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MAXIMAL REGULARITY ESTIMATE FOR A DIFFERENTIAL EQUATION WITH OSCILLATING COEFFICIENTS

The paper considers a second-order differential equation with unbounded coefficients. Sufficient summability conditions with the weight of the solution and its derivatives up to second order are obtained. The equation studied is singular as it is defined in an infinite domain, and its coefficients may be unbounded. Its main feature is the rapid growth of the coefficient at of the first derivative of the solution required, therefore the well-developed theory of the Sturm-Liouville equations is not applicable. The equation studied and its multidimensional generalizations arise in the modeling of the Brownian motion of particles, in problems of biology and financial mathematics. Their well-known representatives are the Ornstein-Uhlenbeck and Fokker-Planck-Kolmogorov equations, which have been actively studied since the first half of the twentieth century. On the other hand, projection methods are well known in applications (e.g., Fourier or Laplace transforms), which reduce partial differential equations with coefficients depending on one variable to ordinary differential equations. Therefore, the present study is important for partial derivative equations with unbounded coefficients. In contrast to previous works, the senior and intermediate coefficients of the equation studied can be strongly fluctuating. In the proof of the main theorems, the authors use their earlier result on the correct solvability of the mentioned equation.

Key words: second order differential equation, linear differential equation, differential equation in an unbounded domain, maximal regularity, oscillating coefficients.

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Тербелмелі коэффициентті бір дифференциалдық теңдеу үшін максималды регулярлық бағасы

Жұмыста коэффициенттері шенелмеген екінші ретті дифференциалдық теңдеу қарастырылған. Шешім мен оның екінші ретке дейінгі туындыларының салмақпен қосындылануы үшін жеткілікті шарттар алынады. Зерттелген теңдеу сингулярлы, өйткені ол шексіз облыста берілген, ал оның коэффициенттері шенелмеген болуы мүмкін. Оның басты ерекшелігі - шешімнің бірінші ретті туындысы алдындағы коэффициенттің жылдам өсуінде жатыр, соның әсерінен Штурм-Лиувилль теңдеулерінің дамыған теориясын қолдану мүмкін емес. Зерттелген теңдеу мен оның көп өлшемді жалпылаулары бөлшектердің броундық қозғалысын модельдеу кезінде, биология және қаржылық математика мәселелерінде туындайды. Олардың белгілі өкілдері - XX ғасырдың бірінші жартысынан бастап белсенді түрде зерттеліп келе жатқан Орнштейн-Уленбек және Фоккер-Планк-Колмогоров теңдеулері. Екінші жағынан, проекциялық әдістерді қолданып (мысалы, Фурье немесе Лаплас түрлендірулерін) коэффициенттері бір айнымалыға тәуелді дербес туындылардағы теңдеулерді қарапайым дифференциалдық теңдеулерге алып келуге болады. Сондықтан, бұл зерттеудің коэффициенттері шенелмеген дербес туындылардағы теңдеулер үшін маңызы бар. Зерттеліп отырған теңдеудің осыған дейін қарастырылғандардан айырмашылығы - оның жоғарғы және аралық коэффициенттері жылдам тербелуі мүмкін. Негізгі теоремаларды дәлелдеу кезінде авторлар өздерінің осы теңдеудің дұрыс шешілуіне қатысты алдыңғы нәтижелерін пайдаланған.

Түйін сөздер: екінші ретті дифференциалдық теңдеу, сызықты дифференциалдық теңдеу, шенелмеген облыстағы дифференциалдық теңдеу, максималды регулярлық, тербелмелі коэффициенттер.

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Оценка максимальной регулярности для дифференциального уравнения с колеблющимися коэффициентами

В работе рассматривается дифференциальное уравнение второго порядка с неограниченными коэффициентами. Получены достаточные условия суммируемости с весом решения и его производных вплоть до второго порядка. Изучаемое уравнение является сингулярным, так как оно задано в бесконечной области, а его коэффициенты могут быть не ограниченными. Главной его особенностью является быстрый рост коэффициента при первой производной искомого решения, из-за чего не применима хорошо развитая теория уравнений Штурма-Лиувилля. Исследуемое уравнение и его многомерные обобщения возникают в моделировании броуновского движения частиц, в задачах биологии и финансовой математики. Их известными представителями являются уравнения Орнштейна-Уленбека и Фоккера-Планка – Колмогорова, которые активно изучаются начиная с первой половины двадцатого века. С другой стороны, в приложениях хорошо известны проекционные методы (например, преобразования Фурье или Лапласа), которые сводят уравнения в частных производных с коэффициентами, зависящими от одной переменной, к обыкновенным дифференциальным уравнениям. Поэтому настоящее исследование важно для уравнений в частных производных с неограниченными коэффициентами. В отличие от предыдущих работ, старший и промежуточный коэффициенты исследуемого уравнения могут быть сильно колеблющимися. При доказательстве основных теорем, авторы пользуются более ранним их результатом о корректной разрешимости указанного уравнения.

