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ALGORITHMIC COMPLEXITY OF LINEAR NONASSOCIATIVE ALGEBRA

One of the central problems of algebraic complexity theory is the complexity of multiplication in algebras. For this, first, the concept of algebra is defined and the class of algebras under study is fixed. Then the concept of the algorithm and its complexity are clarified. in the most general sense, an algebra is a set with operations. An operation is defined, as a rule, as a function of one or more elements of a set, the set of values of which is the original set or some of its subset. Usually, a set of elementary operations is fixed, for example, a Boolean operation on two bits, addition or multiplication of two numbers, after which a computation model is fixed, for example, a sequential algorithm, at each step of which one elementary operation is performed on some inputs and the results of intermediate calculations, the result of which can be used to enter an elementary operation at subsequent steps of the algorithm. The most significant ones are the column-by-column multiplication algorithm, which has quadratic complexity (along the input length) and the row-by-column matrix multiplication algorithm, which has $O(mnp)$ complexity for multiplying $m \times n$ by $n \times p$ matrices. Estimation of the complexity of algebras from other more complex classes is relevant. in this paper, we derive an estimate for the complexity of a nondegenerate symmetric bilinear form over an algebraic closed field for a simple Jordan algebra, as well as an estimate for a Cayley-Dixon body and for a simple Lie algebra over a characteristic field.

Key words: algebra complexity, optimal algorithm, simple algebra, Cayley-Dixon body, Lie algebra, characteristic field.

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Кейбір сызықты ассоциативтік емес алгебраның алгоритмдік күрделілігі

Алгебралық күрделілік теориясының негізгі мәселелерінің бірі - алгебралардағы көбейтудің күрделілігі. Ол үшін, біріншіден, алгебра ұғымы анықталып, зерттелетін алгебралар класы бекітіледі. Содан кейін алгоритм түсінігі және оның күрделілігі нақтыланады. Жалпы мағынада алгебра - амалдардан тұратын жиынтық. Операция, ереже бойынша, жиынның бір немесе бірнеше элементтерінің функциясы ретінде анықталады, оның мәндері жиынтығы бастапқы жиын немесе оның кейбір бөлігі болып табылады. Әдетте, қарапайым операциялардың жиынтығы, мысалы, екі битке логикалық операция, екі санды қосу немесе көбейту, содан кейін есептеу моделі бекітілген, оған мысал - кезекті алгоритм, оның әр сатысында бір элементар амал орындалады. Кейбір кірістер мен аралық есептеулердің нәтижелері, олардың нәтижесі алгоритмнің келесі кезеңдерінде элементар әрекетті енгізу үшін қолданыла алады. Ең маңыздыларына баған бойынша көбейту алгоритмі, және $m \times n$ өлшемді матрицаны $n \times p$ өлшемді матрицаға көбейтетін квадраттық $O(mnp)$ күрделілігі бар (кіріс ұзындығы бойынша) көбейту алгоритмі болып табылады. Басқа күрделі кластардан алгебралардың күрделігін бағалау өте маңызды. Бұл жұмыста қарапайым йордан алгебрасы үшін алгебралық жабық өрістен азайтылмайтын симметриялы билинарлы форманың күрделілігінің бағасы, сонымен қатар Кейли-Диксон денесі мен қарапайым өріс үшін Ли алгебрасы үшін баға алынған.

Түйін сөздер: алгебраның күрделілігі, оңтайлы алгоритм, қарапайым алгебра, Кейли-Диксон денесі, Ли алгебрасы, характеристикалық өріс.

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Алгоритмическая сложность некоторых линейных неассоциативных алгебр

Одной из центральных задач алгебраической теории сложности является сложность умножения в алгебрах. Для этого сначала определяется понятие алгебры и фиксируется класс изучаемых алгебр. Затем уточняется понятие алгоритма и его сложности. В наиболее общем смысле алгеброй называется множество с операциями. Операция определяется, как правило, как функция одного или нескольких элементов множества, множеством значений которой является исходное множество или некоторое его подмножество. Обычно фиксируется некоторое множество элементарных операций, например, булева операция над двумя битами, сложение или умножение двух чисел, после чего фиксируется модель вычислений, например, последовательный алгоритм, на каждом из шагов которого выполняется одна элементарная операция над некоторыми входами и результатами промежуточных вычислений, результат которой может быть использован для входа элементарной операции на последующих шагах алгоритма. К наиболее значимым следует отнести алгоритм умножения чисел "в столбик имеющий квадратичную сложность (по длине входа) и алгоритм умножения матриц "строка на столбец имеющий сложность $O(mnp)$ для умножения матриц размера $m \times n$ на $n \times p$. Оценка сложности алгебр из других более сложных классов актуальна. В данной статье мы выводим оценку сложности невырожденной симметрической билинейной формы над алгебраическим замкнутым полем для простой йордановой алгебры, а также оценку для тела Кэли-Диксона и простой алгебры Ли над характеристическим полем.

Ключевые слова: сложность алгебры, оптимальный алгоритм, простая алгебра, тело Кэли-Диксона, алгебра Ли, характеристическое поле.

