymbol{\tau}}}\le C\|T_{\mathbf{k}}\|_{L_{\mathbf{p}\infty}}\|D_{\mathbf{k}}\|_{L_{\mathbf{r}\boldsymbol{\tau}}},$$

where $\frac{1}{\mathbf{r}}=1+\frac{1}{\mathbf{q}}-\frac{1}{\mathbf{p}}$.

We also note that

$$D^{*_1,*_2}_{k_1,k_2}(t_1,t_2)\le C\min\left(2^{k_1},\frac{1}{t_1}\right)\min\left(2^{k_2},\frac{1}{t_2}\right)$$

and therefore

$$\|D_{\mathbf{k}}\|_{L_{\mathbf{r}\boldsymbol{\tau}}}\asymp 2^{k_1\left(\frac{1}{p_1}-\frac{1}{q_1}\right)+k_2\left(\frac{1}{p_2}-\frac{1}{q_2}\right)}.$$

Let now $\theta_1>0$, $\theta_2=0$. Then

$$T_{\mathbf{k}}(x_1,x_2)=\int_{-\infty}^{\infty}T_{\mathbf{k}}(y_1,x_2)D_{k_1}(x_1-y_1)dy_1,$$

where $D_{k_1}(x_1)=\sum_{2^{k_1-1}\le|m_1|<2^{k_1}} e^{2\pi i x_1 m_1}$.

Further, applying the Lemma 3, we derive

$$\|T_{\mathbf{k}}\|_{L_{(q_1,q_2),(\tau_1,\tau_2)}}\le C\|T_{\mathbf{k}}\|_{L_{(p_1,q_2),(\infty,\tau_2)}}\|D_{k_1}\|_{L_{r_1,\tau_1}}\asymp 2^{k_1\theta_1}\|T_{\mathbf{k}}\|_{L_{(p_1,q_2),(\infty,\tau_2)}},$$

where $\frac{1}{r_1}=1+\frac{1}{q_1}-\frac{1}{p_1}$.

The case $\theta_2>0$, $\theta_1=0$ is considered similarly.

The case $\theta_1=\theta_2=0$ is obvious.

Lemma 5 *Let* $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^2$, $1 \le \mathbf{p}, \mathbf{q} \le \infty$, $0 < \boldsymbol{\tau} \le \infty$. *Let* $\boldsymbol{\alpha} - \dfrac{1}{\mathbf{q}} = \boldsymbol{\beta} - \dfrac{1}{\mathbf{p}}$, $\boldsymbol{\alpha} - \boldsymbol{\beta} = \boldsymbol{\theta} \ge 0$, *then*

$$B^{\boldsymbol{\alpha}\mathbf{s}}_{\mathbf{q}\mathbf{t}} \hookrightarrow B^{\boldsymbol{\beta}\mathbf{s}}_{\mathbf{p}\boldsymbol{\tau}},$$

where $\dfrac{1}{\mathbf{t}} = \dfrac{1 - \operatorname{sgn} \boldsymbol{\theta}}{\boldsymbol{\tau}}$.

Proof. Let $\theta_i > 0$, $i = 1, 2$. Let $f \in B^{\boldsymbol{\alpha}\mathbf{s}}_{\mathbf{q}\mathbf{t}}$. Using Lemma 4, we have

$$\begin{aligned} \|f\|_{B^{\boldsymbol{\beta}\mathbf{s}}_{\mathbf{p}\boldsymbol{\tau}}} &= \left(\sum_{k_2=0}^{\infty} \left(\sum_{k_1=0}^{\infty} \left(2^{\beta_1 k_1 + \beta_2 k_2} \|\Delta_{\mathbf{k}} f\|_{L_{\mathbf{p}\boldsymbol{\tau}}} \right)^{s_1} \right)^{s_2/s_1} \right)^{1/s_2} \\ &\le C \left(\sum_{k_2=0}^{\infty} \left(\sum_{k_1=0}^{\infty} \left(2^{\left(\beta_1 + \frac{1}{q_1} - \frac{1}{p_1}\right) k_1 + \left(\beta_2 + \frac{1}{q_2} - \frac{1}{p_2}\right) k_2} \|\Delta_{\mathbf{k}} f\|_{L_{\mathbf{q}\infty}} \right)^{s_1} \right)^{s_2/s_1} \right)^{1/s_2} \\ &= C \|f\|_{B^{\boldsymbol{\alpha}\mathbf{s}}_{\mathbf{q}\infty}}. \end{aligned}$$

