IRSTI 27.31.15 DOI: : https://doi.org/10.26577/JMMCS.2021.v109.i1.03

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GREEN'S FUNCTIONS AND CORRECT RESTRICTIONS OF THE POLYHARMONIC OPERATOR

In this paper, for completeness of presentation, we give explicitly the Green's functions for the classical problems – Dirichlet, Neumann, and Robin for the Poisson equation in a multidimensional unit ball. There are various ways of constructing the Green's function of the Dirichlet problem for the Poisson equation. For many types of areas, it is built explicitly. Recently, there has been renewed interest in the explicit construction of Green's functions for classical problems. The Green's function of the Dirichlet problem for a polyharmonic equation in a multidimensional ball is constructed in an explicit form, and for the Neumann problem the construction of the Green's function remains an open problem. The paper gives a constructive way of constructing the Green's function of Dirichlet problems for a polyharmonic equation in a multidimensional ball. Finding general well-posed boundary value problems for differential equations is always an urgent problem. In this paper, we briefly outline the theory of restriction and extension of operators and describe well-posed boundary value problems for a polyharmonic operator.

Key words: Poisson equation, polyharmonic equations, Dirichlet problem, Neumann problem, Roben problem, correct restrictions of the operator.

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Полигармоникалық оператор үшін Грин функциялары және тиянақты тарылуы

Бұл жұмыста классикалық – Дирихле, Нейман және Робен есептерінің Грин функциялары айқын түрде көпөлшемді бірлік шарда Пуассон теңдеуі үшін көрсетілген. Пуассон теңдеуі үшін Дирихле есебінің Грин функциясын құрудың әртүрлі тәсілдері бар. Аудандардың көптеген түрлері үшін оның айқын түрі құрылған. Соңғы уақыттарда классикалық есептердің Грин функцияларын айқын түрде құруға деген қызығушылық қайта жандануда. Көпөлшемді шарда полигармоникалық теңдеу үшін Дирихле есебінің Грин функциясы айқын түрде құрылған, ал Нейман есебі үшін Грин функциясын құру ашық проблема болып қала беруде. Жұмыста көпөлшемді шарда полигармоникалық теңдеу үшін Дирихле есебінің Грин функциясын құрудың тиімді әдісі көрсетілген. Дифференциалдық теңдеулер үшін жалпы тиянақты шекаралық есептерді табу әрқашан өзекті мәселе болып табылады. Бұл жұмыста операторлардың тарылуы мен кеңеюі теориясы қысқаша сипатталған және полигармоникалық операторлар үшін тиянақты шеттік есептердің тарылуы сипатталған.

Түйін сөздер: Пуассон теңдеуі, Дирихле есебі, Нейман есебі, Робен есебі, оператордың тиянақты тарылуы

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Функций Грина и корректные сужения полигармонического оператора

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В данной работе приведены в явном виде функций Грина классических задач – Дирихле, Неймана и Робена для уравнения Пуассона в многомерном единичном шаре. Существуют различные способы построения функции Грина задачи Дирихле для уравнения Пуассона. Для многих видов областей она построена в явном виде. В последнее время возобновился интерес к построению в явном виде функций Грина классических задач. Функция Грина задачи Дирихле для полигармонического уравнения в многомерном шаре построена в явном виде, а для задачи Неймана построение функции Грина остается открытой задачей. В работе дан конструктивный способ построения функции Грина задач Дирихле для полигармонического уравнения в многомерном шаре. Нахождение общих корректных краевых задач для дифференциальных уравнений всегда является актуальной задачей. В данной работе кратко изложена теория сужения и расширения операторов и описаны корректные краевые задачи для полигармонического оператора.

Ключевые слова: уравнение Пуассона, полигармонические уравнения, задача Дирихле, задача Неймана, задача Робена, корректные сужения оператора.

Introduction

The need to study boundary value problems for elliptic equations is dictated by numerous practical applications in the theoretical study of the processes of hydrodynamics, electrostatics, mechanics, thermal conductivity, elasticity theory, and quantum physics [1-4]. The distributions of the potential of the electrostatic field are described using the Poisson equation. When studying the vibrations of thin plates of small deflections, biharmonic equations arise.

This work is devoted to the construction of the Green's function of the Dirichlet problem for a polyharmonic equation in a multidimensional ball and to the description of well-posed boundary value problems for polyharmonic operators.

1 Materials and methods

The subject of this research is a constructive way of constructing the Green's function of boundary value problems for a polyharmonic equation in a ball of arbitrary dimension.

The research method is the representation of polyharmonic functions through the sum of harmonic functions with certain weights. When constructing explicitly the Green's function of the Dirichlet problem for a polyharmonic equation in a ball, the method of special expansion of the fundamental solutions of the polyharmonic equation and the method of reflection are essentially used. When describing new well-posed boundary value problems for an inhomogeneous polyharmonic equation in a ball, the method of restricting abstract operators was applied.

