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PARALLEL CUDA IMPLEMENTATION OF THE ALGORITHM FOR SOLVING THE NAVIER-STOKES EQUATIONS USING THE FICTITIOUS DOMAIN METHOD

An important direction of development of numerical modeling methods is the study of approximate methods for solving problems of mathematical physics in complex multidimensional domains. To solve many applied problems in irregular domains, the fictitious domain method is widely used, the idea of which is to solve the problem not in the original, but in a simpler domain. This approach allows to create application software packages for numerical modeling of processes in arbitrary computational domains. In this paper, we develop a computational method for solving the Navier-Stokes equations in the Boussinesq approximation in two-connected domains by the fictitious domain method with continuation by lower coefficients. The problem formulation in the current function, velocity vortex variables is considered. A computational algorithm for solving the auxiliary problem of the fictitious domain method based on the finite difference method is developed. A parallel implementation of the algorithm using the CUDA parallel computation architecture is developed, which was tested on various configurations of the computational mesh. The results of computational experiments for the problem under consideration are presented.

Key words: Navier-Stokes equations, stream function, velocity vortex, fictitious domain method, boundary conditions, CUDA, parallel algorithm, high performance computing.

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Жалған облыстар әдісін пайдаланып Навье-Стокс теңдеулерін шешу алгоритмінің параллельді CUDA жүзеге асырылуы

Сандық модельдеу әдістерін дамытудың маңызды бағыты - математикалық физика есептерін күрделі көп өлшемді облыстарда шешудің жуық әдістерін зерттеу болып табылады. Көптеген қолданбалы есептерді күрделі облыстарда шешу үшін жалған облыстар әдісі кеңінен қолданылады. Оның идеясы есепті бастапқы облыста емес, қарапайым облыста шешуге негізделген. Бұл тәсілдеме еркін есептеу облыстарында үрдістерді сандық модельдеуге арналған қолданбалы бағдарламалар пакеттерін жасауға мүмкіндік береді. Бұл жұмыста кіші коэффициенттері бойынша жалғастырылған жалған облыстар әдісімен екі байланысты облыста Буссинеск жақындатуындағы Навье-Стокс теңдеулерін шешудің есептеу әдісі жасалды. "Ток функциясы, құйын жылдамдығы" айнымалыларындағы есептің қойылымы қарастырылады. Жалған облыстар әдісінің көмекші есебін шешудің ақырлы айырымдық әдісіне негізделген есептеу алгоритмі жасалды. CUDA параллельді есептеу архитектурасын қолдана отырып, параллельдік алгоритм жүзеге асырылды, ол есептеу торының әртүрлі конфигурацияларында сыналды. Қарастырылып отырған есеп үшін есептеу эксперименттерінің нәтижелері келтірілді.

Түйін сөздер: Навье-Стокс теңдеулері, ток функциясы, жылдамдық құйыны, жалған облыстар әдісі, шекаралық шарттар, CUDA, параллель алгоритм, жоғары өнімді есептеулер.

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Параллельная CUDA-реализация алгоритма решения уравнений Навье-Стокса с использованием метода фиктивных областей

Важным направлением развития методов численного моделирования являются исследования приближенных методов решения задач математической физики в сложных многомерных областях. Для решения многих прикладных задач в нерегулярных областях широко применяется метод фиктивных областей, идея которого заключается в решении задачи не в исходной, а в более простой области. Данный подход позволяет создавать пакеты прикладных программ для численного моделирования процессов в произвольных расчетных областях. В настоящей работе разработан вычислительный метод решения уравнений Навье-Стокса в приближении Буссинеска в двухсвязных областях методом фиктивных областей с продолжением по младшим коэффициентам. Рассматривается постановка задачи в переменных "функция тока, вихрь скорости". Разработан вычислительный алгоритм решения вспомогательной задачи метода фиктивных областей на основе конечно-разностного метода. Осуществлена параллельная реализация алгоритма с использованием архитектуры параллельных вычислений CUDA, которая была протестирована на различных конфигурациях вычислительной сетки. Приведены результаты вычислительных экспериментов для рассматриваемой задачи.

