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THE NON-COMMUTATIVE HARDY-LITTLEWOOD MAXIMAL OPERATOR ON NON-COMMUTATIVE LORENTZ SPACES

In this work we study the non-commutative Hardy-Littlewood maximal operator on Lorentz spaces of τ-measurable operators. Non-commutative maximal inequalities were studied, in particular, in [1–3]. Another version of the (non-commutative) Hardy-Littlewood maximal operator was introduced by T. Bekjan [4]. Later J. Shao investigated the Hardy-Littlewood maximal operator on non-commutative Lorentz spaces associated with finite atomless von Neumann algebra (see [5]). Namely, for an operator T affiliated with a semi-finite von Neumann algebra $\mathcal{M}$, the Hardy-Littlewood maximal operator of T is defined by

$$MA(x) = \sup_{r>0} \frac{1}{\tau\left(E_{[x-r,x+r]}\left(|A|\right)\right)} \tau\left(|A| E_{[x-r,x+r]}\left(|A|\right)\right), \ x \geq 0.$$

While the classical Hardy-Littlewood maximal operator of a Lebesgue measurable function $f : \mathbb{R} \to \mathbb{R}$, denoted by $Mf(x)$, is defined as

$$Mf(x) = \sup_{r>0} \frac{1}{m([x-r,x+r])} \int\limits_{[x-r,x+r]} |f(t)| dt,$$

where m is a Lebesgue measure on $(-\infty, \infty)$ [10]. In view of spectral theory, $|A|$ is represented as

$$|A| = \int_{\sigma(|A|)} t dE_t,$$

and $MA(|A|)$ is represented as $MA(x)$. Thus, for the operator A, Bekjan's consideration is that $MA(|A|)$ is defined as the operator analogue of the Hardy-Littlewood maximal operator in the classical case. Our purpose is to investigate the non-commutative Hardy-Littlewood maximal operator M in the sense of T. Bekjan (see [4]). In particular, we obtain boundedness of the non-commutative Hardy-Littlewood maximal operator in non-commutative Lorentz spaces.
Key words: Cesaro operator, Hardy-Littlewood maximal operator, Lorentz space.

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Коммутативті емес Лоренц кеңістіктеріндегі Харди-Литтлвуд коммутативті емес максимал операторы

Бұл жұмыста біз коммутативті емес Харди-Литтлвуд максимал операторын τ-өлшенетін операторлардың Лоренц кеңістіктерінде қарастырамыз. Коммутативті емес максимал теңсіздіктер туралы [1–3] жұмыстарынан көруге болады. Харди-Литтлвуд операторының басқа нұсқасын Т.Бекжан [4] жұмысында келтірді. Кейінірек J.Shao Харди-Литтлвуд максимал операторын шенелген атомсыз фон Нейман алгебрасымен ассоциацияланған коммутативті емес Лоренц кеңістіктерінде зерттеген [5]. Атап айтқанда, жартылай шенелген M фон Нейман алгебрасымен қосылма T операторы үшін Харди-Литтлвуд максимал операторы

$$MA(x) = \sup_{r>0} \frac{1}{\tau\left(E_{[x-r,x+r]}\left(|A|\right)\right)} \tau\left(|A| E_{[x-r,x+r]}\left(|A|\right)\right), \ x \geq 0$$

түрінде анықталады. Лебег бойынша өлшенетін $f : \mathbb{R} \to \mathbb{R}$ функциясы үшін классикалық Харди-Литтлвуд максимал операторын $Mf(x)$ деп белгілеп,

$$Mf(x) = \sup_{r>0} \frac{1}{m([x-r, x+r])} \int\limits_{[x-r,x+r]} |f(t)|dt$$

түрінде анықтаймыз, мұндағы m – Лебег өлшемі [10]. Спектрлік теория бойынша

$$|A| = \int_{\sigma(|A|)} tdE_t$$

болатындығын байқауға болады, сонымен қатар $MA(|A|)$ операторын $MA(x)$ түрінде жазамыз. Сонымен, A операторы үшін Т.Бекжан тұжырымы бойынша $MA(|A|)$ операторы классикалық жағдайдағы Харди-Литтлвуд максимал операторының аналогы ретінде анықталады. Біздің мақсатымыз – Т.Бекжан [4] жұмысындағы M Харди-Литтлвуд максимал операторын зерттеу болып табылады. Нақтырақ айтқанда, біз коммутативті емес Лоренц кеңістіктерінде коммутативті емес Харди-Литтлвуд максимал операторының шенелгендігін аламыз.
Түйін сөздер: Чезаро операторы, Харди-Литтлвуд максимал операторы, Лоренц кеңістігі.