Ключевые слова: дифференциальное уравнение второго порядка, линейное дифференциальное уравнение, дифференциальное уравнение в неограниченной области, максимальная регулярность, колеблющиеся коэффициенты.

1 Introduction

In this paper, we consider the smoothness properties of the solution of a second-order singular differential equation

$$T_0 y = -\rho(x) (\rho(x)y')' + r(x)y' + s(x)y = f(x), \quad (1)$$

where $x \in \mathbb{R} = (-\infty, +\infty)$, ρ is a positive and twice continuously differentiable function, r is a continuously differentiable function, and s is a continuous function, $f \in L_2 = L_2(\mathbb{R})$.

By T_0 we denote the operator mapping from the set of twice continuously differentiable and finite functions $C_0^{(2)}(\mathbb{R})$ to L_2 by the following formula

$$T_0 y = -\rho(x) (\rho(x)y')' + r(x)y' + s(x)y.$$

We denote by T the closure T_0 in L_2 space. The function $y \in D(T)$ such that $Ty = f$ is said to be solution of the equation (1).

The solution $y \in L_2$ of the equation (1) is said to be maximally regular if the following inequality holds

$$\|-\rho(\rho y')'\|_2 + \|ry'\|_2 + \|sy\|_2 \leq C \|f\|_2,$$

where $C > 0$ does not depend on y , $\|\cdot\|_2$ is a norm of L_2 .

Some conditions for existence, uniqueness and maximal regularity of a solution of the equation (1) were obtained in our work [1]. There the relevance in theory and practical

issues of studying this equation in the case when its coefficients can be unbounded functions were also covered, and the case of tending to zero function $\rho(x)$ in the leading term of the equation was also studied. Naturally, the correctness of the equation (1) assumes that there are some relations between its coefficients. The equation (1) is reduced to the well-known Sturm-Liouville equation if the intermediate coefficient $r(x)$ is absent or grows slowly, so that $r(x)\frac{dy}{dx}$ as operator is controlled by the sum of leading and free terms in the left part. When these conditions are not met, the equation (1) is investigated poorly.

The investigated equation (1) and its multidimensional generalizations arise in Brownian particle motion modeling, in biology and financial mathematics problems [2–6]. Their well-known representatives are the Ornstein-Uhlenbeck and Fokker-Planck-Kolmogorov equations, which have been actively studied since the first half of the twentieth century.

In this paper, unlike [1] as well as [7], we will assume that the coefficients $\rho(x)$ and $r(x)$ do not follow the weak fluctuation conditions. Such conditions usually appear when evaluating the norm of the higher derivative of a solution to the second-order singular differential equation. In [8] there is an example of a Sturm-Liouville equation with an oscillating coefficient whose solution is not maximally regular.

The main result of the work is Theorem 2. We have proved the validity of the maximal regularity estimate of a solution of the equation (1) when the mentioned coefficients ρ and r can fluctuate rapidly.

2 Material and Methods

We rely on Lemma 1 obtained in [1], where the theorem of the existence and uniqueness of the solution of the equation (1) is proved and a uniform estimate for the norm of the solution and its first derivative was obtained.

An auxiliary binomial degenerate differential operator associated with the equation (1) was investigated. Applying the method of local estimates developed in the work of M. Otelbaev [9], we obtained a representation of the resolvent of its certain shift. Using this representation we have proved the separability of the above binomial differential operator. Then we applied the closed operator perturbation theorem in [10]. Here, the partition of the real axis chosen by us depends on the dominant intermediate coefficient, which allowed us to consider the case of strongly fluctuating coefficients.