Ключевые слова: уравнения Навье-Стокса, функция тока, вихрь скорости, метод фиктивных областей, граничные условия, CUDA, параллельный алгоритм, высокопроизводительные вычисления.

1 Introduction

The growth of computer technology productivity and the development of parallel computing contributed to the solution of important practical problems of the industry. One example of such problems is the assessment of efficiency and forecasting of oil field development indicators. Due to the complexity of the mathematical models describing these processes, calculations for a single field can last from several hours to several days. Therefore, the issue of developing effective parallel algorithms that can significantly speed up calculations becomes relevant.

Along with the classical model of fluid flow in porous media based on Darcy's law, a number of other models are widely used in the study of fluid flows in oil reservoirs, such as the models of N. E. Zhukovsky [1], Forchheimer [2], Navier-Stokes [1, 2]. The use of these models is associated with a violation of the Darcy law under certain conditions, the need for a detailed study of flow processes near wells [3], etc.

The aim of this paper is to construct an algorithm for the numerical implementation of the model of an incompressible fluid motion using the CUDA parallel computing software and hardware architecture. The initial boundary value problem for the Navier-Stokes equations in the current function, velocity vortex variables in a two-dimensional two-connected domain is considered. To solve this problem, we consider an approximate method based on the fictitious domain method with continuation by lower coefficients. The discretization of the obtained equations is carried out by the finite difference method, but the obtained results will be used in parallel implementation of finite element methods in subsequent works. In conclusion,

the results of computational experiments for various mesh configurations and acceleration analysis of the calculations are presented.

2 Literature review

Let us first give a literature review of recent works devoted to solving problems of fluid motion in complex domains by the fictitious domain method. This method is applied for solving a wide class of problems in computational fluid dynamics, including the problem of flow around obstacles of a viscous incompressible fluid with boundary slip condition using the Navier law [4], the two-phase Stokes problem with surface tension forces [5], the problem of the non-Newtonian incompressible fluid motion [6], the problem of flow with arbitrary particle density [7], the problem of modeling the interaction of movable or deformable structure with an internal or surrounding fluid flow [8, 9], simulations of superquadric particles in fluid flows [10]. In [11], the fictitious domain method with H^1-penalty for the Stokes problem with the Dirichlet boundary condition is studied. [12] is devoted to the application of the fictitious domain method in the numerical simulation of a pulse oscillation converter.

The papers [13, 14] are devoted to the study of the fictitious domain method for problems with discontinuous coefficients. In [15,16], an elliptic equation with strongly varying coefficients is considered. The interest in the study of such equations is caused by the fact that equations of this type are obtained using the fictitious domain method. A special method for the numerical solution of an elliptic equation with strongly varying coefficients is proposed. A theorem is proved for estimating the convergence rate of the developed iterative process. A computational algorithm is developed and numerical calculations are performed to illustrate the effectiveness of the proposed method. In [17], modifications of well-known iterative methods for solving grid problems are constructed that arise is the fictitious domain method. The possibilities of the fictitious domain method are illustrated by examples of solving the problems of ideal and viscous incompressible fluid, fluid flow in porous media under a hydraulic structure.

Parallel implementations of the fictitious domain method are also known. For example, in [18], a parallel fictitious domain method is constructed for the three-dimensional Helmholtz equation, in [19] - for modeling particle-loaded flows and turbulent flow in a channel, in [20] - for biomechanics problems.

Currently, high-performance computing is widely used in the field of scientific research. Computer technologies and fluid dynamics models are developing every day, which allow to evaluate and analyze various technological processes. In this regard, the efficiency of solving scientific problems increases. Supercomputing technologies are widely used in many industries. Calculations that are performed on graphics devices significantly speed up the calculation of these "large" problems due to their unique architecture [21, 22].

Many papers have been devoted to the study of the applicability of the CUDA parallel computing architecture to various applied problems which allows increasing computing performance through the use of graphics processors. Its numerous applications to computational fluid dynamics problems are known, including the problems of the oil industry [21, 23], the problems of the motion of a viscous incompressible fluid [24], the problems of underground hydrogen storage [25], and others [22, 26].