Other cases are checked similarly.

Lemma 6 *Let* $\boldsymbol{\alpha}, \widetilde{\boldsymbol{\alpha}} \in \mathbb{R}^2$, $1 \le \mathbf{p}, \widetilde{\mathbf{p}} < \infty$, $\boldsymbol{\theta}' = \widetilde{\boldsymbol{\alpha}} - \boldsymbol{\alpha} \ge 0$, $\boldsymbol{\theta}'' = \widetilde{\mathbf{p}} - \mathbf{p} \ge 0$, $0 < \mathbf{q}, \boldsymbol{\tau} \le \infty$. *Then*

$$B^{\widetilde{\boldsymbol{\alpha}}\widetilde{\mathbf{q}}}_{\widetilde{\mathbf{p}}\widetilde{\boldsymbol{\tau}}} \hookrightarrow B^{\boldsymbol{\alpha}\mathbf{q}}_{\mathbf{p}\boldsymbol{\tau}},$$

where $\dfrac{1}{\widetilde{\mathbf{q}}} = \dfrac{1 - \operatorname{sgn} \boldsymbol{\theta}'}{\mathbf{q}}$, $\dfrac{1}{\widetilde{\boldsymbol{\tau}}} = \dfrac{1 - \operatorname{sgn} \boldsymbol{\theta}''}{\boldsymbol{\tau}}$.

Proof. The proof follows from the embeddings $l^{\widetilde{\alpha}_i}_{\widetilde{q}_i} \hookrightarrow l^{\alpha_i}_{q_i}$ and $L_{\widetilde{\mathbf{p}}\widetilde{\boldsymbol{\tau}}} \hookrightarrow L_{\mathbf{p}\boldsymbol{\tau}}$.

Lemma 7 *Let* $\boldsymbol{\alpha}, \widetilde{\boldsymbol{\alpha}} \in \mathbb{R}^2$, $1 \le \mathbf{p}, \widetilde{\mathbf{p}} < \infty$, $0 < \mathbf{q}, \boldsymbol{\tau} \le \infty$, $\boldsymbol{\delta}' = \left(\widetilde{\boldsymbol{\alpha}} - \dfrac{1}{\widetilde{\mathbf{p}}}\right) - \left(\boldsymbol{\alpha} - \dfrac{1}{\mathbf{p}}\right) \ge 0$, $\boldsymbol{\theta}' = \widetilde{\boldsymbol{\alpha}} - \boldsymbol{\alpha} \ge 0$. *Then*

$$B^{\widetilde{\boldsymbol{\alpha}}\widetilde{\mathbf{q}}}_{\widetilde{\mathbf{p}}\widetilde{\boldsymbol{\tau}}} \hookrightarrow B^{\boldsymbol{\alpha}\mathbf{q}}_{\mathbf{p}\boldsymbol{\tau}},$$

where $\dfrac{1}{\widetilde{\boldsymbol{\tau}}} = \dfrac{(1 - \operatorname{sgn} \boldsymbol{\delta}')(1 - \operatorname{sgn} \boldsymbol{\theta}')}{\mathbf{t}}$, $\dfrac{1}{\widetilde{\mathbf{q}}} = \dfrac{1 - \operatorname{sgn} \boldsymbol{\delta}' \operatorname{sgn} \boldsymbol{\theta}'}{\mathbf{q}}$.