1 Introduction

The complexity $L(A)$ of a finite-dimensional algebra A is a multiplication number (non-scalar:), divisions of the optimal algorithm, computing the production of two elements of algebra.

In the work [1] for associated algebra the results are obtained:

1)$L(A) \geq L(A/radA) + 2 * dim(radA)$,

where $radA$ is a radical of algebra.

2)$L(A) \geq 2 * dimA - 1$,

where A is a simple algebra. More general: for simple algebra A and arbitrary algebra B it is proved, that

$$L(A \oplus B) \geq 2 * dimA - 1 + L(B)$$

As a result, for arbitrary finite-dimensional associative algebra A the final estimation is obtained:

3)$L(A) \geq 2 * dimA - t$,

where t is a number of maximal two-sided ideal of algebra A. Naturally, the issue about complexity of algebra from other classes is arisen.

We note, that the 1-st result, as a sequence of structural theorems [2], are true for jordan alternative algebra.

The 2-nd result for jordan algebra, in general, is not true. We show in §2 of this work, that for simple jordan algebra $B(f) = K * 1 + V$ nondegenerate symmetric bilinear form f over algebraic closed field K the complexity

$$L(B(f)) = 2 * dimB(f) - 2.$$

In §3 we show for keli-Dixon body C, that

$$15 \leq Z(C) \leq 30$$

In §4 we show for simple Li algebra $sl(2,K)$ over filed K the characteristics $\neq 2$, that

$$L(sl(2,K)) = 5.$$

2 The main definitions

We present some definitions from [1] below. Let K be infinite filed, $x_1, \ldots, x_n$ are variables over K.

Definition 1 *The sequence of rational functions $g_1, ..., g_r \in K(x_1, ..., x_n)$ are called by computing sequence, if for any number $\rho \leq r$ there exist*

$$u_\rho, v_\rho \in K + Kx_1 + ... + Kx_n + Kg_1 + ...Kg_{\rho-1},$$

such, that

$$g_\rho = u_\rho * v_\rho$$

or

$$g_\rho = u_\rho / v_\rho, v_\rho \neq 0.$$

Definition 2 *Let $f_1, ..., f_q \in K(x_1, ..., x_n)$. The complexity $L(f_1, ..., f_q)$ of set $f_1, ..., f_q$ is a least r with property: there exists a computing sequence $g_1, ..., g_r$ such, that for all $i \leq q$*

$$f_i \in K + Kx_1 + ... + Kx_n + Kg_1 + ...Kg_r$$

Let E, W be finite-dimensional vector K - spaces with basis, accordingly $e_1, ..., e_r$ and $\hat{e}_1, ...\hat{e}_q$.

Definition 3 *Mapping $f : E \to W$ is called by quadratic, if there exist quadratic forms $f_1, ..., f_q$ of $K[x_1, ..., x_n]$ such that for all $\xi_1, ..., \xi_n \in K$*

$$f(\sum_{i=1}^{n} \xi_i e_i) = \sum_{j=1}^{q} f_j(\xi_1, ..., \xi_n)\hat{e}_j$$

$L(f) = L(f_1, ..., f_q)$ is called by complexity f, where $f_1, ..., f_q$ are considered as elements $K(x_1, ..., x_n)$.

Let A is a finite-dimensional algebra with unit, $e_1, ..., e_n$ is a basis of vector space.

$$e_i * e_j = \sum_{m=1}^{n} \tau_{ijm} e_m,$$

where $\tau_{ijm} \in K, i, j, = 1, ..., n$. Then we get

$$(\sum_{i=1}^{n} \xi_i e_i) * (\sum_{j=1}^{n} \eta_j e_j) = \sum_{m=1}^{n} (\sum_{i,j=1}^{n} \tau_{ijm} \xi_i \eta_j) e_m$$

The elements $x, y \in A$ are considered as vector-columns with coordinates $x_1, ..., x_n$ and $y_1, ..., y_n$ accordingly. Then for $\sum_{i,j=1}^{n} \tau_{ijm} x_i y_j$ we get the following:

$$\sum_{i,j=1}^{n} \tau_{ijm} x_i y_j = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}^t \begin{pmatrix} \tau_{11m} & \tau_{12m} & \tau_{1nm} \\ \tau_{21m} & \tau_{22m} & \tau_{2nm} \\ \tau_{n1m} & \tau_{n2m} & \tau_{nnm} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = x^t T_m y,$$

where x^t is a vector-line, $T_m \in M_n(K), m = 1, \ldots, n$. For any $x, y \in A$ we get

$$xy = \sum_{m=1}^{n} (x^t T_m y) e_m$$

Let $T = \{T_1, \ldots, T_n\}$. We consider $M \subseteq M_n(K)$ - subset, for which power $|M| = r, r > 0, linM$ is a linear membrane M.

Definition 4 *M is called by the algorithm of the length r of algebra A, if the conditions are held:*
1) M consists of linearly independent matrixes,
2) for any $m \in M, rang = 1$,
3) $T \leq linM$.

Algorithm M is called by optimal, if its length does not exceed of the length of any algorithm of algebra A.