3 Material and methods

3.1 The formulation of the problem

To model convective flows, we consider the Navier-Stokes equations in the Boussinesq approximation [27] in the two-dimensional domain $\overline{D} = D \cup \partial D$:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} = \frac{1}{\text{Re}}\Delta u, \tag{1}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial p}{\partial y} = \frac{1}{\text{Re}}\Delta v - \text{Gr}\theta, \tag{2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{3}$$

$$\frac{\partial \theta}{\partial t} + u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} = \frac{1}{\text{RePr}}\Delta\theta, \quad (x,y) \in D, \ t \in (0,T] \tag{4}$$

with the following initial and boundary conditions:

$$u = u_0(x,y), \quad v = v_0(x,y), \quad \theta = \theta_0(x,y), \quad (x,y) \in \overline{D}, \ t = 0, \tag{5}$$

$$u = a_x(x,y,t), \quad v = a_y(x,y,t), \quad \theta = \xi(x,y,t), \quad (x,y) \in \partial D, \ t = [0,T], \tag{6}$$

where u, v are components of the velocity, p is the pressure, θ is the temperature, Re is the Reynolds number, Gr is the Grashof number, Pr is the Prandtl number, $\partial D = \gamma_1 \cup \gamma_2$ is the boundary of the domain $\overline{D}$.

We introduce the current function ψ and the velocity vortex ω, which are related with the velocity components u, v by the following relations:

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}, \quad \omega = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}. \tag{7}$$

The problem (1)-(6) in the variables ψ, ω is written as follows [12]:

$$\frac{\partial \omega}{\partial t} + \frac{\partial \psi}{\partial y}\frac{\partial \omega}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \omega}{\partial y} = \frac{1}{\text{Re}}\Delta\omega + \text{Gr}\frac{\partial \theta}{\partial x}, \tag{8}$$

$$\Delta\psi = \omega, \tag{9}$$

$$\frac{\partial \theta}{\partial t} + \frac{\partial \psi}{\partial y}\frac{\partial \theta}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \theta}{\partial y} = \frac{1}{\text{RePr}}\Delta\theta, \quad (x,y) \in D, \ t \in (0,T], \tag{10}$$

$$\omega = \alpha(x,y), \quad \theta = \varphi(x,y), \quad (x,y) \in \overline{D}, \ t = 0, \tag{11}$$

$$\psi = \xi_1(x,y,t), \quad \frac{\partial \psi}{\partial \vec{n}} = \eta_1(x,y,t), \quad (x,y) \in \gamma_1, \ t \in (0,T], \tag{12}$$

$$\psi = \xi_2(x,y,t) + \lambda(t), \quad \frac{\partial \psi}{\partial \vec{n}} = \eta_2(x,y,t), \quad (x,y) \in \gamma_2, \ t \in (0,T], \tag{13}$$

$$\theta = \beta_l(x,y,t), \quad (x,y) \in \gamma_1, \ l = 1,2, \ t \in (0,T], \tag{14}$$

α, φ, ξ_i, η_i, β_i, $i = 1, 2$ are given functions.

Introduce a uniform mesh in $\overline{D}$:

$$D_h = \{(x_i, y_j), \ x_i = (i-1)h_1, \ y_j = (j-1)h_2, \ i = 1, ..., n_1,$$

$$j = 1, ..., n_2, \ h_1 = \frac{l_1}{n_1 - 1}, \ h_2 = \frac{l_2}{n_2 - 1}\right\}.$$

Assume that the inner subdomain D_0 is a rectangle:

$$D_{0h} = \{(x, y), \ x_{k1} \leq x \leq x_{k2}, \ y_{m1} \leq y \leq y_{m2}\}.$$

Consider the domain D_1 covering the domain D_0, that is $D_0 \subset D_1$. $D_1 = \{(x, y), \ x_{k3} \leq x \leq x_{k4}, \ y_{m3} \leq y \leq y_{m4}\}$.