Proof. Let the conditions of the lemma be satisfied. Then there are $\bar{\boldsymbol{\alpha}}$ and $\bar{\mathbf{p}}$ such that

$$\widetilde{\boldsymbol{\alpha}} - \frac{1}{\widetilde{\mathbf{p}}} = \bar{\boldsymbol{\alpha}} - \frac{1}{\bar{\mathbf{p}}}, \quad \widetilde{\boldsymbol{\alpha}} \ge \boldsymbol{\alpha} \ge \boldsymbol{\alpha}, \quad \bar{\mathbf{p}} \ge \mathbf{p}.$$

Moreover, for $\delta_i > 0$, $\theta_i > 0$, $\widetilde{\alpha}_i > \bar{\alpha}_i > \alpha$, $\bar{p}_i > p$ Lemma 6 implies the embedding

$$B^{\bar{\boldsymbol{\alpha}}\bar{\mathbf{q}}}_{\bar{\mathbf{p}}\bar{\boldsymbol{\tau}}} \hookrightarrow B^{\boldsymbol{\alpha}\mathbf{q}}_{\mathbf{p}\mathbf{t}},$$

where

$$\frac{1}{\bar{\mathbf{q}}} = \frac{1 - \operatorname{sgn}(\bar{\boldsymbol{\alpha}} - \boldsymbol{\alpha})}{\mathbf{q}}, \quad \frac{1}{\bar{\boldsymbol{\tau}}} = \frac{1 - \operatorname{sgn}(\bar{\mathbf{p}} - \mathbf{p})}{\boldsymbol{\tau}}.$$

Applying Lemma 5, we have

$$B^{\widetilde{\boldsymbol{\alpha}}\widetilde{\mathbf{q}}}_{\widetilde{\mathbf{p}}\widetilde{\boldsymbol{\tau}}} \hookrightarrow B^{\bar{\boldsymbol{\alpha}}\bar{\mathbf{q}}}_{\bar{\mathbf{p}}\bar{\boldsymbol{\tau}}},$$

where

$$\widetilde{\boldsymbol{\alpha}} - \frac{1}{\widetilde{\mathbf{q}}} = \bar{\boldsymbol{\alpha}} - \frac{1}{\bar{\mathbf{q}}} \geq \boldsymbol{\alpha} - \frac{1}{\mathbf{p}}.$$

$$\frac{1}{\widetilde{\boldsymbol{\tau}}} = \frac{(1 - \operatorname{sgn}(\widetilde{\boldsymbol{\alpha}} - \bar{\boldsymbol{\alpha}}))}{\bar{\boldsymbol{\tau}}} = \frac{(1 - \operatorname{sgn}(\widetilde{\boldsymbol{\alpha}} - \bar{\boldsymbol{\alpha}}))(1 - \operatorname{sgn}(\bar{\mathbf{p}} - \mathbf{p}))}{\boldsymbol{\tau}},$$

$$\frac{1}{\widetilde{\mathbf{q}}} = \frac{1}{\bar{\mathbf{q}}} = \frac{1 - \operatorname{sgn}(\bar{\boldsymbol{\alpha}} - \boldsymbol{\alpha})}{\mathbf{q}}.$$

We note that $(\bar{\alpha}_i - \alpha_i) > 0$ if and only if $\delta_i' > 0$ и $\theta_i' > 0$, i.e.

$$\operatorname{sgn}(\bar{\alpha}_i - \alpha_i) = \operatorname{sgn} \delta_i' \cdot \operatorname{sgn} \theta_i'.$$

And $\widetilde{\alpha}_i - \bar{\alpha}_i = 0$, $\bar{p}_i - p_i = 0$ if and only if $\delta_i = 0$, $\theta_i' = 0$, which means

$$(1 - \operatorname{sgn}(\bar{\boldsymbol{\alpha}} - \boldsymbol{\alpha}))(1 - \operatorname{sgn}(\bar{\mathbf{p}} - \mathbf{p})) = (1 - \operatorname{sgn} \boldsymbol{\delta}')(1 - \operatorname{sgn} \boldsymbol{\theta}').$$