Multiplication in algebra A is a bilinear mapping $f : A \oplus A \to A$. For any $x, y \in A$ we get

$$f(x+y) = xy = \sum_{m=1}^{n} x^t T_m y e_m = \sum_{m=1}^{n} f_m(x_1, ..., x_n, y_1, ..., y_n) e_m = \sum_{m=1}^{n} f_m(x, y) e_m,$$

where $x^t T_m y = f_m(x_1, ..., x_n, y_1, ..., y_n) = f_m(x, y) \in k[x_1, ..., x_n, y_1, ..., y_n]$- are quadratic forms, $m = 1, ..., n$. In fact, $f : A \bigoplus A \to A$ is quadratic mapping.

Definition 5 *Complexity of algebra A is called complexity of quadratic mapping $f : A \bigoplus A \to A$, defined by the rule:*

$$f(x+y) = xy.$$

We denote the complexity of algebra A by $L(A)$.

Proposal 1 *$L(A)$ is equal to the optimal algorithm of algebra A.*

Proof. If M is optimal algorithm of the length r of algebra A, then

$$f_m(x_1, \ldots, x_n, y_1, \ldots, y_n) = \sum_{j=1}^{r} d_{m_j} x^t M_j y,$$

where $M_j \in M, j = 1, \ldots, r$. Any matrix C of rang 1 is represented in the form

$$C = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} (b_1, \ldots, b_n).$$

Therefore for each matrix M_j there exist such vector-lines u_j, v_j that $M_j = u_j^t v_j$; moreover

$$x^t M_j y = x^t u_j^t v_j y = u_j(x) v_j(y), j = 1, \dots, r,$$

where $u_j(x) = (x^t, u_j), v_j(y) = (y^t, v_j)$ are linear forms.

As a result we get

$$f_m(x, y) = \sum_{j=1}^{r} d_{mj} u_j(x) v_j(y)$$

hence, that $L(A) = L(f) \leq r$.

Let $L(f) = L(f_1, \dots, f_n) = k$. In the work [1] is proved, that the set of quadratic forms is optimally calculated without division. There exists

$$u_\rho \in K + Kx_1 + \dots + Kx_n,$$

$$v_\rho \in K + Ky_1 + \dots + Ky_n,$$

$\rho = 1, \dots, k$, such that

$$f_m(x, y) = \sum_{\rho=1}^{k} d_{m\rho} u_\rho(x) v_\rho(y).$$

If $u_\rho, v_\rho, \rho = 1, \dots, k$, are linear dependent, then the number k can be decreased, consequently u_ρ, v_ρ are linear independent.

Let $u_\rho(x) = (u_\rho, x^t), v_\rho(y) = (v_\rho, y^t)$, for some vectors u_ρ, v_ρ.

Then the matrix $M_\rho = u_\rho^t v_\rho$ has a rang 1, and we get

$$u_\rho(x) v_\rho(y) = x^t M_\rho y.$$

Then $f_m(x, y) = \sum_{j=1}^{k} d_{mj} x^t M_j y = x^t (\sum_{j=1}^{k} d_{mj} M_j) y$, on the other hand $f_m(x, y) = x^t T_m y$, hence $T_m \in lin(M_1, \dots, M_k)$. We have constructed the algorithm, the length of which is equal to $L(A)$ Now it is clear, that $L(A) \geq r$ and finally, we get, that $L(A) = r$.

Example 1 *Let C are field of complex numbers. For any $x, y \in C$ we get*

$$xy = (x_1 y_1 - x_2 y_2) + (x_1 y_2 - x_2 y_2) i,$$

i.e. $xy = x^t \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} y + x^t \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} yi,$

For C $T = \left\{ T_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, T_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$. *We take*

$$M = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}.$$

Then $L(C) \leq 3$. We suppose, that $M = \{M_1, M_2\}$ is an algorithm of the length 2 for C. Then there exist $d_1, d_2, d_3, d_4 \in R$ such that $d_1 d_4 - d_2 d_3 \neq 0$ and

$$\begin{cases} d_1 M_1 + d_2 M_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\ d_3 M_1 + d_4 M_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \end{cases}$$

Let $d_1 \neq 0$, *then* $M_1 = \frac{1}{d_1 d_4 - d_2 d_3}\begin{pmatrix} -d_3 & d_1 \\ d_1 & d_3 \end{pmatrix}$ *and* $d_1^2 + d_3^2 = 0$, *i.e.* $d_1 = 0$, *contradiction,* $L(C) = 3$.