Consider the fictitious domain method for solving the problem (8)-(14):

$$\frac{\partial \omega^\varepsilon}{\partial t} + \frac{\partial \psi^\varepsilon}{\partial y}\frac{\partial \omega^\varepsilon}{\partial x} - \frac{\partial \psi^\varepsilon}{\partial x}\frac{\partial \omega^\varepsilon}{\partial y} = \frac{1}{\mathrm{Re}}\Delta \omega^\varepsilon + \mathrm{Gr}\frac{\partial \theta^\varepsilon}{\partial x} - \mathrm{div}\,(k\,(x,y)\,\nabla \psi), \tag{15}$$

$$\Delta \psi^\varepsilon = \omega^\varepsilon, \tag{16}$$

$$\frac{\partial \theta^\varepsilon}{\partial t} + \frac{\partial \psi^\varepsilon}{\partial y}\frac{\partial \theta^\varepsilon}{\partial x} - \frac{\partial \psi^\varepsilon}{\partial x}\frac{\partial \theta^\varepsilon}{\partial y} = \frac{1}{\mathrm{RePr}}\Delta \theta^\varepsilon, \tag{17}$$

$$\psi^\varepsilon\big|_{\gamma_1} = 0, \quad \frac{\partial \psi^\varepsilon}{\partial \vec{n}}\bigg|_{\gamma_1} = 0, \tag{18}$$

$$\theta^\varepsilon = \beta_l\,(x, y, t), \quad (x, y) \in \gamma_l, \ l = 1, 2, \ t \in (0, T],$$

where

$$k\,(x, y) = \left\{ \begin{array}{ll} 1, & (x, y) \in D_0, \\ 0, & (x, y) \in \overline{D}\backslash D_0. \end{array} \right.$$

For the numerical solution of the obtained problem (15)-(18), the following explicit scheme and the iterative method of successive over-relaxation are constructed. For simplicity, we exclude the superscript ε. Replace the differential problem with its difference analog of the following form:

$$\frac{\omega_{ij}^{n+1} - \omega_{i,j}^n}{\tau} + \Lambda_{1,h}\omega_{i,j}^n + \Lambda_{2,h}\omega_{i,j}^n = \frac{1}{\mathrm{Re}}\Lambda_{11,h}\omega_{i,j}^n + \frac{1}{\mathrm{Re}}\Lambda_{22,h}\omega_{i,j}^n + \mathrm{Gr}\,\Phi_h\theta_{i,j}^n - \Lambda_{12,h}\psi_{i,j}^n, \tag{19}$$

$$\Lambda_{11,h}\psi_{i,j}^{n+1} + \Lambda_{22,h}\psi_{i,j}^{n+1} = \omega_{i,j}^{n+1}, \tag{20}$$

$$u_{i,j}^{n+1} = \frac{\psi_{i,j+1/2}^{n+1} - \psi_{i,j-1/2}^{n+1}}{h_2}, \tag{21}$$

$$v_{i,j}^{n+1} = -\frac{\psi_{i+1/2,j}^{n+1} - \psi_{i-1/2,j}^{n+1}}{h_1}, \tag{22}$$

$$\frac{\theta_{i,j}^{n+1} - \theta_{i,j}^n}{\tau} + \Lambda_{1,h}\theta_{i,j}^n + \Lambda_{2,h}\theta_{i,j}^n = \frac{1}{\mathrm{Re}\,\mathrm{Pr}}\Lambda_{11,h}\theta_{i,j}^n + \frac{1}{\mathrm{Re}\,\mathrm{Pr}}\Lambda_{22,h}\theta_{i,j}^n. \tag{23}$$