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Некоммутативный максимальный оператор Харди-Литтлвуда в некоммутативных пространствах Лоренца

В данной работе мы исследуем некоммутативный максимальный оператор Харди-Литтлвуда в симметричных пространствах τ-измеримых операторов. Некоммутативные максимальные неравенства были рассмотрены, в частности, в [1–3]. Другая версия некоммутативного максимального оператора Харди-Литтлвуда представлена Т. Бекжаном [4]. Позже Дж. Шао занимался исследованиями максимального оператора Харди-Литтлвуда в некоммутативных пространствах Лоренца ассоциацированной с ограниченной безатомной алгеброй фон Неймана [5]. А именно, для оператора T, аффилированного с полуограниченной алгеброй фон Неймана M, максимальный оператор Харди-Литтлвуда определяется как

$$MA(x) = \sup_{r>0} \frac{1}{\tau\left(E_{[x-r,x+r]}(|A|)\right)} \tau\left(|A|E_{[x-r,x+r]}(|A|)\right), \ x \geq 0.$$

В то время как классический максимальный оператор Харди-Литтлвуда измеримых по Лебегу функций $f : \mathbb{R} \to \mathbb{R}$, обозначаемый через $Mf(x)$, определяется как

$$Mf(x) = \sup_{r>0} \frac{1}{m([x-r, x+r])} \int\limits_{[x-r,x+r]} |f(t)|dt$$

где m–мера Лебега [10]. С точки зрения спектральной теории, $|A|$ представим в виде

$$|A| = \int_{\sigma(|A|)} tdE_t,$$

а $MA(|A|)$ представим в виде $MA(x)$. Таким образом, для оператора A, рассуждение Т. Бекжана говорит о том, что $MA(|A|)$ определяется как аналог максимального оператора Харди-Литтлвуда в классическом случае [4]. Нашей целью является исследовать некоммутативный максимальный оператор M в смысле Т. Бекжана. В частности, мы получим ограниченность некоммутативного максимального оператора Харди-Литтлвуда в некоммутативных пространствах Лоренца.

Ключевые слова: Оператор Чезаро, максимальный оператор Харди-Литтлвуда, пространства Лоренца.

1 Introduction

The Hardy-Littlewood maximal operator M is an important sub-linear operator with numerous applications in real analysis and harmonic analysis. It takes a locally integrable function $f:\mathbb{R}^d\to\mathbb{C}$ and returns another function Mf that, at each point $x\in\mathbb{R}^d$, gives the maximum average value that f can have on balls centered at that point. More precisely,

$$Mf(x)=\sup_{r>0}\frac{1}{|B(x,r)|}\int\limits_{B(x,r)}|f(y)|dy \tag{1}$$

where $B(x,r)$ is the ball of radius r centred at x, and $|E|$ denotes the d-dimensional Lebesgue measure of $E\subset\mathbb{R}^d$. There is an uncountable number of papers devoted to investigation of the Hardy-Littlewood function defined by the formula (1). For instance, see [7, 8, 10] and references therein.

2 Materials and methods

Let $(\mathbb{R}_+,m)$ denote the measure space $\mathbb{R}_+=(0,\infty)$ equipped with Lebesgue measure m. Let $L(\mathbb{R}_+,m)$ be the space of all measurable real-valued functions on $\mathbb{R}_+$ equipped with Lebesgue measure m i.e. functions which coincide almost everywhere are considered identical. Define $S(\mathbb{R}_+,m)$ to be the subset of $L(\mathbb{R}_+,m)$ which consists of all functions x such that $m(t:|x(t)|>s)<\infty$ for some $s>0$. For $x\in S(\mathbb{R}_+)$ we denote by $\mu(x)$ the decreasing rearrangement of the function $|x|$. That is,