3 Complexity of $B(f)$-jordan algebra of nondegenerated symmetric bilinear form

$B(f) = K1 + V$ is jordan algebra [2] of symmetric nondegenerated bilinear form $f : V \times V \to K$, where K is a field, V is the vector space over K dimensionality n. Multiplication in $B(f)$ is defined:

$$(x_0 * 1 + x)(y_0 * 1 + y) = (x_0 y_0 + f(x, y))(x_0 y_0 + y_0 x).$$

Theorem 1 *a) If K is algebraic closed field, then*

$$L(B(f)) = 2n.$$

b) If $K = \Re$ is a field of the real numbers, then

$$L(B(f)) = \begin{cases} 2n+1, if \quad f \quad is \quad negative \quad defined; \\ \quad 2n, in \quad the \quad other \quad case. \end{cases}$$

Proof. By choosing the canonical basis $l_1, \dots, l_n$ in V with respect to f for any $a, b \in B(f)$ we obtain, that

$$ab = (x_0 1 + x)(y_0 1 + y) = (x_0 y_0 + x_1 y_1 + \cdots + x_k y_k -$$

$$-x_{k+1} y_{k+1} - \cdots - x_n y_n)1 + (x_0 y_1 + x_0 y_1) l_1 + \cdots + (x_0 y_n + x_n y_0) l_n,$$

where $K = 0, 1, \dots, n$ for $K = \Re$ and $k = n$, if the field K is algebraic closed. If $K = 0$, then we take

$$f_0 = x_0 y_0, f_i = \Big(\sqrt{\frac{1}{n}} x_0 + x_i\Big)\Big(\sqrt{\frac{1}{n}} y_0 + y_i\Big),$$

$$g_i = \Big(\sqrt{\frac{1}{n}} x_0 + x_i\Big)\Big(\sqrt{\frac{1}{n}} y_0 - y_i\Big), i = 1, \dots, n.$$

Then

$$ab = \Big(2f_0 - \sum_{i=1}^{n} \frac{f_i + g_i}{2}\Big)1 + \sum_{i=1}^{n} \frac{f_i - g_i}{2} l_i.$$

If $0 < k < n$, then we take

$$f_i = \Big(\sqrt{\frac{2}{k}} x_0 + x_i\Big)\Big(\sqrt{\frac{2}{k}} y_0 + y_i\Big), g_i = \Big(\sqrt{\frac{2}{k}} x_0 - x_i\Big)\Big(\sqrt{\frac{2}{k}} y_0 - y_i\Big), i = 1, \dots, k,$$

$$f_i = \Big(\sqrt{\frac{2}{n-k}} x_0 + x_i\Big)\Big(\sqrt{\frac{2}{n-k}} y_0 + y_i\Big), i = k+1, \dots, n,$$

$$g_i = \Big(\sqrt{\frac{2}{n-k}} x_0 - x_i\Big)\Big(\sqrt{\frac{2}{n-k}} y_0 - y_i\Big), i = k+1, \dots, n.$$

Then

$$ab = \Big(\sum_{i=1}^{k} \frac{f_i + g_i}{2} - \sum_{i=k+1}^{n} \frac{f_i + g_i}{2}\Big)1 + \sum_{i=1}^{k} \sqrt{\frac{k}{2}} \frac{f_i - g_i}{2} l_i + \sum_{i=k+1}^{k} \sqrt{n-k} \frac{f_i - g_i}{2} l_i.$$

If $k = n$, then

$$f_i = \Big(\sqrt{\frac{1}{n}} x_0 + x_i\Big)\Big(\sqrt{\frac{1}{n}} y_0 + y_i\Big),$$

$$g_i = \Big(\sqrt{\frac{1}{n}} x_0 - x_i\Big)\Big(\sqrt{\frac{1}{n}} y_0 - y_i\Big), i = 1, \dots, n$$

and we get

$$ab = \sum_{i=1}^{k} \frac{f_i + g_i}{2} 1 + \sum_{i=1}^{n} \sqrt{n} \frac{f_i - g_i}{2} l_i.$$

Thus, we obtain, that

$$L(B(f)) \leq 2n, \tag{1}$$

Under $0 < K < n$ or algebraic closed K,

$$L(B(f)) \leq 2n + 1, \quad K = 0. \tag{2}$$

Now the elements $a, b \in B(f)$ are considered as vector-columns.

Then we get

$$ab = a^t \begin{pmatrix} E_{k+1} & 0 \\ 0 & -E_{n-k} \end{pmatrix} b \cdot 1 + \sum_{i=2}^{n+1} a^t (E_{1i} + E_{2i}) b \cdot l_{i-1} =$$

$$= a^t C_0 b + \sum_{i=2}^{n+1} a^t C_{i-1} b \cdot l_{i-1},$$

where $C_0, C_1, \dots, C_n \in M_{n+1}(K)$. Let $M \subseteq M_{n+1}(K)$ is the algorithm of the length r for $B(f)$. Then we get

$$C_i = \sum_{j=0}^{r} d_{ij} x_j,$$

where $i = 0, 1, \dots, n, x_j \in M, d_{ij} \in K$. We consider a matrix of order $n \times r$,

$$\begin{pmatrix} d_{11} & \cdots & d_{1n} & \cdots & d_{1r} \\ d_{21} & \cdots & d_{2n} & \cdots & d_{2r} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ d_{n1} & \cdots & d_{nn} & \cdots & d_{nr} \end{pmatrix}.$$

Since $C_1, \dots, C_n$ are linear independent, that there exists a minor M_1 of order $n \times n$, that $|M_1| \neq 0$. Without limitation of generality, we can assume, that

$$M_1 = \begin{pmatrix} d_{11} & \cdots & d_{1n} \\ \cdots & \cdots & \cdots \\ d_{n1} & \cdots & d_{nn} \end{pmatrix}.$$