3 Auxiliary statements

Consider the equation

$$l_0 y = -\rho(\rho y)' + r y' = F(x), \quad (2)$$

Let $D(l_0) = C_0^{(2)}(\mathbb{R})$, and l is a closure of the operator l_0 by the norm of L_2 . A function $y \in D(l)$ such that $ly = f$ is said to be a solution of the equation (2). Let $u(x)$ and $v(x) \neq 0$ are some real continuous functions. We denote

$$\gamma_{u,v} = \max \left(\sup_{x>0} \left(\int_0^x u^2(t) dt \right)^{\frac{1}{2}} \left(\int_x^{+\infty} v^{-2}(t) dt \right)^{\frac{1}{2}}, \sup_{\tau<0} \left(\int_{\tau}^0 u^2(t) dt \right)^{\frac{1}{2}} \left(\int_{-\infty}^{\tau} v^{-2}(t) dt \right)^{\frac{1}{2}} \right).$$

In [1, reopema 3.1] the following statement is proved.

Lemma 1 *Let $\rho(x) > 0$ is a twice continuously differentiable function, and $r(x) \geq 1$ is a continuously differentiable function. Let*

$$\frac{r}{\rho^2} \geq 1, \quad \gamma_{1,\sqrt{r}} < +\infty, \quad (3)$$

and there also exists $a \in \mathbb{R}$ such that

$$\sup_{x<a} \left\{ \rho(x) \exp \left(- \int_x^a \frac{r(t)}{\rho^2(t)} dt \right) \right\} < +\infty. \quad (4)$$

Then for any $F \in L_2$ the equation (2) has a unique solution y , and for y the following estimate holds

$$\|\sqrt{r}y'\|_2 + \|y\|_2 \leq C \|f\|_2.$$

When the condition (3) holds, the following inequality was also proved in [1]:

$$\|\sqrt{r}y'\|_2 \leq \left\| \frac{1}{\sqrt{r}}ly \right\|_2, \quad (5)$$

where $y \in D(l)$.

4 Main results

We use the following theorem in the proof of the main result which is Theorem 2. Meanwhile Theorem 1 is of independent interest.

Theorem 1 *Let $0 < \rho(x) < +\infty$ is a twice continuously differentiable function and $r(x) \geq 1$ is a continuously differentiable function for which the conditions (3) and (4) of Lemma 1 are satisfied. Suppose, moreover*

$$\sup_{|x-\eta| \leq \frac{k(\eta)}{r(\eta)}} \frac{\rho(x)}{\rho(\eta)} < +\infty, \quad \sup_{|x-\eta| \leq \frac{k(\eta)}{r(\eta)}} \frac{r(x)}{r(\eta)} < +\infty, \quad (6)$$

where $k(\eta) \geq 4$ is continuous and $\lim_{|\eta| \rightarrow +\infty} k(\eta) = +\infty$. Then the following estimate holds for the solution y of the equation (2):

$$\|-\rho(\rho y')'\|_2 + \|ry'\|_2 + \|y\|_2 \leq C_1 \|f\|_2. \quad (7)$$

Proof. By virtue of lemma 2.1 [9] and the condition (6) there is a cover of $\{\Delta_j\}_{j=1}^{+\infty}$ of the set \mathbb{R} (i. e. $\bigcup_{j=1}^{+\infty} \Delta_j = \mathbb{R}$) such that each interval $\Delta_j = (a_j, b_j)$, where

$$b_j - a_j \leq \frac{k \left(\frac{b_j - a_j}{2} \right)}{2r \left(\frac{b_j - a_j}{2} \right)},$$

can intersect with the others no more than ξ times. There exists also a set of functions $\{\varphi_j\}_{j=1}^{+\infty}$ such that

$$\sum_{j=1}^{+\infty} \varphi_j^2(x) = 1, \quad \varphi_j \in C_0^\infty(\Delta_j).$$