The difference analogs of the corresponding differential operators are as follows:

$$\Lambda_{1,h}\omega_{i,j}^n = \frac{1}{2}\left[\left(u_{i+1/2,j}^n - \left|u_{i+1/2,j}^n\right|\right)\frac{\omega_{i+1,j}^n - \omega_{i,j}^n}{h_1} + \left(u_{i-1/2,j}^n + \left|u_{i-1/2,j}^n\right|\right)\frac{\omega_{i,j}^n - \omega_{i-1,j}^n}{h_1}\right],$$

$$\Lambda_{2,h}\omega_{i,j}^n = \frac{1}{2}\left[\left(v_{i,j+1/2}^n - \left|v_{i,j+1/2}^n\right|\right)\frac{\omega_{i,j+1}^n - \omega_{i,j}^n}{h_2} + \left(v_{i,j-1/2}^n + \left|v_{i,j-1/2}^n\right|\right)\frac{\omega_{i,j}^n - \omega_{i,j-1}^n}{h_2}\right],$$

$$\Lambda_{11,h}\omega_{i,j}^n = \frac{\omega_{i+1,j}^n - 2\omega_{i,j}^n + \omega_{i-1,j}^n}{h_1^2},$$

$$\Lambda_{22,h}\omega_{i,j}^n = \frac{\omega_{i,j+1}^n - 2\omega_{i,j}^n + \omega_{i,j-1}^n}{h_2^2},$$

$$\Lambda_{12,h}\psi_{i,j}^n = \left[k_{i+1/2,j}\frac{\psi_{i+1,j}^n - \psi_{i,j}^n}{h_1^2} - k_{i-1/2,j}\frac{\psi_{i,j}^n - \psi_{i-1,j}^n}{h_1^2}\right] +$$

$$+ \left[k_{i,j+1/2}\frac{\psi_{i,j+1}^n - \psi_{i,j}^n}{h_1^2} - k_{i,j-1/2}\frac{\psi_{i,j}^n - \psi_{i,j-1}^n}{h_1^2}\right],$$

$$\Phi_h\theta_{i,j}^n = \frac{\theta_{i+1,j}^n - \theta_{i-1,j}^n}{2h_1}.$$

The numerical implementation algorithm is performed as follows: first, the values of $\omega_{i,j}^{n+1}$ are calculated using the formula (19); then the values of $\psi_{i,j}^{n+1}$ are found by (20). The obtained values of $\psi_{i,j}^{n+1}$ are used to determine the values of $u_{i,j}^{n+1}$ and $v_{i,j}^{n+1}$ using the formulas (21), (22); after that, using the new values of $u_{i,j}^{n+1}$ and $v_{i,j}^{n+1}$, the values of $\theta_{i,j}^{n+1}$ are calculated using (23). The iterative process is continued until the following condition is met:

$$\max_{\substack{1\le i\le n_1 \\ 1\le j\le n_2}}\left|\omega_{i,j}^{n+1} - \omega_{i,j}^n\right| < \varepsilon.$$

The algorithm described above is implemented using the CUDA parallel computing architecture. The grids are divided into blocks, and each block copies the data to the shared memory, after which each node of the individual block performs the calculation and saves the calculated data to the global memory. In each subdomain, it is required to use data from the neighboring subdomain, i.e. it is necessary to copy the boundary data from the global memory, therefore, the size of each subdomain will be increased. The first stage (19), the recalculation stage (21), (22) and the temperature calculation stage (23) are parallelizable, since an explicit scheme is used to implement these stages. The second stage (20) is calculated in global memory, since the neighboring values of the same iterative process are needed to determine the current function.

4 Results and discussion

The method given above is used to numerically solve the test problem (1)-(6) for the Navier-Stokes equations describing the motion of a viscous incompressible fluid in the current function, velocity vortex variables in the Boussinesq approximation.

Below, temperature distributions and current functions are presented as numerical results. The results are obtained for different cavity sizes, temperature conditions at the boundary, and values that determine the flow of dimensionless parameters, the Grashof Gr and Prandtl Pr numbers.