Then from the system of equations

$$\begin{cases} d_{11}x_1 + \cdots + d_{1n}x_n = C_1 - \sum_{j=n+1}^{r} d_{1j}x_j = B_1 \\ d_{n1}x_2 + \cdots + d_{nn}x_n = C_n - \sum_{j=n+1}^{r} d_{nj}x_j = B_n \end{cases}$$

We find x_i;

$$x_i = \frac{1}{|M_i|}\sum_{j=1}^{n}(-1)^{i+j}M_1^{ij}B_j, i = 1, \ldots, n.$$

where M_1^{ij} is a minor of the element d_{ij} of the matrix M_1. By substituting $x_i, i = 1, \ldots, n$ to $C_0 = \sum_{j=1}^{r} d_{0j}x_i$ we obtain

$$C_0 = \frac{1}{|M_1|}(\gamma_1 B_1 - \gamma_2 B_2 + \cdots + (-1)^{n+1}\gamma_n B_n) + \sum_{j=n+1}^{r} d_{0j}x_j,$$

where

$$\gamma_1 = \begin{vmatrix} d_{01} & \cdots & d_{0n} \\ d_{21} & \cdots & d_{2n} \\ \cdots & \cdots & \cdots \\ d_{n1} & \cdots & d_{nn} \end{vmatrix}, \ldots, \gamma_n = \begin{vmatrix} d_{01} & \cdots & d_{0n} \\ d_{21} & \cdots & d_{2n} \\ \cdots & \cdots & \cdots \\ d_{n-11} & \cdots & d_{n-1n} \end{vmatrix}.$$

Hence

$$\Delta = C_0 + \sum_{i=1}^{n}\frac{(-1)^i\gamma_i}{|M_1|}C_i = \sum_{j=n+1}^{r}\frac{d_j}{|M_1|}x_i, \tag{3}$$

where

$$d_j = \begin{vmatrix} d_{01} & \cdots & d_{0n} & d_{0j} \\ d_{21} & \cdots & d_{2n} & d_{1j} \\ \cdots & \cdots & \cdots & \cdots \\ d_{n-11} & \cdots & d_{n-1n} & d_{nj} \end{vmatrix}, j = n+1, ..., r. \tag{4}$$

We assume $\delta_1 = \frac{(-1)^i\gamma_i}{|M_1|}, i = 1, \ldots, n$. Then in (3) for Δ we get

$$\Delta = \begin{pmatrix} 1 & \delta_1 & \cdots & \delta_k & \delta_{k+1} & \cdots & \delta_n \\ \delta_1 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \delta_k & 0 & \cdots & 1 & 0 & \cdots & 0 \\ \delta_{k+1} & 0 & \cdots & 0 & -1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \delta_n & 0 & \cdots & 0 & 0 & \cdots & -1 \end{pmatrix}. \tag{5}$$

Let $T_{12}(-\delta_1), \ldots, T_{1k+1}(-\delta_k), T_{1k+2}(\delta_{k+1}), \ldots, T_{1n+1}(\delta_n) \in M_{n+1}(K)$ be transfection. We assume

$$U = T_{1n+1}(\delta_n) \ldots T_{12}(-\delta_1). \tag{6}$$

Then

$$U\Delta U^t = \begin{pmatrix} 1-\delta_1^2-\cdots-\delta_k^2+\cdots+\delta_n^2 & 0 & 0 \\ 0 & E_k & 0 \\ 0 & 0 & -E_{n-k} \end{pmatrix}$$

hence we get, that $rang\Delta \geq n$. It is shown, that in (3) $r \geq 2n$. Together with (1) we obtain

$$L(B(f)) = 2n,$$

if $0 < k < n$ or K is algebraic closed. If $k = 0$ and $K = R$ is a field of the real numbers, then $rang\Delta = n + 1$.

Consequently $r \geq 2n + 1$. Together with (2) it leads, that

$$L(B(f)) = 2n + 1.$$

The theorem is proved.

4 Complexity of the Keli-Dixon body

Let C be a Keli-Dixon body, $1, l_1, \ldots, l_7$ is a basis C of the multiplication table: Let $x, y \in C$.

Table 1 – The multiplication table

0	l_1	l_2	l_3	l_4	l_5	l_6	l_7
l_1	-1	l_3	$-l_2$	l_5	$-l_4$	$-l_7$	l_6
l_2	$-l_3$	-1	l_1	l_6	l_7	$-l_4$	$-l_5$
l_3	l_2	$-l_1$	-1	l_7	$-l_6$	l_5	$-l_4$
l_4	$-l_5$	$-l_6$	$-l_7$	-1	l_1	l_2	l_3
l_5	l_4	$-l_7$	l_6	$-l_1$	-1	$-l_3$	l_2
l_6	l_7	l_4	$-l_5$	$-l_2$	l_3	-1	$-l_1$
l_7	$-l_6$	l_5	l_4	$-l_3$	$-l_2$	l_1	-1

Using the table, we compute the production x, y by its coordinates.

