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IDENTIFICATION OF THE COEFFICIENTS OF EQUATION FOR A VIBRATING ROD IN ACOUSTIC DIAGNOSTICS

The work is devoted to the study solving some inverse problem of identifying the coefficients of Sturm-Liouville operator. Inverse problems in vibration are concerned with constructing a vibrating system of a particular type, e.g., a string, a rod, that has specified properties. During the operation of the technical design, the dynamic characteristics can be changed by changing the boundary connection. Often these compounds are not directly accessible and their states can be judged from indirect information. In acoustic diagnostics, often the available information is the natural frequencies. Thus, by the set of natural frequencies it is necessary to estimate the state of the boundary connections. In this work an algorithm for constructive determination of coefficients of Sturm-Liouville operator is given. A straightforward solution of the inverse problem for Sturm-Liouville equation in a rod is presented.

Key words: Sturm-Liouville equation, inverse problems, natural frequency, identification of the coefficients, acoustic diagnostics, longitudinal vibrations, oscillator equation.

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Акустикалық диагностикада білікшенің тербеліс теңдеуінің коэффициенттерін идентификациялау

Бұл жұмыс Штурм-Лиувилль операторының коэффициенттерін анықтаудың кері есептерін шешуге арналған. Тербеліске байланысты кері мәселелер белгілі бір типтегі тербелмелі жүйенің құрылысына байланысты, мысалы, белгілі бір қасиеттері бар стержень, білікше. Техникалық жобаны пайдалану кезінде динамикалық сипаттамаларды қосылыстың шекарасын өзгерту арқылы өзгертуге болады. Көбінесе бұл қосылыстарға тікелей қол жетімді емес, олардың жағдайына жанама ақпарат арқылы баға беруге болады. Акустикалық диагностикада жиі кездесетін ақпарат бар. Осылайша, табиғи жиіліктер жиынтығына сәйкес шекаралық байланыстардың жағдайын бағалау қажет. Бұл жұмыста Штурм-Лиувилль операторының коэффициенттерін конструктивті түрде анықтау алгоритмі ұсынылған. Білікшедегі Штурм-Лиувилль теңдеуі үшін кері есептің шешімі ұсынылған.

Түйін сөздер: Штурм-Лиувилль теңдеуі, кері есептер, табиғи жиілік, коэффициентті идентификациялау, акустикалық диагностика, бойлық тербелістер, осциллятор теңдеуі.

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Идентификация коэффициентов уравнения колебаний струны в акустической диагностике

Работа посвящена исследованию решения обратной задачи идентификации коэффициентов оператора Штурма-Лиувилля. Обратные задачи, связанные с колебанием, связаны с конструированием колебательной системы определенного типа, например, струны, стержня, который имеет определенные свойства. Во время работы технического проекта, динамические характеристики могут быть изменены путем изменения границы соединения. Часто эти соединения не являются непосредственно доступными, и их состояния можно судить по косвенной информации. В акустической диагностике часто доступной информацией являются собственные частоты. Таким образом, по множеству собственных частот необходимо оценить состояние граничных связей. В данной работе приведен алгоритм конструктивного определения коэффициентов оператора Штурма-Лиувилля. Представлено решение обратной задачи для уравнения Штурма-Лиувилля для струны.

Ключевые слова: Уравнение Штурма-Лиувилля, обратные задачи, собственная частота, идентификация коэффициентов, акустическая диагностика, продольные колебания, уравнение осциллятора.

1 Introduction

The Sturm-Liouville equation can appear in three different forms. The one

$$(S(x)u'(x))' + \rho^2 S(x)u(x) = 0, \quad 0 \le x \le l \tag{1}$$

occurs in the longitudinal or torsional vibrations of a straight rod of cross section $S(x)$.

The work is devoted to the study solving some inverse problem of identifying the coefficients of Sturm-Liouville operator. Inverse problems in vibration are concerned with constructing a vibrating system of a particular type, e.g., a string, a rod, that has specified properties. During the operation of the technical design, the dynamic characteristics can be changed by changing the boundary connection. Often these compounds are not directly accessible and their states can be judged from indirect information. In acoustic diagnostics, often the available information is the natural frequencies. Thus, by the set of natural frequencies it is necessary to estimate the state of the boundary connections. It turned out that the problems of technical diagnosis from the mathematical point of view are the so-called inverse problems of mathematical physics. The question of identifying the coefficients of the of Sturm-Liouville equation has been studied widely in the works [1–3]. Different approach to the reconstruction from the spectra is discussed in the works [4–9]. In this work an algorithm for constructive determination of coefficients of Sturm-Liouville operator is presented.