Let $\rho_j(x)$, $r_j(x)$ and $F_j(x)$ ($j = 1, 2, \dots$) are restrictions on Δ_j of the functions $\rho(x)$, $r(x)$ and $F(x)$, respectively, and $\lambda \geq 0$. Consider the following problem

$$l_{0,j,\lambda} y = -\rho_j(x) (\rho_j(x) y')' + [r_j(x) + \lambda] y' = F_j(x), \quad (8)$$

$$y(a_j) = y(b_j) = 0. \quad (9)$$

We define the solution to the problem (8), (9) as the function $y(x)$, for which there exists the sequence $\{y_k(x)\}_{k=1}^{+\infty}$ from the set $C_0^{(2)}(\Delta_j)$ of twice continuously differentiable and finite in Δ_j functions such that $\|y_k - y\|_{L_2(\Delta_j)} \rightarrow 0$ and $\|l_{0,j,\lambda} y_k - f_j\|_{L_2(\Delta_j)} \rightarrow 0$ as $k \rightarrow +\infty$. We denote by $l_{j,\lambda}$ ($j = 1, 2, \dots$) the closure of the operator $l_{0,j,\lambda}$ with $D(l_{0,j,\lambda}) = C_0^{(2)}(\Delta_j)$ in the space $L_2(\Delta_j)$. The function $y \in L_2(\Delta_j)$ is said to be the solution of the problem (8), (9) if $y \in D(l_{j,\lambda})$ and $l_{j,\lambda} y = F_j$. It follows from the general theory of differential equations that for any $F_j \in L_2(\Delta_j)$ the solution to the problem (8), (9) exists.

Let us introduce the following notation: $z = y'$ ($y \in D(l_{0,j,\lambda})$), $L_{0,j,\lambda} z = -\rho_j(\rho_j z)' + (r_j + \lambda)z$, $\|\cdot\|_{2,\Delta_j} = \|\cdot\|_{L_2(\Delta_j)}$. Let $z \in D(L_{0,j,\lambda})$. Integrating by parts we obtain

$$\int_{\Delta_j} z L_{0,j,\lambda} z dx = \int_{\Delta_j} z (-\rho_j(\rho_j z)' + (r_j + \lambda)z) dx = \int_{\Delta_j} (r_j + \lambda) z^2 dx = \left\| \sqrt{r_j + \lambda} z \right\|_{2,\Delta_j}^2. \quad (10)$$

On the other hand, according to Hölder's inequality

$$\begin{aligned} \int_{\Delta_j} z L_{0,j,\lambda} z dx &\leq \left(\int_{\Delta_j} |(r_j + \lambda)^{-\frac{1}{2}} L_{0,j,\lambda} z|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Delta_j} |(r_j + \lambda)^{\frac{1}{2}} z|^2 dx \right)^{\frac{1}{2}} = \\ &= \left\| \frac{1}{\sqrt{r_j + \lambda}} L_{0,j,\lambda} z \right\|_{2,\Delta_j} \left\| \sqrt{r_j + \lambda} z \right\|_{2,\Delta_j}. \quad (11) \end{aligned}$$

From (10) and (11) it follows that

$$\left\| \sqrt{r_j + \lambda} z \right\|_{2, \Delta_j} \leq \left\| \frac{1}{\sqrt{r_j + \lambda}} L_{0,j,\lambda} z \right\|_{2, \Delta_j}, \quad z \in D(L_{0,j,\lambda}). \quad (12)$$

Since $z = y'$, we have

$$\left\| \sqrt{r_j + \lambda} y' \right\|_{2, \Delta_j} \leq \left\| \frac{1}{\sqrt{r_j + \lambda}} l_{0,j,\lambda} y \right\|_{2, \Delta_j}, \quad y \in C_0^{(2)}(\Delta_j). \quad (13)$$

Further, according to the well-known Friedrichs's inequality

$$\|y\|_{2, \Delta_j} \leq C \left\| \sqrt{r_j + \lambda} y' \right\|_{2, \Delta_j}, \quad y \in C_0^{(2)}(\Delta_j). \quad (14)$$

By virtue of (13) and (14)

$$\left\| \sqrt{r_j + \lambda} y' \right\|_{2, \Delta_j} + \|y\|_{2, \Delta_j} \leq (C + 1) \|F_j\|_{2, \Delta_j}, \quad y \in C_0^{(2)}(\Delta_j).$$

According to (12)

$$\left\| \sqrt{r_j + \lambda} z \right\|_{2, \Delta_j} \leq \frac{1}{\inf_{x \in \Delta_j} \sqrt{r_j + \lambda}} \|L_{0,j,\lambda} z\|_{2, \Delta_j}.$$