$$xy = (x_1y_1 - x_2y_2 - x_3y_3 - x_4y_4 - x_5y_5 - x_6y_6 - x_7y_7 - x_8y_8) \cdot 1$$

$$+(x_1y_2 + x_2y_1 + x_3y_4 - x_4y_3 - x_5y_6 - x_6y_5 - x_7y_8 + x_8y_7) \cdot e_1$$

$$+(x_1y_3 - x_2y_4 + x_3y_1 + x_4y_2 + x_5y_7 + x_6y_8 - x_7y_5 - x_8y_6) \cdot e_2$$

$$+(x_1y_4 + x_2y_3 - x_3y_2 + x_4y_1 + x_5y_8 - x_6y_7 + x_7y_6 - x_8y_6) \cdot e_3$$

$$+(x_1y_5 - x_2y_6 - x_3y_7 - x_4y_8 + x_5y_1 + x_6y_2 + x_7y_3 + x_8y_4) \cdot e_4$$

$$+(x_1y_6 + x_2y_5 - x_3y_8 + x_4y_7 - x_5y_2 + x_6y_1 - x_7y_4 + x_8y_3) \cdot e_5$$

$$+(x_1y_7 + x_2y_8 + x_3y_5 - x_4y_6 - x_5y_3 + x_6y_4 + x_7y_1 - x_8y_2) \cdot e_6$$

$$+(x_1y_8 - x_2y_7 + x_3y_6 + x_4y_5 - x_5y_4 - x_6y_3 + x_7y_2 + x_8y_1) \cdot e_7$$

$$= C_1 \cdot 1 + C_2 \cdot e_1 + C_3 \cdot e_2 + C_4 \cdot e_3 + C_5 \cdot e_4 + C_6 \cdot e_5 + C_7 \cdot e_6 + C_8 \cdot e_7.$$

Now we assume

$$f_1 = (x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8)(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8),$$

$$f_2 = (x_1 + x_2 + x_3 + x_4 - x_5 - x_6 - x_7 - x_8)(y_1 + y_2 + y_3 + y_4 - y_5 - y_6 - y_7 - y_8),$$

$$f_3 = (x_1 - x_2 + x_3 - x_4 + x_5 - x_6 + x_7 - x_8)(y_1 - y_2 + y_3 - y_4 + y_5 - y_6 + y_7 - y_8),$$

$$f_4 = (x_1 - x_2 + x_3 - x_4 - x_5 + x_6 - x_7 + x_8)(y_1 - y_2 + y_3 - y_4 - y_5 + y_6 - y_7 + y_8),$$

$$f_5 = (x_1 + x_2 - x_3 - x_4 + x_5 + x_6 - x_7 - x_8)(y_1 + y_2 - y_3 - y_4 + y_5 + y_6 - y_7 - y_8),$$

$$f_6 = (x_1 + x_2 - x_3 - x_4 - x_5 - x_6 + x_7 + x_8)(y_1 + y_2 - y_3 - y_4 - y_5 - y_6 + y_7 + y_8),$$

$$f_7 = (x_1 - x_2 - x_3 + x_4 + x_5 - x_6 - x_7 + x_8)(y_1 - y_2 - y_3 + y_4 + y_5 - y_6 - y_7 + y_8),$$

$$f_8 = (x_1 - x_2 - x_3 + x_4 - x_5 + x_6 + x_7 - x_8)(y_1 - y_2 - y_3 + y_4 - y_5 + y_6 + y_7 - y_8),$$

$$f_9 = x_1y_1, f_{10} = x_4y_3, f_{11} = x_6y_5, f_{12} = x_7y_8, f_{13} = x_2y_4,$$

$$f_{14} = x_7y_5, f_{15} = x_8y_6, f_{16} = x_3y_2, f_{17} = x_6y_7,$$

$$f_{18} = x_8y_5, f_{19} = x_2y_6, f_{20} = x_3y_7, f_{21} = x_4y_8,$$

$$f_{22} = x_3y_8, f_{23} = x_5y_2, f_{24} = x_7y_4,$$

$$f_{25} = x_4y_6, f_{26} = x_5y_3, f_{27} = x_8y_2, f_{28} = x_2y_7, f_{29} = x_6y_3, f_{30} = x_5y_4$$

Then

$$C_1 = \frac{1}{8}(16f_9 - f_1 - f_2 - f_3 - f_4 - f_5 - f_6 - f_7 - f_8),$$

$$C_2 = \frac{1}{8}(f_1 + f_2 + f_5 + f_6 - f_3 - f_4 - f_7 - f_8) - 2(f_{10} + f_{11} + f_{12}),$$

$$C_3 = \frac{1}{8}(f_1 + f_2 + f_3 + f_4 - f_5 - f_6 - f_7 - f_8) - 2(f_{13} + f_{14} + f_{15}),$$

$$C_4 = \frac{1}{8}(f_1 + f_2 + f_7 + f_8 - f_3 - f_4 - f_5 - f_6) - 2(f_{16} + f_{17} + f_{18}),$$

$$C_5 = \frac{1}{8}(f_1 - f_4 + f_5 - f_8 - f_2 + f_3 - f_6 + f_7) - 2(f_{19} + f_{20} + f_{21}),$$

$$C_6 = \frac{1}{8}(f_1 + f_4 + f_5 + f_8 - f_2 - f_3 - f_6 - f_7) - 2(f_{22} + f_{23} + f_{24}),$$

$$C_7 = \frac{1}{8}(f_1 + f_4 + f_5 + f_8 - f_2 - f_4 - f_5 - f_7) - 2(f_{25} + f_{26} + f_{27}),$$

$$C_8 = \frac{1}{8}(f_1 + f_4 + f_6 + f_7 - f_2 - f_3 - f_5 - f_8) - 2(f_{28} + f_{29} + f_{30}),$$