Theorem 1 *Let $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbb{R}^2$, $1 < \mathbf{p}, \mathbf{r}, \mathbf{h} < \infty$, $0 < \boldsymbol{\tau}, \boldsymbol{\mu}, \boldsymbol{\nu}, \mathrm{q}, \boldsymbol{\xi}, \boldsymbol{\eta} \leq \infty$. In order for the inequality*

$$\|f * g\|_{B^{\boldsymbol{\alpha}\mathbf{q}}_{\mathbf{p}\boldsymbol{\tau}}([0,1]^2)} \leq C \|f\|_{B^{\boldsymbol{\beta}\boldsymbol{\eta}}_{\mathbf{r}\boldsymbol{\mu}}([0,1]^2)} \|g\|_{B^{\boldsymbol{\gamma}\boldsymbol{\xi}}_{\mathbf{h}\boldsymbol{\nu}}([0,1]^2)} \tag{11}$$

to hold for $f \in B^{\boldsymbol{\beta}\boldsymbol{\eta}}_{\mathbf{r}\boldsymbol{\mu}}([0,1]^2)$ and $g \in B^{\boldsymbol{\gamma}\boldsymbol{\xi}}_{\mathbf{h}\boldsymbol{\nu}}([0,1]^2)$ it is necessary and sufficient that the following conditions are met:

$$\boldsymbol{\theta} = \boldsymbol{\beta} + \boldsymbol{\gamma} - \boldsymbol{\alpha} \geq 0; \tag{12}$$

$$\boldsymbol{\delta} = \boldsymbol{\beta} + \boldsymbol{\gamma} - \boldsymbol{\alpha} + 1 + \frac{1}{\mathbf{p}} - \frac{1}{\mathbf{r}} - \frac{1}{\mathbf{h}} \geq 0; \tag{13}$$

$$\frac{(1 - \operatorname{sgn} \boldsymbol{\delta})(1 - \operatorname{sgn} \boldsymbol{\theta})}{\boldsymbol{\tau}} \leq \frac{1}{\boldsymbol{\mu}} + \frac{1}{\boldsymbol{\nu}}; \tag{14}$$

$$\frac{(1 - \operatorname{sgn} \boldsymbol{\delta} \operatorname{sgn} \boldsymbol{\theta})}{\mathbf{q}} \leq \frac{1}{\boldsymbol{\xi}} + \frac{1}{\boldsymbol{\eta}}. \tag{15}$$

Proof. Let conditions (12)-(15) be satisfied. In this case, there are $\widetilde{\mathbf{p}}, \widetilde{\boldsymbol{\alpha}}$ such that $\boldsymbol{\beta} + \boldsymbol{\gamma} \geq \widetilde{\boldsymbol{\alpha}} \geq \boldsymbol{\alpha}$, $\widetilde{\mathbf{p}} \geq \mathbf{p}$ и $\boldsymbol{\beta} + \boldsymbol{\gamma} - \widetilde{\boldsymbol{\alpha}} + 1 + \frac{1}{\widetilde{\mathbf{p}}} - \frac{1}{\mathbf{r}} - \frac{1}{\mathbf{h}} = 0$. Moreover, if $\delta_i > 0$, $\theta_i > 0$, then $\beta_i + \gamma_i > \widetilde{\alpha}_i > \alpha_i$, $\widetilde{p}_i > p$. Applying Lemma 7, we have

$$B^{\widetilde{\boldsymbol{\alpha}}\widetilde{\mathbf{q}}}_{\widetilde{\mathbf{p}}\widetilde{\boldsymbol{\tau}}} \hookrightarrow B^{\boldsymbol{\alpha}\mathbf{q}}_{\mathbf{p}\boldsymbol{\tau}},$$