Hence, taking into account of the condition (6) and the choice of Δ_j , we have

$$\begin{aligned} \|(r_j + \lambda)z\|_{2, \Delta_j} &\leq \sup_{x \in \Delta_j} \sqrt{r_j + \lambda} \left\| \sqrt{r_j + \lambda} z \right\|_{2, \Delta_j} \leq \\ &\leq C \inf_{x \in \Delta_j} \sqrt{r_j + \lambda} \left\| \sqrt{r_j + \lambda} z \right\|_{2, \Delta_j} \leq C \|L_{0,j,\lambda} z\|_{2, \Delta_j}, \quad z \in D(L_{0,j,\lambda}). \end{aligned}$$

Then

$$\|-\rho_j(\rho_j z)'\|_{2, \Delta_j} + \|(r_j + \lambda)z\|_{2, \Delta_j} \leq C_1 \|L_{0,j,\lambda} z\|_{2, \Delta_j}, \quad z \in D(L_{0,j,\lambda}). \quad (15)$$

Due to (8) и (14) we have

$$\|-\rho_j(\rho_j y')'\|_{2, \Delta_j} + \|(r_j + \lambda)y'\|_{2, \Delta_j} + \|y\|_{2, \Delta_j} \leq C_1 \|l_{0,j,\lambda} y\|_{2, \Delta_j}, \quad y \in C_0^{(2)}(\Delta_j). \quad (16)$$

Since the $l_{j,\lambda}$ is closed, the inequality (16) holds for all $y \in D(l_{j,\lambda})$, in particular, for a solution of the problem (8), (9).

Let $L_{j,\lambda}$ is an operator from the set $D(L_{j,\lambda}) = \{z \in L_2(\Delta_j) : \exists y \in D(l_{j,\lambda}), z = y'\}$ in $L_2(\Delta_j)$ by the following formula

$$L_{j,\lambda} z = -\rho_j(x)(\rho_j(x)z)'+(r_j(x)+\lambda)z.$$

Since $R(L_{j,\lambda}) = R(l_{j,\lambda}) = L_2(\Delta_j)$, and for all $z \in D(L_{j,\lambda})$ the inequality (15) holds, the operator $L_{j,\lambda}$ is bounded invertible. We define the following operators for $f \in L_2$:

$$B_\lambda f := - \sum_{j=-\infty}^{+\infty} \rho^2(x) \varphi_j'(x) L_{j,\lambda}^{-1} \varphi_j f, \quad M_\lambda f := \sum_{j=-\infty}^{+\infty} \varphi_j(x) L_{j,\lambda}^{-1} \varphi_j f.$$

For any point $x \in \mathbb{R}$ the sums in the right-hand sides are consisted of no more that $\xi + 1$ terms, thus B_λ and M_λ are well-defined.

Let $L_\lambda z = -\rho(x)(\rho(x)z)' + (r(x) + \lambda)z$ for any $z = y'$, where $y \in D(l)$. Consider the operator $L_\lambda M_\lambda$. The operators L_λ and $L_{j,\lambda}$ are same in the interval Δ_j , therefore taking into account the properties of φ_j ($j \in \mathbb{Z}$), we get

$$L_\lambda M_\lambda f = \sum_{j=-\infty}^{+\infty} L_{j,\lambda}(\varphi_j L_{j,\lambda}^{-1} \varphi_j f) = \sum_{j=-\infty}^{+\infty} (\varphi_j L_{j,\lambda} L_{j,\lambda}^{-1} \varphi_j f - \rho^2 \varphi_j' L_{j,\lambda}^{-1} \varphi_j f) = (E + B_\lambda) f,$$

i. e.

$$L_\lambda M_\lambda = E + B_\lambda. \quad (17)$$

Let's estimate the norm of the operator B_λ .

$$\begin{aligned} \|B_\lambda f\|_2^2 &= \int_{-\infty}^{+\infty} \left| \sum_{j=-\infty}^{+\infty} \rho^2 \varphi_j' L_{j,\lambda}^{-1} \varphi_j f \right|^2 dx \leq \\ &\leq \sum_{j=-\infty}^{+\infty} \int_{\Delta_j} |\rho^2 \varphi_j' L_{j,\lambda}^{-1} \varphi_j f|^2 dx \leq \sum_{k=-\infty}^{+\infty} \int_{\Delta_k} \left| \sum_{j=-\infty}^{+\infty} \rho^2 \varphi_j' L_{j,\lambda}^{-1} \varphi_j f \right|^2 dx \leq \\ &\leq C_3(\xi + 1) \sum_{k=-\infty}^{+\infty} \int_{\Delta_k} |\rho^2(x) \varphi_k'(x) L_{j,\lambda}^{-1} \varphi_k(x) f(x)|^2 dx. \end{aligned}$$