Hence we get, that $L(C) \leq 30$. Let $L(C) = r$ and the elements $x, y \in C$ are considered as vector-columns, then $C_i = x^t C_i y, i = 1, \ldots, 8$ where $C_i \in M_8(K)$ and there exists an algorithm M of the length r for which

$$C_i = \sum_{j=1}^{r} d_{ij} x_i, i = 1, \ldots, 8,$$

where $x_j \in M \subseteq M_8(K), i = 1, \ldots, r, d_{ij} \in K$. For Keli-Dixon body C the matrix Δ has the form

$$\Delta = \begin{pmatrix} 1 & \delta_1 & \delta_2 & \delta_3 & \delta_4 & \delta_5 & \delta_6 & \delta_7 \\ \delta_1 & -1 & \delta_3 & -\delta_2 & \delta_5 & -\delta_4 & -\delta_7 & \delta_6 \\ \delta_2 & -\delta_3 & -1 & \delta_1 & \delta_6 & \delta_7 & -\delta_4 & -\delta_5 \\ \delta_3 & \delta_2 & -\delta_1 & -1 & \delta_7 & -\delta_6 & \delta_5 & -\delta_4 \\ \delta_4 & -\delta_5 & -\delta_6 & -\delta_7 & -1 & \delta_4 & \delta_2 & \delta_3 \\ \delta_5 & \delta_4 & -\delta_7 & \delta_6 & -\delta_1 & -1 & -\delta_3 & \delta_2 \\ \delta_6 & \delta_7 & \delta_4 & -\delta_5 & -\delta_2 & \delta_3 & -1 & -\delta_1 \\ \delta_7 & -\delta_6 & \delta_5 & \delta_4 & -\delta_3 & -\delta_2 & \delta_1 & -1 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.$$

Then $detA_{11} \cdot det(A_{22} - A_{21}A_{11}^{-1}A_{12}) = d_1^4(1 + d_1 d_2) \neq 0$, where $d_1 = (1 + d_1^2 + d_2^2 + d_3^2 + d_4^2)$, $d_2 = d_4^2 + d_5^2 + d_6^2 + d_7^2$. On the other hand $\Delta = \sum_{j=8}^{r} \frac{d_j}{|M_1|} x_j$, consequently $r - 7 \geq 8$, i.e. $L(C) \geq 15$. By this way it is proved the following theorem.

Theorem 2 *C is a Keli-Dixon body. Then* $15 \leq L(C) \leq 30$.

5 Complexity of Li algebra

In this section we calculate the complexity of the least simple Li algebra $sl(2, K)$ over field K of characteristics $\neq 2$. We mention, that $sl(2, K)$ consists of 2×2 matrixes over K with zero trace: as its basis the elements can be taken $l_1 = l_{11} - l_{22}, l_2 = l_{12}, l_3 = l_{21}$ where l_{ij} are ordinary matrix units. For any $x = \sum_{i=1}^{3} x_i l_i = \sum_{i=1}^{3} y_i l_i$ we have

$$[x, y] = 2(x_1y_2 - x_2y_1)l_2 - 2(x_1y_3 - x_3y_1)l_3 + (x_2y_3 - x_2y_3)l_1 =$$

$$= C_1 l_1 + 2C_2 l_2 + 2C_3 l_3,$$

where

$$C_1(x, y) = x_2y_3 - x_3y_2, C_2(x, y) = x_1y_2 - x_2y_1, C_3(x, y) = x_3y_1 - x_1y_2.$$

We assume

$$f_1 = (x_1 - x_2 - x_3)(y_1 - y_2 + y_3),$$

$$f_2 = (x_1 + x_2 + x_3)(y_1 + y_2 - y_3),$$

$$f_3 = (x_1 + x_2 - x_3)(y_1 + y_2 + y_3),$$

$$f_4 = (x_1 - x_2 + x_3)(y_1 - y_2 - y_3),$$

$$f_5 = x_1 y_2.$$

Then

$$C_1 = -\frac{1}{4}(f_1 + f_2 - f_3 - f_4),$$

$$C_3 = \frac{1}{2}(f_2 - f_1) + C_1$$

$$C_2 = \frac{1}{2}(f_2 - f_1) - C_3 - 2f_5.$$

Therefore

$$sl(2,K) \leq 5.$$

Let $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$; then we get

$$C_i = xC_iy^t, i = 1, 2, 3,$$

where

$$C_1 = l_{23} - l_{32}, C_2 = l_{12} - l_{21}, C_3 = l_{31} - l_{13}.$$

Let $L(sl(2,K)) = r$ then there exists an algorithm of the length r for $sl(2,K)$ i.e. there exist the matrixes $X_1, \ldots, X_r$ of $M_3(K)$ rang 1 such that