$$\mu(t,x)=\inf\{s\geq 0: m(\{|x|>s\})\leq t\},\ t>0.$$

We say that y is submajorized by x in the sense of Hardy–Littlewood–Polya (written $y\prec\prec x$) if

$$\int\limits_0^t\mu(s,y)ds\leq\int\limits_0^t\mu(s,x)ds,\ t\geq 0.$$

Let $\mathcal{M}$ be a semifinite von Neumann algebra on a separable Hilbert space H equipped with a faithful normal semifinite trace τ. A closed and densely defined operator A affiliated with $\mathcal{M}$ is called τ-measurable if $\tau(E_{|A|}(s,\infty))<\infty$ for sufficiently large s. We denote the set of all τ-measurable operators by $S(\mathcal{M},\tau)$. Let $Proj(\mathcal{M})$ denote the lattice of all projections in $\mathcal{M}$. For every $A\in S(\mathcal{M},\tau)$, we define its singular value function $\mu(A)$ by setting

$$\mu(t,A)=\inf\{\| A(1-P)\|_{\mathcal{L}_\infty(\mathcal{M})}: P\in Proj(\mathcal{M}),\ \tau(P)\leq t\},\ t>0,$$

where the norm $\|\cdot\|_{\mathcal{L}_\infty(\mathcal{M})}$ is the usual operator (uniform) norm. Equivalently, for positive self-adjoint operators $A\in S(\mathcal{M},\tau)$, we have

$$n_A(s)=\tau(E_A(s,\infty)),\ \mu(t,A)=\inf\{s>0: n_A(s)<t\},\ t>0.$$

An operator in $S(\mathcal{M},\tau)$ is called τ-compact if $\mu(\infty, A) = 0$. This notion is a direct generalization of the ideal of compact operators on a Hilbert space H. For more details on generalised singular value functions and τ-compact operators, we refer the reader to [9] and [13]. Let $\mathcal{L}_{loc}(\mathcal{M},\tau)$ be the set of all τ-measurable operators such that

$$\tau(|A|E_{|A|}(I)) < +\infty,$$

for all bounded intervals $I \subset [0, +\infty)$.

Definition 1 *[4, Definition 1]. For $A \in \mathcal{L}_{loc}(\mathcal{M},\tau)$, we define the maximal operator of A by*

$$MA(x) = \sup_{r>0} \frac{1}{\tau\left(E_{[x-r,x+r]}(|A|)\right)} \tau\left(|A|E_{[x-r,x+r]}(|A|)\right),\ x \geq 0,$$

(let $\frac{0}{0} = 0$). M is called the non-commutative Hardy-Littlewood maximal operator.

$MA(|A|)$ is represented as $MA(x)$. Then for A, we consider $MA(|A|)$ as the operator analogue of the Hardy-Littlewood maximal operator in the classical case. Hence roughly speaking, $MA(|A|)$ stands in relation to A as $Mf(x)$ stands in relation to f in classical analysis, i.e.

$$MA(|A|) = \int_{\sigma(|A|)} MA(\lambda) dE_\lambda(|A|),$$

where $\sigma(|A|)$ is the spectrum of $|A|$.

Define $L_{1,\infty}(\mathbb{R}_+) = \{f \in S(\mathbb{R}_+) : \sup_{t>0} t\mu(t,f) < \infty\}$ equipped with the quasi-norm

$$\|f\|_{1,\infty} = \sup_{t>0} t\mu(t,f).$$

It is well-known that this space is a quasi-Banach space (see [8]).

Let $C : L_1(\mathbb{R}_+) \to L_{1,\infty}(\mathbb{R}_+)$ be the Cesaro operator defined by

$$(Cf)(t) := \frac{1}{t} \int_0^t f(s)ds,\ f \in L_1(\mathbb{R}_+),\ t > 0.$$

In [14], it was proved the following result.