Due to (12)

$$\|\rho^2 \varphi_k' L_\lambda^{-1} \varphi_k f\|_{2,\Delta_k} \leq \frac{C_4 \sup_{x \in \Delta_k} \rho^2(x)}{\inf_{x \in \Delta_k} (r_k(x) + \lambda)} \|\varphi_k f\|_{2,\Delta_k} \leq \frac{C_5}{1 + \lambda} \|\varphi_k f\|_{2,\Delta_k},$$

therefore, using the properties of the functions $\varphi_k(x)$ ($k \in \mathbb{Z}$), we have

$$\begin{aligned} \sum_{k=-\infty}^{+\infty} \int_{\Delta_k} |\rho^2 \varphi_k' L_{k,\lambda}^{-1} \varphi_k f|^2 dx &\leq \frac{C_5^2}{(1 + \lambda)^2} \sum_{k=-\infty}^{+\infty} \int_{\Delta_k} \varphi_k^2 |f|^2 dx = \\ &= \frac{C_5^2}{(1 + \lambda)^2} \sum_{k=-\infty}^{+\infty} \int_{\mathbb{R}} \varphi_k^2 |f|^2 dx = \frac{C_5^2}{(1 + \lambda)^2} \int_{\mathbb{R}} \left(\sum_{k=-\infty}^{+\infty} \varphi_k^2 \right) |f|^2 dx = \left(\frac{C_5}{1 + \lambda} \right)^2 \|f\|_2^2. \end{aligned}$$

Thus,

$$\|B_\lambda f\|_2^2 \leq C_2(\xi + 1) \frac{C_5^2}{(1 + \lambda)^2} \|f\|_2^2, \quad f \in L_2.$$

Hence $\|B_\lambda\| \rightarrow 0$ as $\lambda \rightarrow +\infty$, so there exists $\lambda_0 > 0$ such that $\|B_\lambda\| \leq \frac{1}{2}$ for all $\lambda \geq \lambda_0$. It follows from Lemma 1 that the operator L_λ^{-1} , the inverse of L_λ , exists and is bounded in L_2 . From (17), by virtue of the well-known Banach theorem, it follows that

$$L_\lambda^{-1} = M_\lambda(E + B_\lambda)^{-1}, \quad \|(E + B_\lambda)^{-1}\| \leq 2, \quad \lambda \geq \lambda_0. \quad (18)$$

Let us now prove the estimate (7). According to (5) and (18), for $z \in D(L_\lambda)$, we have

$$\begin{aligned} \|(r + \lambda)z\|_2^2 &= \|(r + \lambda)L_\lambda^{-1}f\|_2^2 \leq 2\|(r + \lambda)M_\lambda f\|_2^2 \leq \\ &\leq C_7 \sum_{j=-\infty}^{+\infty} \|(r_j + \lambda)\varphi_j L_{j,\lambda}^{-1} \varphi_j f\|_{2,\Delta_j}^2 \leq C_7 \sum_{j=-\infty}^{+\infty} \sup_{x \in \Delta_j} (r_j + \lambda) \|L_{j,\lambda}^{-1} \varphi_j f\|_{2,\Delta_j}^2 \leq \\ &\leq C_7 \sum_{j=-\infty}^{+\infty} \sup_{x \in \Delta_j} (r + \lambda) \frac{1}{\inf_{x \in \Delta_j} (r + \lambda)} \|\varphi_j f\|_{2,\Delta_j}^2 \leq C_8 \sum_{j=-\infty}^{+\infty} \|\varphi_j f\|_{2,\Delta_j}^2 \leq C_9 \|f\|_2^2. \end{aligned}$$

So

$$\|(r + \lambda)z\|_2 \leq C_9 \|f\|_2.$$

Hence, assuming $\lambda = 0$ and $z = y'$, we obtain an estimate (7). The theorem is proved.