where

$$\frac{1}{\widetilde{\widetilde{\boldsymbol{\tau}}}} = \frac{1-\operatorname{sgn}\left(\left(\widetilde{\boldsymbol{\alpha}}-\frac{1}{\widetilde{\mathbf{p}}}\right)-\left(\boldsymbol{\alpha}-\frac{1}{\mathbf{p}}\right)\right)(1-\operatorname{sgn}(\widetilde{\boldsymbol{\alpha}}-\boldsymbol{\alpha}))}{\boldsymbol{\tau}},$$

$$\frac{1}{\widetilde{\widetilde{\mathbf{q}}}} = \frac{1-\operatorname{sgn}\left(\left(\widetilde{\boldsymbol{\alpha}}-\frac{1}{\widetilde{\mathbf{p}}}\right)-\left(\boldsymbol{\alpha}-\frac{1}{\mathbf{p}}\right)\right)\operatorname{sgn}(\widetilde{\boldsymbol{\alpha}}-\boldsymbol{\alpha})}{\mathbf{q}}.$$

We note, that

$$\left(1-\operatorname{sgn}\left(\left(\widetilde{\boldsymbol{\alpha}}-\frac{1}{\widetilde{\mathbf{p}}}\right)-\left(\boldsymbol{\alpha}-\frac{1}{\mathbf{p}}\right)\right)\right)(1-\operatorname{sgn}(\widetilde{\boldsymbol{\alpha}}-\boldsymbol{\alpha})) = (1-\operatorname{sgn}\boldsymbol{\delta})(1-\operatorname{sgn}\boldsymbol{\theta}),$$

$$1-\operatorname{sgn}\left(\left(\widetilde{\boldsymbol{\alpha}}-\frac{1}{\widetilde{\mathbf{p}}}\right)-\left(\boldsymbol{\alpha}-\frac{1}{\mathbf{p}}\right)\right)\operatorname{sgn}(\widetilde{\boldsymbol{\alpha}}-\boldsymbol{\alpha}) = 1-\operatorname{sgn}\boldsymbol{\delta}\operatorname{sgn}\boldsymbol{\theta},$$

that is

$$\frac{1}{\widetilde{\widetilde{\boldsymbol{\tau}}}} = \frac{(1-\operatorname{sgn}\boldsymbol{\delta})(1-\operatorname{sgn}\boldsymbol{\theta})}{\boldsymbol{\tau}}, \quad \frac{1}{\widetilde{\widetilde{\mathbf{q}}}} = \frac{1-\operatorname{sgn}\boldsymbol{\delta}\operatorname{sgn}\boldsymbol{\theta}}{\mathbf{q}}.$$

Therefore

$$\begin{aligned}\|f*g\|_{B^{\boldsymbol{\alpha}\mathbf{q}}_{\mathbf{p}\boldsymbol{\tau}}} &\le C\|f*g\|_{B^{\widetilde{\boldsymbol{\alpha}}\widetilde{\mathbf{q}}}_{\widetilde{\mathbf{p}}\widetilde{\boldsymbol{\tau}}}} = \\ &= C\left(\sum_{k_2=0}^{\infty}\left(\sum_{k_1=0}^{\infty}\left(2^{\widetilde{\alpha}_1k_1+\widetilde{\alpha}_2k_2}\|\Delta_{\mathbf{k}}(f*g)\|_{L_{\widetilde{p}\widetilde{\boldsymbol{\tau}}}}\right)^{\widetilde{q}_1}\right)^{\widetilde{q}_2/\widetilde{q}_1}\right)^{1/\widetilde{q}_2}.\end{aligned}$$