Figures 1-4 show the results of solving the problem by the method of fictitious domain with continuation by the lowest coefficients.
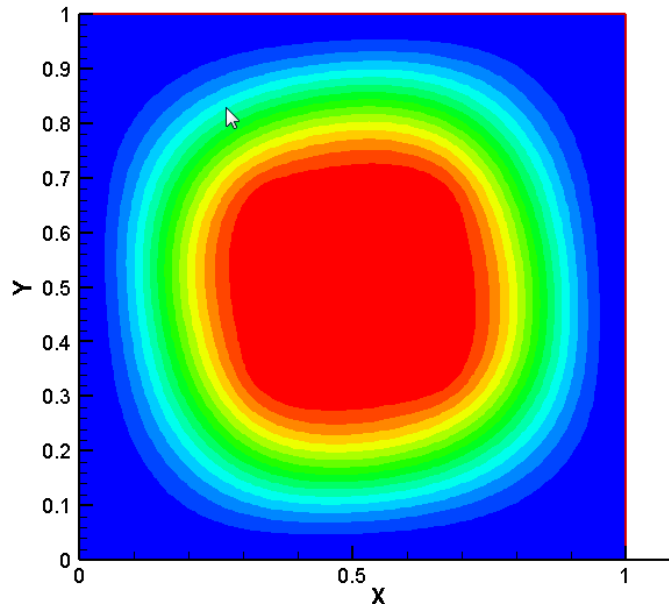

Figure 1: Isolines of the current function. The cavity size is 1.0×1.0; $\theta = 0.5$, Pr $= 5.39$, Gr $= 100$ on internal borders

The parallel algorithm of this problem was implemented using the CUDA architecture. When implementing the parallel algorithm on CUDA, two optimization methods were used:

1. Computational data was copied to the internal subdomains, then it was copied from global memory to shared memory. At the end of the optimization process, the boundary data is copied from the global memory. In such cases, the size of the subdomain remains unchanged [27].

2. In our case, it is impossible to avoid re-copying data at the border from the global memory. In these cases, columns and rows are copied at the boundaries of the subdomain. Therefore, the two-dimensional decomposition must be changed to a one-dimensional one. As a result, we do not make repeated copies.

A uniform grid with dimensions of 128x128, 256x256, 512x512, 1024x1024 and 2048x2048 is used in the calculations. All data was represented as single-precision real numbers. The computational experiment was conducted on a personal computer with an Intel Core i7-3770 3.40 GHz quad-core processor and an Nvidia GeForce GTX 550 Ti graphics card. The test result is shown in Figure 5. During the calculation, the following optimal block size was chosen: 16x16 (the number of threads in one block is 256). Figure 6 shows the performance gain compared to the sequential algorithm, depending on the mesh size.
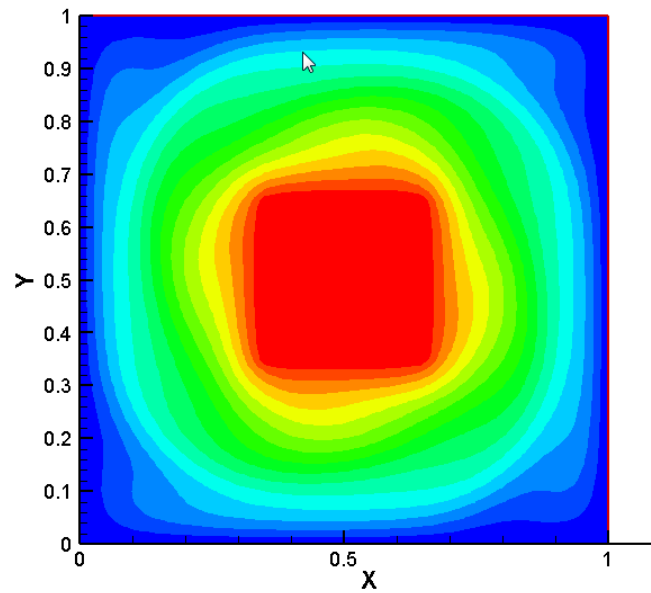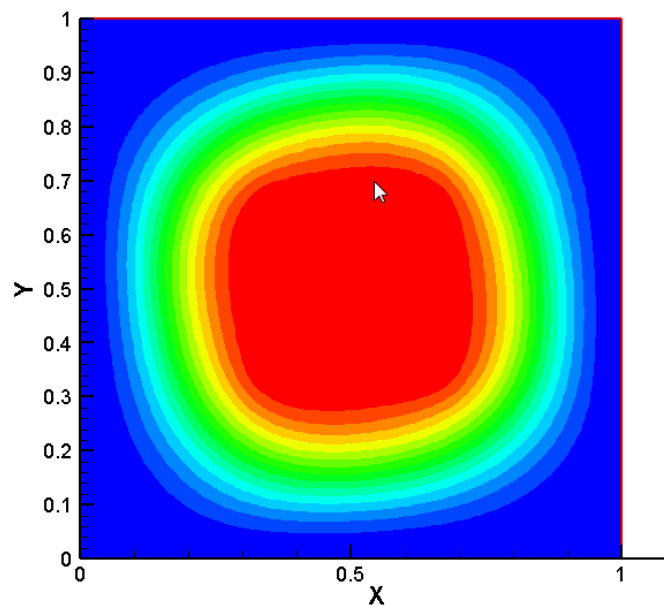
Figure 2: Isotherms. The cavity size is 1.0×1.0; $\theta = 1$, $\mathrm{Pr} = 5.39$, $\mathrm{Gr} = 100$ on internal borders