$$C_i = \sum_{i=1}^{r} \alpha_{ij} X_j, i = 1, 2, 3. \tag{7}$$

The procedure is repeated for $B(f)$, and we get

$$\Delta = C_2 + \delta_1 C_3 + \delta_2 C_1 = \sum_{j=2}^{r} \frac{\alpha_j}{|M_1|} X_j, \tag{8}$$

where

$$|M_1| = \begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{31} & \alpha_{32} \end{vmatrix}, \alpha_j = \begin{vmatrix} \alpha_{21} & \alpha_{22} & \alpha_{2j} \\ \alpha_{11} & \alpha_{12} & \alpha_{1j} \\ \alpha_{31} & \alpha_{32} & \alpha_{3j} \end{vmatrix} .j = 3, \ldots, r.$$

Since $det\Delta = \delta_1\delta_2 - \delta_1\delta_2 = 0$, then $rang\Delta = 2$, i.e. $r \geq 4$. Let $r = 4, C_i = \sum_{j=1}^{4} \alpha_{ij} X_j, i = 1, 2, 3$. Since C_1, C_2, C_3 are linear independent, we can suppose, that

$$\begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{vmatrix} \neq 0.$$

Let $|M_{23}| = \alpha_{11}\alpha_{32} - \alpha_{12}\alpha_{31} \neq 0$. Then under $i = 1, 3$ we find X_1 из (7).

$$X_1 = \frac{1}{|M_{23}|}(\alpha_{32}C_3 - \alpha_{12}C_1 + \gamma_3^1 X_3 + \gamma_4^1 X_4) = \gamma_1 C_3 + \gamma_2 C_1 + \gamma_3 X_3 + \gamma_4 X_4$$

Let

$$R = \begin{pmatrix} 1 & 0 & \delta_2 \\ 0 & 1 & \delta_1 \\ 0 & 0 & 1 \end{pmatrix}, L = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ \delta_2 & \delta_1 & 1 \end{pmatrix},$$

then from (8) we obtain

$$L\Delta R = d_3' L X_3 R + d_4' L X_4 R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, d_3 \neq 0, d_4 \neq 0.$$

Now we show, that if X, Y is a matrix of rang 1 and

$$X+Y=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{9}$$

then X, Y have the form

$$\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let

$$X=\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}(x_1,x_2,x_3), Y=\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}(y_1,y_2,y_3).$$

Then from (9) we get

$$\begin{cases} a_1x_1+b_1y_1=1 \\ a_2x_1+b_2y_1=0 \\ a_3x_1+b_3y_1=0 \end{cases}, \begin{cases} a_1x_1+b_1y_1=0 \\ a_2x_1+b_2y_1=1 \\ a_3x_1+b_3y_1=0 \end{cases}, \begin{cases} a_1x_1+b_1y_1=0 \\ a_2x_1+b_2y_1=0 \\ a_3x_1+b_3y_1=0 \end{cases}.$$

It leads us,

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0, \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = 0, \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} = 0.$$

Hence we get, that $x_3 = y_3 = 0$. If $a_3 \neq 0 \neq b_3$, then $a_1b_2 - b_1a_2 = b_1b_2\frac{a_3}{b_3} - b_1b_2\frac{a_3}{b_3} = 0$, consequently $a_3 = b_3 = 0$. Now, let

$$d_3'LX_3R=\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 0 \end{pmatrix}_1, d_4'LX_4R=\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 0 \end{pmatrix}_2.$$

Then

$$Y_1=LX_1R=\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ \delta_2 & \delta_1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & -\gamma_1 \\ 0 & 0 & \gamma_2 \\ \gamma_1 & -\gamma_2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & \delta_2 \\ 0 & 1 & \delta_1 \\ 0 & 0 & 1 \end{pmatrix}+\frac{\gamma_3}{d_3'}(d_3'LX_3R)+$$

$$+\frac{\gamma_4}{d_4'}(d_4'LX_4R)=\begin{pmatrix} 0 & 0 & \gamma_2 \\ 0 & 0 & -\gamma_1 \\ \gamma_1 & -\gamma_2 & 0 \end{pmatrix}+\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 0 \end{pmatrix}=\begin{pmatrix} * & * & \gamma_2 \\ * & * & -\gamma_1 \\ \gamma_1 & -\gamma_2 & 0 \end{pmatrix}.$$

Since $rang Y_1 = 1$ we obtain, that $\gamma_1 = \gamma_2 = 0$, i.e. $|M_{23}| = 0$, is the contradiction. By this way the following theorem is proved.

Theorem 3 *Let $sl(2, K)$ is Li algebra 2×2 with zero trace over the field K of characteristics $\neq 2$. Then $L(sl(2, K)) = 5$.*

6 Conclusion

A dependence on the field of the complexity algebra is presented in the paper. For example, there is a difference between the complexity of nondegenerate bilinear form of algebra in the field of the real numbers and complex numbers.

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