We consider the oscillator equation of vibrating rod (1) with boundary conditions

$$u(0) = 0, \quad u'(l) = 0, \tag{2}$$

where l is the length of the rod. We denote by $A(x)$ the cross section area of a rod. And by $u(x)$ we denote the longitudinal deviation of the point x. And parameter ρ describes the natural frequency of the rod:

$$0 = \rho_0 < \rho_1 < \ldots < \rho_n = \frac{\pi n}{2l}.$$

We study the longitudinal vibration of uniform rod that has three parts. We identify the cross section area of the any part of the rod by the given natural frequency of the rod. In this case, each parts of $S(x)$ satisfy the following conditions

$$S(x) = S_1 \quad \text{for} \quad 0 < x < \frac{l}{3},$$

$$S(x) = S_2 \quad \text{for} \quad \frac{l}{3} < x < \frac{2l}{3},$$

$$S(x) = S_3 \quad \text{for} \quad \frac{2l}{3} < x < l.$$

Hence, we write the vibrating rod equation for each part of the rod as follows

$$S_1 u''(x) + \rho^2 S_1 u(x) = 0 \quad \text{for} \quad 0 < x < \frac{l}{3},$$

$$S_2 u''(x) + \rho^2 S_2 u(x) = 0 \quad \text{for} \quad \frac{l}{3} < x < \frac{2l}{3},$$

$$S_3 u''(x) + \rho^2 S_3 u(x) = 0 \quad \text{for} \quad \frac{2l}{3} < x < l,$$

Lemma 1 *The equation* (1) *is equivalent to the system of equation*

$$
\begin{cases}
u'(x) = \dfrac{i\rho}{S(x)} \cdot v(x), \\
v'(x) = i\rho S(x) \cdot u(x).
\end{cases}
\tag{3}
$$

Proof. We introduce the following notations

$$
\begin{cases}
S_1 u'(x) = i\rho\, v(x) \\
i\rho v'(x) = i\rho S_1 u(x)
\end{cases}
\quad \text{for} \quad 0 < x < \frac{l}{3}
$$

$$
\begin{cases}
S_2 u'(x) = i\rho\, v(x) \\
i\rho v'(x) = i\rho S_2 u(x)
\end{cases}
\quad \text{for} \quad \frac{l}{3} < x < \frac{2l}{3}
$$

$$
\begin{cases}
S_3 u'(x) = i\rho\, v(x) \\
i\rho v'(x) = i\rho S_3 u(x)
\end{cases}
\quad \text{for} \quad \frac{2l}{3} < x < l.
$$

We will rewrite the last system of equations in the form (3) or

$$
\begin{bmatrix} \dfrac{d}{dx}u(x) \\ \dfrac{d}{dx}v(x) \end{bmatrix}
=
\begin{bmatrix} 0 & \dfrac{-i\rho}{S(x)} \\ -i\rho S(x) & 0 \end{bmatrix}
\begin{bmatrix} u(x) \\ v(x) \end{bmatrix}
$$

The proof of the lemma is complete.

Lemma 2 *The following boundary value problem*

$$U'(x) = i\rho U(x),$$
$$V'(x) = -i\rho V(x)$$

(4)

$$U(0) = 0, \quad U(l) = 0,$$
$$V(0) = 0, \quad V(l) = 0.$$

(5)

on the interval $0 < x < l$ is equivalent to the problem (1), (2).

Proof. The function $S(x)u'(x)$ will be continuous on the interval $[0, l]$, since there exists $(S(x)u'(x))'$. Consequently, the function $v(x) = \dfrac{S(x)u'(x)}{i\rho}$ is continuous on the interval $[0, l]$. Therefore the conditions $v\left(\dfrac{l}{3} - 0\right) = v\left(\dfrac{l}{3} + 0\right)$, $v\left(\dfrac{2l}{3} - 0\right) = v\left(\dfrac{2l}{3} + 0\right)$ hold.

The function $u(x)$ will be continuous on the interval $[0, l]$, since there exists $u'(x)$. Therefore the conditions $u\left(\dfrac{l}{3} - 0\right) = u\left(\dfrac{l}{3} + 0\right)$, $u\left(\dfrac{2l}{3} - 0\right) = u\left(\dfrac{2l}{3} + 0\right)$ hold.