Theorem 1 *For every $A \in \mathcal{L}_{loc}(\mathcal{M},\tau)$, we have*

$$\mu(t, MA(|A|)) \leq 16 \cdot (C\mu(A))(t),\ \forall t > 0.$$

Definition 2 *[6, Definition 1.1, p. 49]. A function φ defined on the semiaxis $[0,\infty)$ is said to be quasiconcave if*

(i) $\varphi(t) = 0 \Leftrightarrow t = 0$;

(ii) $\varphi(t)$ *is positive and increasing for* $t > 0$;

(iii) $\frac{\varphi(t)}{t}$ *is decreasing for* $t > 0$.

Observe that every nonnegative concave function on $[0, \infty)$ that vanishes at origin is quasiconcave. The reverse, however, is not always true. However, we may replace, if necessary, a quasiconcave function φ by its least concave majorant $\widetilde{\varphi}$ such that

$$\frac{1}{2}\widetilde{\varphi} \leq \varphi \leq \widetilde{\varphi}$$

(see [7, Proposition 5.10, p.71]).

Let Ω denote the set of increasing concave functions $\varphi : [0, \infty) \to [0, \infty)$ for which $\lim\limits_{t \to 0+} \varphi(t) = 0$(or simply $\varphi(+0) = 0$). For the function φ in Ω, the Lorentz space $\Lambda_\varphi(\mathbb{R}_+)$ is defined by setting

$$\Lambda_\varphi(\mathbb{R}_+) := \left\{ x \in S(\mathbb{R}_+) : \int\limits_{\mathbb{R}_+} \mu(s, x) d\varphi(s) < \infty \right\},$$

and equipped with the norm

$$\|x\|_{\Lambda_\varphi(\mathbb{R}_+)} := \int\limits_{\mathbb{R}_+} \mu(s, x) d\varphi(s).$$

Let ψ be a quasiconcave function on $[0, \infty)$. Define the Marcinkiewicz space $M_\psi(\mathbb{R}_+)$ by setting

$$M_\psi(\mathbb{R}_+) := \left\{ f \in S(\mathbb{R}_+) : \|f\|_{M_\psi(\mathbb{R}_+)} < \infty \right\}$$

equipped with the norm

$$\|f\|_{M_\psi(\mathbb{R}_+)} := \sup_{t>0} \frac{1}{\psi(t)} \int\limits_0^t \mu(s, f) ds.$$

The space $(L_1 + L_\infty)(\mathbb{R}_+) = L_1(\mathbb{R}_+) + L_\infty(\mathbb{R}_+)$ consists of functions which are sums of bounded measurable and summable functions $x \in S(\mathbb{R}_+)$ equipped with the norm given by

$$\|x\|_{(L_1+L_\infty)(\mathbb{R}_+)} = \inf \left\{ \|x_1\|_{L_1(\mathbb{R}_+)} + \|x_2\|_{L_\infty(\mathbb{R}_+)} : x = x_1 + x_2, x_1 \in L_1(\mathbb{R}_+), x_2 \in L_\infty(\mathbb{R}_+) \right\}.$$

For more details on Lorentz spaces, we refer the reader to [7, Chapter I and II] and [6, Chapter II].

As in the commutative case, for a function φ in Ω define the corresponding non-commutative Lorentz space by setting

$$\Lambda_\varphi(\mathcal{M}) := \left\{ A \in S(\mathcal{M}, \tau) : \int\limits_{\mathbb{R}_+} \mu(s, A) d\varphi(s) < \infty \right\},$$

and equipped with the norm

$$\|A\|_{\Lambda_\varphi(\mathcal{M})} := \int_{\mathbb{R}_+} \mu(s,A)d\varphi(s).$$

3 Main results.

Let φ be an increasing concave function on $[0,\infty)$ such that $\varphi(+0)=0$ and satisfying

$$\varphi(t) \geq ct\log(1+1/t),\ t>0. \tag{2}$$

The following is the main result of this paper.

Theorem 2 *Let φ be an increasing concave function such that $\varphi(+0)=0$ and satisfying* (2) *and let ψ be an increasing concave function on $[0,\infty)$ such that $\psi(+0)=0$ and*

$$\int_t^\infty \frac{\psi(s)}{s^2}ds \leq \frac{\varphi(t)}{t},\ t>0. \tag{3}$$

Then the non-commutative Hardy-Littlewood maximal operator

$$MA(\cdot) : \Lambda_\varphi(\mathcal{M}) \to \Lambda_\psi(\mathcal{M})$$

is bounded.