Using Lemma 1, Lemma 2, and Hölder's inequality, we derive

$$\begin{aligned}\|f*g\|_{B^{\boldsymbol{\alpha}\mathbf{q}}_{\mathbf{p}\boldsymbol{\tau}}} &\le C\left(\sum_{k_2=0}^{\infty}\left(\sum_{k_1=0}^{\infty}\left(2^{\widetilde{\alpha}_1k_1+\widetilde{\alpha}_2k_2}\|\Delta_{\mathbf{k}}f*\Delta_{\mathbf{k}}g\|_{L_{\widetilde{p}\widetilde{\boldsymbol{\tau}}}}\right)^{\widetilde{q}_1}\right)^{\widetilde{q}_2/\widetilde{q}_1}\right)^{1/\widetilde{q}_2} \\ &\le C\left(\sum_{k_2=0}^{\infty}\left(\sum_{k_1=0}^{\infty}\left(2^{\widetilde{\alpha}_1k_1+\widetilde{\alpha}_2k_2}\|\Delta_{\mathbf{k}}f\|_{L_{\mathbf{r}\widetilde{\mu}}}\|\Delta_{\mathbf{k}}g\|_{L_{\mathbf{h}\widetilde{\nu}}}\right)^{\widetilde{q}_1}\right)^{\widetilde{q}_2/\widetilde{q}_1}\right)^{1/\widetilde{q}_2} \\ &\le C\left(\sum_{k_2=0}^{\infty}\left(\sum_{k_1=0}^{\infty}\left(2^{\beta_1k_1+\beta_2k_2}\|\Delta_{\mathbf{k}}f\|_{L_{\mathbf{r}\widetilde{\mu}}}\right)^{\widetilde{\eta}_1}\right)^{\widetilde{\eta}_2/\widetilde{\eta}_1}\right)^{1/\widetilde{\eta}_2} \\ &\times\left(\sum_{k_2=0}^{\infty}\left(\sum_{k_1=0}^{\infty}\left(2^{\gamma_1k_1+\gamma_2k_2}\|\Delta_{\mathbf{k}}g\|_{L_{\mathbf{h}\widetilde{\nu}}}\right)^{\widetilde{\xi}_1}\right)^{\widetilde{\xi}_2/\widetilde{\xi}_1}\right)^{1/\widetilde{\xi}_2} = \|f\|_{B^{\boldsymbol{\beta}\widetilde{\boldsymbol{\eta}}}_{\mathbf{r}\widetilde{\boldsymbol{\mu}}}}\|g\|_{B^{\boldsymbol{\gamma}\widetilde{\boldsymbol{\xi}}}_{\mathbf{h}\widetilde{\boldsymbol{\nu}}}},\end{aligned}$$

where $\widetilde{\boldsymbol{\mu}} \ge \boldsymbol{\mu}$, $\widetilde{\boldsymbol{\nu}} \ge \boldsymbol{\nu}$, $\widetilde{\boldsymbol{\eta}} \ge \boldsymbol{\eta}$, $\widetilde{\boldsymbol{\xi}} \ge \boldsymbol{\xi}$ and

$$\frac{1}{\widetilde{\boldsymbol{\tau}}} = \frac{1}{\widetilde{\boldsymbol{\mu}}}+\frac{1}{\widetilde{\boldsymbol{\nu}}} \le \frac{1}{\boldsymbol{\mu}}+\frac{1}{\boldsymbol{\nu}},$$

$$\frac{1}{\widetilde{\mathbf{q}}} = \frac{1}{\widetilde{\boldsymbol{\eta}}} + \frac{1}{\widetilde{\boldsymbol{\xi}}} \le \frac{1}{\boldsymbol{\eta}} + \frac{1}{\boldsymbol{\nu}}.$$

And therefore, considering that

$$B^{\boldsymbol{\beta}\widetilde{\boldsymbol{\eta}}}_{\mathbf{r}\widetilde{\boldsymbol{\mu}}} \hookrightarrow B^{\boldsymbol{\beta}\boldsymbol{\eta}}_{\mathbf{r}\boldsymbol{\mu}}, \quad B^{\boldsymbol{\gamma}\widetilde{\boldsymbol{\xi}}}_{\mathbf{h}\widetilde{\boldsymbol{\nu}}} \hookrightarrow B^{\boldsymbol{\gamma}\boldsymbol{\xi}}_{\mathbf{h}\boldsymbol{\nu}},$$

we obtain the inequality (11).