Now we introduce the following substitutions $\mu^2 = S(x)$. Then

$$\begin{cases} u(x) = \dfrac{2U(x)}{\mu} + \dfrac{2V(x)}{\mu}, \\ v(x) = 2\mu U(x) - 2\mu V(x). \end{cases}$$

The following equalities

$$U(x) = \frac{1}{2}\left(\mu_1 u(x) + \frac{1}{\mu_1}v(x)\right),$$
$$V(x) = \frac{1}{2}\left(\mu_1 u(x) - \frac{1}{\mu_1}v(x)\right)$$

(6)

hold for $0 < x < \dfrac{l}{3}$. Hence for part $0 < x < \dfrac{l}{3}$ we have the following equations

$$U'(x) = \frac{1}{2}\left(\mu_1 u'(x) + \frac{1}{\mu_1}v'(x)\right) = \frac{1}{2}\left(\mu_1\frac{i\rho v(x)}{S_1} + \frac{1}{\mu_1}i\rho S_1 u(x)\right),$$
$$V'(x) = \frac{1}{2}\left(\mu_1 u'(x) - \frac{1}{\mu_1}v'(x)\right) = \frac{1}{2}\left(\mu_1\frac{i\rho v(x)}{S_1} - \frac{1}{\mu_1}i\rho S_1 u(x)\right).$$

(7)

Since $S_1 = \mu_1^2$, then the last system of equations can be represented as follows

$$U'(x) = \frac{1}{2}i\rho\left(\frac{v(x)}{\mu_1} + \mu_1 u(x)\right) = i\rho U,$$

$$V'(x) = \frac{1}{2}i\rho\left(\frac{v(x)}{\mu_1} - \mu_1 u(x)\right) = -i\rho V.$$

By arguing as in the previous computations, for $\dfrac{l}{3} < x < \dfrac{2l}{3}$ we have the following equations

$$U'(x) = \frac{1}{2}i\rho\left(\frac{v(x)}{\mu_2} + \mu_2 u(x)\right) = i\rho U,$$

$$V'(x) = \frac{1}{2}i\rho\left(\frac{v(x)}{\mu_2} - \mu_2 u(x)\right) = -i\rho V,$$

where $\mu_2 = \sqrt{S_2}$.

In a similar way for $\frac{2l}{3} < x < l$ we have the following equations

$$U'(x) = \frac{1}{2}i\rho\left(\frac{v(x)}{\mu_3} + \mu_3 u(x)\right) = i\rho U,$$

$$V'(x) = \frac{1}{2}i\rho\left(\frac{v(x)}{\mu_3} - \mu_3 u(x)\right) = -i\rho V,$$

where $\mu_3 = \sqrt{S_3}$.

Noted above equalities imply that the system of equations (4) is equivalent to equation (1) for all parts of the interval $0 < x < l$. Thus, equation (1) is equivalent to the system of equation (4). And if we substitute the boundary conditions (2) into the equations (3), (6), (7), then conditions (2) can be rewritten in the following form

$$U(0) = 0, \quad U(l) = 0,$$

$$V(0) = 0, \quad V(l) = 0.$$

Thus, the proof of the lemma is complete.

In the case then the interval $[0, l]$ is divided into three parts, we state the connection between two boundary values

$$\begin{bmatrix} U(+0) \\ V(+0) \end{bmatrix} \quad and \quad \begin{bmatrix} U(l-0) \\ V(l-0) \end{bmatrix}$$

of the interval $[0, l]$ in the following theorem.

Theorem 1 *For the boundary value problem (4), (5) the following equality*

$$\begin{bmatrix} U(+0) \\ V(+0) \end{bmatrix} = \begin{bmatrix} e^{-i\rho\frac{l}{3}} & 0 \\ 0 & e^{i\rho\frac{l}{3}} \end{bmatrix} \gamma_1 \begin{bmatrix} e^{-i\rho\frac{l}{3}} & 0 \\ 0 & e^{i\rho\frac{l}{3}} \end{bmatrix} \gamma_2 \begin{bmatrix} e^{-i\rho\frac{l}{3}} & 0 \\ 0 & e^{i\rho\frac{l}{3}} \end{bmatrix} \begin{bmatrix} U(l-0) \\ V(l-0) \end{bmatrix} \quad (8)$$

holds.