Figure 3: Isolines of the current function. The cavity size is 1.0×1.0; $\theta = 0.5$, $\mathrm{Pr} = 5.39$, $\mathrm{Gr} = 100$ on internal borders

5 Conclusion

Thus, the paper deals with the numerical implementation of the Navier-Stokes equations in a two-dimensional two-connected domain using the CUDA parallel computing architecture.
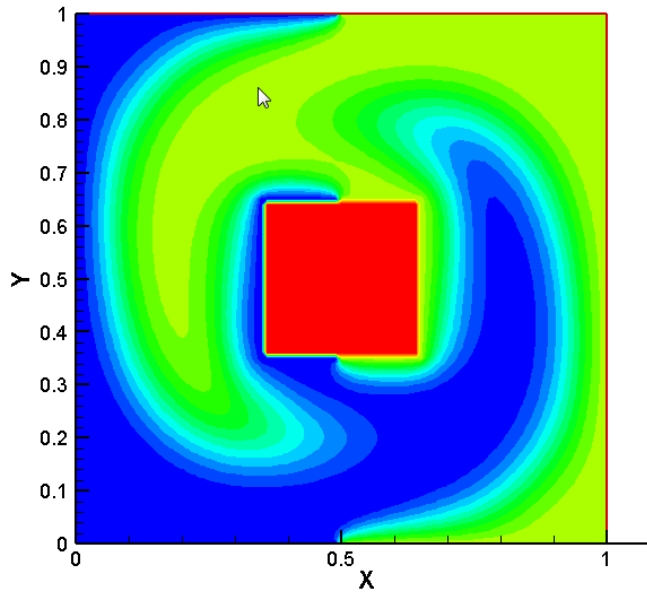
Figure 4: Isotherms. The cavity size is 1.0×1.0; the temperature on the right part of the border is $\theta = 0.5$, the temperature on the left part of the border is $\theta = -0.5$, Pr = 5.39, Gr = 100
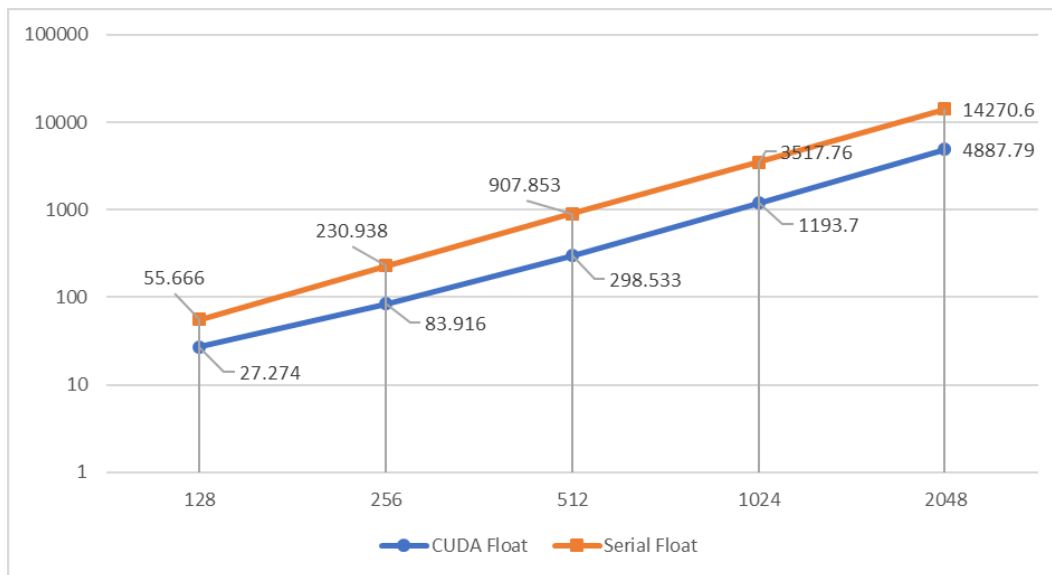