Conversely, we show that the conditions (12)-(15) are necessary.

Let (11) hold.

Let $\mathbf{k} \in \mathbb{N}^2$. We consider the functions $f_1(x_1, x_2) = e^{2\pi i(2^{k_1 x_1} + 2^{k_2 x_2})}$, $g_1(x_1, x_2) = e^{2\pi i(2^{k_1 x_1} + 2^{k_2 x_2})}$. Then

$$(f_1 * g_1)(x_1, x_2) = e^{2\pi i(2^{k_1 x_1} + 2^{k_2 x_2})}$$

From the inequality (11) we have

$$2^{\alpha_1 k_1 + \alpha_2 k_2} \le C 2^{\beta_1 k_1 + \beta_2 k_2 + \gamma_1 k_1 + \gamma_2 k_2}.$$

Given the correct choice of $\mathbf{k} \in \mathbb{N}^2$, we derive

$$\alpha_i \le \beta_i + \gamma_i. \tag{16}$$

Let $\mathbf{k} \in \mathbb{N}^2$. We consider the functions

$$f_2(x_1, x_2) = \sum_{m_2 = 2^{k_2 - 1}}^{2^{k_2} - 1} \sum_{m_1 = 2^{k_1 - 1}}^{2^{k_1} - 1} e^{2\pi i(2^{m_1 x_1} + 2^{m_2 x_2})} = g_2(x_1, x_2).$$

Then $f_2 * g_2 = f_2 = g_2$. Using Hardy-Littlewood theorem, we have

$$\|f_2 * g_2\|_{B^{\boldsymbol{\alpha}\mathbf{q}}_{\mathbf{p}\boldsymbol{\tau}}} = 2^{\alpha_1 k_1 + \alpha_2 k_2} \|\Delta_{\mathbf{k}} f_2\|_{L_{\mathbf{p}\boldsymbol{\tau}}} \asymp 2^{\left(\alpha_1 + \frac{1}{p_1'}\right) k_1 + \left(\alpha_2 + \frac{1}{p_2'}\right) k_2}$$

$$\|f\|_{B^{\boldsymbol{\beta}\mathbf{q}}_{\mathbf{r}\boldsymbol{\mu}}} \asymp 2^{\left(\beta_1 + \frac{1}{r_1'}\right) k_1 + \left(\beta_2 + \frac{1}{r_2'}\right) k_2},$$

$$\|g\|_{B^{\boldsymbol{\gamma}\boldsymbol{\xi}}_{\mathbf{h}\boldsymbol{\nu}}} \asymp 2^{\left(\gamma_1 + \frac{1}{h_1'}\right) k_1 + \left(\gamma_2 + \frac{1}{h_2'}\right) k_2}.$$

From the inequality (11), since $\mathbf{k}$ is arbitrary, we have

$$\alpha_i - \beta_i - \gamma_i \le 1 + \frac{1}{p_i} - \frac{1}{r_i} - \frac{1}{h_i}, \quad i = 1, 2,$$

that is, the condition (13) is necessary.

From (13) and (16) follows (12).

The condition (15) makes sense when $\delta_i = 0$ $(i = 1, 2)$. We consider the functions

$$f_3(x_1, x_2) = \sum_{k=0}^{N} \sum_{m=2^{k-1}}^{2^k - 1} 2^{-\left(\frac{1}{r_i'} + \beta_i\right) k} e^{2\pi i m x_i},$$

$$g_3(x_1,x_2)=\sum_{k=0}^{N}\sum_{m=2^{k-1}}^{2^k-1}2^{-\left(\frac{1}{h_i'}+\gamma_i\right)k}e^{2\pi i m x_i},$$

then

$$(f_3*g_3)(x_1,x_2)=\sum_{k=0}^{N}\sum_{m=2^{k-1}}^{2^k-1}2^{-\left(\left(\frac{1}{r_i'}+\frac{1}{h_i'}\right)+(\beta_i+\gamma_i)\right)k}e^{2\pi i m x_i}.$$