Proof. As shown in the proof of the lemma 2 that

$$\begin{cases} U\left(\frac{l}{3} - 0\right) = U(+0)e^{i\rho\frac{l}{3}}, \\ V\left(\frac{l}{3} - 0\right) = V(+0)e^{i\rho\frac{l}{3}}. \end{cases}$$

Consequently, it follows that

$$\begin{bmatrix} U\left(\frac{l}{3} - 0\right) \\ V\left(\frac{l}{3} - 0\right) \end{bmatrix} = \begin{bmatrix} e^{i\rho\frac{l}{3}} & 0 \\ 0 & e^{-i\rho\frac{l}{3}} \end{bmatrix} \begin{bmatrix} U(+0) \\ V(+0) \end{bmatrix}. \quad (9)$$

Equality (9) implies that

$$\begin{bmatrix} U(+0) \\ V(+0) \end{bmatrix} \cdot = \begin{bmatrix} e^{-i\rho\frac{l}{3}} & 0 \\ 0 & e^{i\rho\frac{l}{3}} \end{bmatrix} \cdot \begin{bmatrix} U\left(\dfrac{l}{3}-0\right) \\ V\left(\dfrac{l}{3}-0\right) \end{bmatrix}. \tag{10}$$

In a similar way we write

$$\begin{cases} U\left(\dfrac{2l}{3}-0\right) = U\left(\dfrac{l}{3}+0\right) e^{i\rho\frac{l}{3}}, \\ V\left(\dfrac{2l}{3}-0\right) = V\left(\dfrac{l}{3}+0\right) e^{-i\rho\frac{l}{3}}. \end{cases}$$

The last system of equation we rewrite as follows

$$\begin{bmatrix} U\left(\dfrac{2l}{3}-0\right) \\ V\left(\dfrac{2l}{3}-0\right) \end{bmatrix} = \begin{bmatrix} e^{i\rho\frac{l}{3}} & 0 \\ 0 & e^{-i\rho\frac{l}{3}} \end{bmatrix} \begin{bmatrix} U\left(\dfrac{l}{3}+0\right) \\ V\left(\dfrac{l}{3}+0\right) \end{bmatrix}.$$

By arguing in a similar way for we write

$$\begin{cases} U\left(l-0\right) = U\left(\dfrac{2l}{3}+0\right) e^{i\rho\frac{l}{3}}, \\ V\left(l-0\right) = V\left(\dfrac{2l}{3}+0\right) e^{-i\rho l} \end{cases}$$

$$\begin{bmatrix} U(l-0) \\ V(l-0) \end{bmatrix} = \begin{bmatrix} e^{i\rho\frac{l}{3}} & 0 \\ 0 & e^{-i\rho\frac{l}{3}} \end{bmatrix} \begin{bmatrix} U\left(\dfrac{2l}{3}+0\right) \\ V\left(\dfrac{2l}{3}+0\right) \end{bmatrix}. \tag{11}$$

On the other hand the following equalities

$$\begin{bmatrix} U(+0) \\ V(+0) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \mu_1 & \dfrac{1}{\mu_1} \\ \mu_1 & -\dfrac{1}{\mu_1} \end{bmatrix} \cdot \begin{bmatrix} u(+0) \\ v(+0) \end{bmatrix} \tag{12}$$

$$\begin{bmatrix} U\left(\dfrac{l}{3}+0\right) \\ V\left(\dfrac{l}{3}+0\right) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \mu_2 & \dfrac{1}{\mu_2} \\ \mu_2 & -\dfrac{1}{\mu_2} \end{bmatrix} \cdot \begin{bmatrix} u\left(\dfrac{l}{3}+0\right) \\ v\left(\dfrac{l}{3}+0\right) \end{bmatrix} \tag{13}$$

$$\begin{bmatrix} U\left(\dfrac{2l}{3}+0\right) \\ V\left(\dfrac{2l}{3}+0\right) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \mu_3 & \dfrac{1}{\mu_3} \\ \mu_3 & -\dfrac{1}{\mu_3} \end{bmatrix} \cdot \begin{bmatrix} u\left(\dfrac{2l}{3}+0\right) \\ v\left(\dfrac{2l}{3}+0\right) \end{bmatrix} \tag{14}$$

hold. Therefore, by applying the proof of the lemma 2, we obtain

$$\begin{bmatrix} U\left(\dfrac{l}{3}-0\right) \\ V\left(\dfrac{l}{3}-0\right) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \mu_1 & \dfrac{1}{\mu_1} \\ \mu_1 & -\dfrac{1}{\mu_1} \end{bmatrix} \cdot \begin{bmatrix} u\left(\dfrac{l}{3}-0\right) \\ v\left(\dfrac{l}{3}-0\right) \end{bmatrix} =$$

$$= \frac{1}{2}\begin{bmatrix} \mu_1 & \dfrac{1}{\mu_1} \\ \mu_1 & -\dfrac{1}{\mu_1} \end{bmatrix} \cdot \begin{bmatrix} u\left(\dfrac{l}{3}+0\right) \\ v\left(\dfrac{l}{3}+0\right) \end{bmatrix} \tag{15}$$