Figure 5: Execution time of a parallel algorithm on different computational meshes

The results of computational experiments show that the use of parallel algorithms using CUDA for this kind of tasks gives a good acceleration.

Further research will focus on the creation of parallel algorithms and acceleration of calculations related to the solution of nonlinear problems of multiphase fluid flows in porous
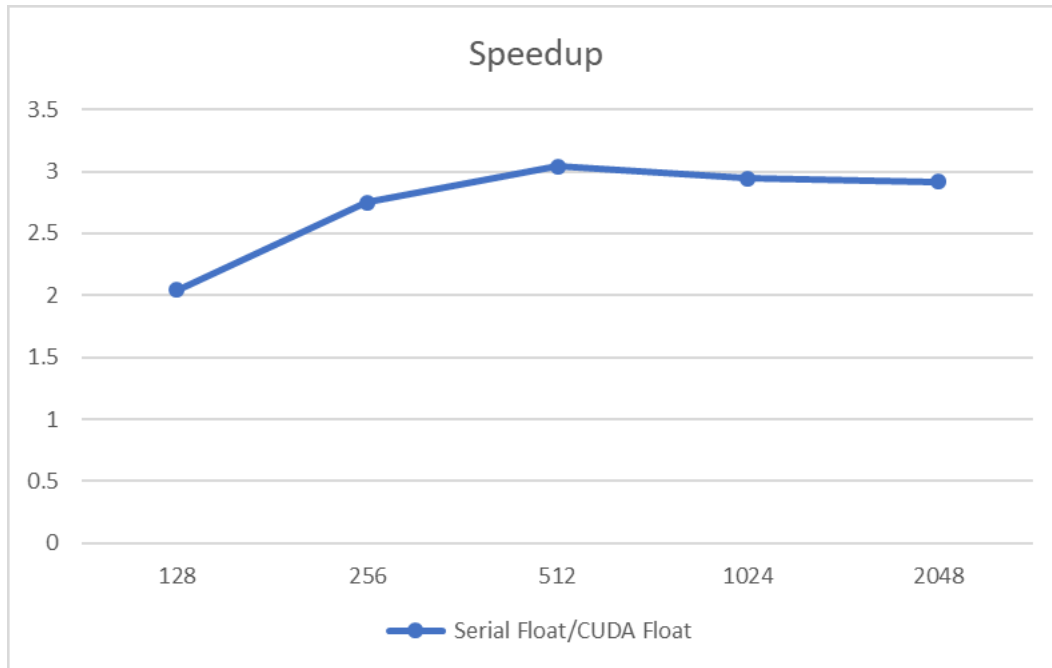
Figure 6: Parallel algorithm acceleration on CUDA

media considered in [28] by finite element methods using the CUDA software and hardware architecture of parallel computing.

6 Acknowledgement

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