Then we have

$$\|f*g\|_{B_{\mathbf{p}\boldsymbol{\tau}}^{\boldsymbol{\alpha}\mathbf{q}}}\asymp\left(\sum_{k=0}^{N}\left(2^{\left(\alpha_i+\frac{1}{q_i'}-\frac{1}{h_i'}-\frac{1}{r_i'}-\beta_i-\gamma_i\right)k}\right)^{\tau_i}\right)^{1/\tau_i}=N^{\frac{1}{\tau_i}},$$

$$\|f\|_{B_{\mathbf{r}\boldsymbol{\mu}}^{\boldsymbol{\beta}\mathbf{q}}}\asymp N^{\frac{1}{\eta_i}},$$

$$\|g\|_{B_{\mathbf{h}\boldsymbol{\nu}}^{\boldsymbol{\gamma}\boldsymbol{\xi}}}\asymp N^{\frac{1}{\xi_i}}.$$

Therefore, $\dfrac{1}{\tau_i}\leq\dfrac{1}{\eta_i}+\dfrac{1}{\xi_i}$ follows from (11).

The condition (14) makes sense when $\delta_i=0$, $\theta_i=0$. This means that $\alpha_i=\beta_i+\gamma_i$, $1+\dfrac{1}{p_i}=\dfrac{1}{r_i}+\dfrac{1}{h_i}$.

Let $k\in\mathbb{N}$.

$$f_4(x_1,x_2)=2^{-\beta_i k_i}\sum_{m=2^{k-1}}^{2^k-1}\left(m-2^{k-1}\right)^{\frac{1}{r_i'}}e^{2\pi i m x_i},$$

$$g_4(x_1,x_2)=2^{-\gamma_i k_i}\sum_{m=2^{k-1}}^{2^k-1}\left(m-2^{k-1}\right)^{\frac{1}{h_i'}}e^{2\pi i m x_i}.$$

Then

$$\|f\|_{B_{\mathbf{r}\boldsymbol{\mu}}^{\boldsymbol{\beta}\boldsymbol{\eta}}}\asymp\left(\sum_{m=1}^{2^k-1}m^{-1}\right)^{\frac{1}{\mu_i}}\asymp k^{\frac{1}{\mu_i}},$$

$$\|g\|_{B_{\mathbf{h}\boldsymbol{\nu}}^{\boldsymbol{\gamma}\boldsymbol{\xi}}}\asymp k^{\frac{1}{\nu_i}},$$

$$\|f*g\|_{B_{\mathbf{p}\boldsymbol{\tau}}^{\boldsymbol{\alpha}\mathbf{q}}}\asymp k^{\frac{1}{\tau_i}}.$$

Therefore $\dfrac{1}{\tau_i}\leq\dfrac{1}{m_i}+\dfrac{1}{\nu_i}$.

3 Conclusion

In conclusion, we note that in the article we investigate the boundedness of the norm of the convolution operator in anisotropic Besov spaces. We proved a criterion for the fulfillment of the inequality

$$\|f * g\|_{B^{\boldsymbol{\alpha}\mathbf{q}}_{\mathbf{p}\boldsymbol{\tau}}([0,1]^2)} \leq C\|f\|_{B^{\boldsymbol{\beta}\boldsymbol{\eta}}_{\mathbf{r}\boldsymbol{\mu}}([0,1]^2)}\|g\|_{B^{\boldsymbol{\gamma}\boldsymbol{\xi}}_{\mathbf{h}\boldsymbol{\nu}}([0,1]^2)}$$

in terms of the corresponding parameters. The resulting theorem:
1) summarizes the result of Burenkov and Batyrov [21];
2) it implies the classical O'Neil inequalities.

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