$$\begin{bmatrix} U\left(\dfrac{2l}{3}-0\right) \\ V\left(\dfrac{2l}{3}-0\right) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \mu_2 & \dfrac{1}{\mu_2} \\ \mu_2 & -\dfrac{1}{\mu_2} \end{bmatrix} \cdot \begin{bmatrix} u\left(\dfrac{2l}{3}-0\right) \\ v\left(\dfrac{2l}{3}-0\right) \end{bmatrix} =$$

$$= \frac{1}{2}\begin{bmatrix} \mu_2 & \dfrac{1}{\mu_2} \\ \mu_2 & -\dfrac{1}{\mu_2} \end{bmatrix} \cdot \begin{bmatrix} u\left(\dfrac{2l}{3}+0\right) \\ v\left(\dfrac{2l}{3}+0\right) \end{bmatrix} \tag{16}$$

$$\begin{bmatrix} U(l-0) \\ V(l-0) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \mu_3 & \dfrac{1}{\mu_3} \\ \mu_3 & -\dfrac{1}{\mu_3} \end{bmatrix} \cdot \begin{bmatrix} u(l+0) \\ v(l+0) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \mu_3 & \dfrac{1}{\mu_3} \\ \mu_3 & -\dfrac{1}{\mu_3} \end{bmatrix} \cdot \begin{bmatrix} u(l-0) \\ v(l-0) \end{bmatrix}.$$

By substituting equality (13) into (15), we have

$$\begin{bmatrix} U\left(\dfrac{l}{3}-0\right) \\ V\left(\dfrac{l}{3}-0\right) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \mu_1 & \dfrac{1}{\mu_1} \\ \mu_1 & -\dfrac{1}{\mu_1} \end{bmatrix} \left(\frac{1}{2}\begin{bmatrix} \mu_2 & \dfrac{1}{\mu_2} \\ \mu_2 & -\dfrac{1}{\mu_2} \end{bmatrix} \right)^{-1} \begin{bmatrix} U\left(\dfrac{l}{3}+0\right) \\ V\left(\dfrac{l}{3}+0\right) \end{bmatrix}, \tag{17}$$

here

$$\frac{1}{2}\begin{bmatrix} \mu_1 & \dfrac{1}{\mu_1} \\ \mu_1 & -\dfrac{1}{\mu_1} \end{bmatrix} \left(\frac{1}{2}\begin{bmatrix} \mu_1 & \dfrac{1}{\mu_1} \\ \mu_1 & -\dfrac{1}{\mu_1} \end{bmatrix} \right)^{-1} = \gamma_1 = \frac{S_2 - S_1}{S_1 + S_2}.$$

Also by substituting equality (14) into (16), we obtain

$$\begin{bmatrix} U\left(\dfrac{2l}{3}-0\right) \\ V\left(\dfrac{2l}{3}-0\right) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \mu_2 & \dfrac{1}{\mu_2} \\ \mu_2 & -\dfrac{1}{\mu_2} \end{bmatrix} \left(\frac{1}{2}\begin{bmatrix} \mu_3 & \dfrac{1}{\mu_3} \\ \mu_3 & -\dfrac{1}{\mu_3} \end{bmatrix} \right)^{-1} \begin{bmatrix} U\left(\dfrac{2l}{3}+0\right) \\ V\left(\dfrac{2l}{3}+0\right) \end{bmatrix}, \tag{18}$$

where

$$\frac{1}{2}\begin{bmatrix} \mu_2 & \dfrac{1}{\mu_2} \\ \mu_2 & -\dfrac{1}{\mu_2} \end{bmatrix} \left(\frac{1}{2}\begin{bmatrix} \mu_3 & \dfrac{1}{\mu_3} \\ \mu_3 & -\dfrac{1}{\mu_3} \end{bmatrix} \right)^{-1} = \gamma_2 = \frac{S_3 - S_2}{S_3 + S_2}.$$

Substitutions of (17) and (18) into (10) gives formula (8).

The proof of the theorem is complete.

By taking into account the fact $\exp(i\rho\Delta) = z^{\frac{1}{2}}$, formula (8) implies the polynomial with respect to z. By virtue of $\gamma_j = \dfrac{S_j - S_{j-1}}{S_j + S_{j-1}}$, formula $\rho = \dfrac{z - \gamma}{1 - \overline{\gamma}z}$ implies that the unknown cross section areas A_1, A_2, A_3.

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