Theorem 2 *Let the functions ρ and r satisfy the conditions of Theorem 1, and s is a continuous function such that $\gamma_{s,r} < +\infty$. Then for any $f \in L_2$ the equation (1) has a unique solution y , and for y the following estimate holds*

$$\|-\rho(\rho y')'\|_2 + \|r y'\|_2 + \|(1 + |s|)y\|_2 \leq C \|f\|_2, \quad (19)$$

Proof. Let $x = at$ in (1), where $a > 0$. Let us introduce the following notations

$$\tilde{y}(t) = y(at), \quad \tilde{\rho}(t) = \rho(at), \quad \tilde{r}(t) = r(at), \quad \tilde{s}(t) = s(at), \quad \tilde{f}(t) = a^{-1}F(at).$$

Then the equation (1) is transformed to the following form

$$-\tilde{\rho}(\tilde{\rho}\tilde{y}')' + \tilde{r}\tilde{y} + a^{-1}\tilde{s}\tilde{y} = \tilde{f}. \quad (20)$$

Let us denote by l_a the closure in L_2 of the operator $-\tilde{\rho}(\tilde{\rho}\tilde{y}')' + \tilde{r}\tilde{y}$, defined in $C_0^{(2)}(\mathbb{R})$. The function $\tilde{y}(t) \in D(l_a)$ is said to be a solution of the equation (20) if it satisfies the equality $l_a \tilde{y} = \tilde{f}$. Clearly, if the function $y(x)$ is a solution to the equation (1), then $\tilde{y}(t)$ is a solution to the equation (20) and vice versa.

It is easy to show that $\tilde{\rho}$ and \tilde{r} satisfy the conditions of Theorem 1, so

$$\|-\tilde{\rho}(\tilde{\rho}\tilde{y}')'\|_2 + \|\tilde{r}\tilde{y}'\|_2 + \|\tilde{y}\|_2 \leq C_{l_a} \|l_a \tilde{y}\|_2, \quad \forall \tilde{y} \in D(l_a).$$

According to lemma 2.1 [1] and the last inequality, we have

$$\|a^{-1}\tilde{s}\tilde{y}\|_2 \leq 2a^{-1}\gamma_{s,r} \|\tilde{r}\tilde{y}'\|_2 \leq 2a^{-1}\gamma_{s,r} C_{l_a} \|l_a \tilde{y}\|_2.$$

Let us choose $a = 4\gamma_{s,r} C_{l_a}$, then $\|a^{-1}\tilde{s}\tilde{y}\|_2 \leq \frac{1}{2} \|l_a \tilde{y}\|_2$. According to the theorem 1.16 in [10] (chapter 4) we get that the operator $l_a + a^{-1}\tilde{s}E$ (where E is an identity operator) is reversible and its range coincides with L_2 . This means that the equation (20) is uniquely solvable for any $\tilde{f} \in L_2$, then the equation (1) also has a unique solution $y(x) = \tilde{y}(a^{-1}x)$ for any $f \in L_2$.

Applying Lemma 2.1 [1], we obtain the estimate

$$\|s y\|_2 \leq 2\gamma_{s,r} \|r y'\|_2,$$

from which, taking into account (7), it follows (19). The inequality (19) implies uniqueness of the solution of the equation (1). The theorem is proved.

Example 1 Consider the following equation

$$-(1 + 5 \sin^2 e^x) [(1 + 5 \sin^2 e^x) y']' + (4 + x^8 + e^{2x} \sin^3 e^x) y' + (x^3 + e^x \sin e^x) y = f(x). \quad (21)$$

It is easy to check that for $k(x) = 4 + x^2$ the coefficients of the equation (21) satisfy the conditions of Theorem 2, so for any $f \in L_2$ the equation (21) has a unique solution y and for y the following estimate holds

$$\begin{aligned} & \left\| -(1 + 5 \sin^2 e^x) [(1 + 5 \sin^2 e^x) y']' \right\|_2 + \left\| (4 + x^8 + e^{2x} \sin^3 e^x) y' \right\|_2 + \\ & \quad + \left\| (1 + x^3 + e^x \sin e^x) y \right\|_2 \leq C \|f\|_2. \end{aligned}$$

5 Conclusion

A singular second-order differential equation with an unbounded variable coefficient at the first derivative of the unknown function is investigated in the paper. Only positiveness is required from the leading coefficient, i.e. the equation can degenerate near of infinity. In addition, we have studied the case of rapidly fluctuating coefficients. We have obtained conditions for the summability with weight of a strong solution of the considered equation and its derivatives up to the second order. The obtained result theoretically extends the class of coercive solvable differential equations of the second order. They can find application in stochastic analysis, modeling problems in biology and financial mathematics.

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