Proof. It is known that if φ satisfies (2), then Cesaro operator (see [11, Proposition 4.4])

$$C : \Lambda_\varphi(\mathbb{R}_+) \to \Lambda_\psi(\mathbb{R}_+)$$

is bounded if and only if (3) holds. Moreover, $\Lambda_\psi(\mathbb{R}_+)$ is minimal among such symmetric Banach function spaces. That means, there exists a constant $c>0$ such that

$$\|Cf\|_{\Lambda_\psi(\mathbb{R}_+)} \leq c\|f\|_{\Lambda_\varphi(\mathbb{R}_+)},\ \forall f\in\Lambda_\varphi(\mathbb{R}_+). \tag{4}$$

Hence, by Theorem 1 and by (4), for any $A\in\Lambda_\varphi(\mathcal{M})$ we have

$$\|MA(|A|)\|_{\Lambda_\psi(\mathcal{M})} := \|\mu(MA(|A|))\|_{\Lambda_\psi(\mathbb{R}_+)} \leq 16\|C\mu(A)\|_{\Lambda_\psi(\mathbb{R}_+)}$$

$$\leq 16c\|\mu(A)\|_{\Lambda_\psi(\mathbb{R}_+)} = 16c\|A\|_{\Lambda_\varphi(\mathcal{M})},$$

where c is an absolute constant independent of other parametrs. This concludes the proof.

Corollary 1 *Let $\varphi_\alpha(t) = t\log^\alpha\left(1+t^{-1/\alpha}\right)$, $\alpha \geq 1$. Then the non-commutative Hardy-Littlewood maximal operator*

$$MA(\cdot) : \Lambda_{\varphi_\alpha}(\mathcal{M}) \to \Lambda_{\varphi_{\alpha+1}}(\mathcal{M})$$

is bounded.

Proof. It was proved in [11] that the function $\varphi_\alpha(t)$ satisfies

$$\int_t^\infty \frac{\varphi_\alpha(s)}{s^2} ds \le \frac{\varphi_{\alpha+1}(t)}{t}, \ t > 0.$$

Therefore, the assertion follows from Theorem 2.

Moreover, we illustrate two examples of Lorentz spaces which the Hardy-Littlewood maximal operator bounded on.

Example 1 *Let* $\varphi(t) = max\{1, t\}, \ t > 0$. *Then*

$$MA(\cdot) : \Lambda_\varphi(\mathcal{M}) \to M_\psi(\mathcal{M})$$

is bounded, where $\psi(t) = \log(1 + t)$, $t > 0$. *Here*

$$M_\psi(\mathcal{M}) = \{A \in S(\mathcal{M}, \tau) : \mu(A) \in M_\psi(\mathbb{R}_+).\}$$

Indeed, the Cesaro operator C *is bounded from* $\Lambda_\varphi(\mathbb{R}_+)$ *into* $M_\psi(\mathbb{R}_+)$ *(see Example 2.9 in [12]). Therefore, by Theorem 1 we obtain the desired result.*

Example 2 *Let* $\varphi(t) = t \log(1 + \frac{1}{t})$, $t > 0$. *Then*

$$MA(\cdot) : \Lambda_\varphi(\mathcal{M}) \to (L_1 + L_\infty)(\mathcal{M})$$

is bounded. Here, $(L_1 + L_\infty)(\mathcal{M}) = \{A \in S(\mathcal{M}, \tau) : \mu(A) \in (L_1 + L_\infty)(\mathbb{R}_+)\}$ *As in Example 1, it was shown in [12, Example 2.10] that* $C : \Lambda_\varphi(\mathbb{R}_0) \to (L_1 + L_\infty)(\mathbb{R}_+)$ *is bounded. Hence, the assertion follows from Theorem 1.*

4 Conclusion

In this paper, we investigated boundedness of Hardy-Littlewood maximal operator (in the sense of T. Bekzhan) on non-commutative Lorentz spaces. As an illustration we showed several examples of non-commutative Lorentz spaces in which the maximal Hardy-Littlewood operator is bounded.

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