There are various ways to construct the Green Function of the Dirichlet problem for the Poisson equation. For many types of domains, it is constructed explicitly. And for the Neumann problem in multidimensional domains, the construction of the Green function is an open problem. For the ball, the Green function of the internal and external Neumann problem is constructed explicitly only for the two-dimensional and three-dimensional cases. In the general case, for a multidimensional ball, the explicit form of the Green function of the Neumann and Robin problems for the Poisson equation is constructed recently in [5,6].

2 Results and discussion

Note that recently there has been renewed interest in the explicit construction of Green's functions for classical problems. In [7-9], the Green function of the Dirichlet problem for a polyharmonic equation in a multidimensional ball is constructed explicitly. In [10], the Green harmonic functions of the Dirichlet, Neumann, and Robin problems are used to construct the Green functions of the biharmonic Dirichlet, Neumann, and Robin problems in a two-dimensional circle. Similar results in the class of inhomogeneous biharmonic and triharmonic functions in the sector were obtained in [11-13]. Note also that the construction of explicit Green functions of the Robin problem in a circle, when the parameter in the boundary condition is equal to one, is devoted to the work [14,15]. The results of these studies are based on the classical theory of integral representations for analytic, harmonic, and polyharmonic functions on the plane.

Finding general correct boundary value problems for differential equations is always an urgent problem. The abstract theory of operator contraction and expansion originates from the work of John von Neumann [16], in which a method for constructing self-adjoint extensions of a symmetric operator was described and a theory of extension of symmetric operators with finite defect indices was developed in detail. Many problems for partial differential equations lead to operators with infinite defect indices.

In [17,18] considered extensions of the minimal operator, rejecting its symmetry, and described the areas of definition of the extension that have certain solvability properties, here are investigated to general boundary value problems for general second-order elliptic differential equations. In [19] found a correct problem that is not contained among the problems described [18]. This type of problem for ordinary differential equations was studied in [20].

In the early 80s of the last century, M. Otelbaev and his students [21-23] constructed an abstract theory that allows us to describe all correct contractions of a certain maximum operator and separately - all correct extensions of a certain minimum operator, independently of each other, in terms of the inverse operator. This theory was extended to the case of Banach spaces [24].

In [25] certain estimates are obtained for the deviation upon domain perturbation of singular number of correct restrictions of elliptic differential operators.

Thus, this paper is devoted to the construction of the Green function of the classical Dirichlet, Neumann and Robin problems for the Poisson equation in a multidimensional ball, a constructive way to construct the Green function of the Dirichlet problem for a polyharmonic equation in a multidimensional ball, and the description of correct boundary value problems for polyharmonic operators.

3 Green's function of the Dirichlet, Neumann, and Robin problem for the Poisson equation in a multidimensional unit ball

Let $\Omega \subseteq \mathbb{R}^n, n \geq 2$ be a bounded region with a smooth boundary $\partial\Omega$. Consider in domain Ω following the Dirichlet problem for the Poisson equation

$$-\Delta u(x) = f(x),\; x \in \Omega,\;\; u(x) = \varphi(x),\; x \in \partial\Omega. \tag{1}$$

The classical solution $u(x) \in C^2(\Omega) \cap C(\overline{\Omega})$ of the Dirichlet problem (1) exists, is unique, and is represented by the Green's function $G_D(x,y)$ in the following form [1]

$$u(x) = \int_\Omega G_D(x,y) f(y) dy - \int_{\partial\Omega} \frac{\partial G_D(x,y)}{\partial n_y} \varphi(y) dS_y, \tag{2}$$

where $\dfrac{\partial}{\partial n_y}$ – the external normal of $\partial\Omega$, and is calculated by the formula

$$\frac{\partial}{\partial n_y} = \sum_{k=1}^{n} (n_y)_k \frac{\partial}{\partial x_k},\ n_y \equiv \overrightarrow{n}_y = \{(n_y)_1, (n_y)_2, ..., (n_y)_n\},\ |n_y| = 1.$$

The Green function of the Dirichlet problem (1) is defined as follows

$$-\Delta G_D(x,y) = \delta(x-y),\ x, y \in \Omega,$$

$$G_D(x,y) = 0,\ x \in \partial\Omega,\ x \in \Omega,$$

where $\delta(x-y)$ is the Dirac delta function.

In particular, when $\Omega = \{x \in \mathbb{R}^n : |x| < 1\}$ is a unit ball, the Green function of the Dirichlet problem (1) can be constructed by the reflection method and has the form

$$G_D(x,y) = \frac{1}{\omega_n} \left[\varepsilon_n(x-y) - \varepsilon_n\Big(x|y| - \frac{y}{|y|} \Big) \right], \tag{3}$$

where $\omega_n = \dfrac{2\pi^{n/2}}{\Gamma(n/2)}$ – the surface area of a unit ball, $\varepsilon_n(x-y)$ is the fundamental solution of the Laplace equation [2,3]

$$\varepsilon_n(x-y) = \begin{cases} \ln \dfrac{1}{|x-y|}, & n = 2,\ |x-y| = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}, \\ \dfrac{1}{n-2} |x-y|^{2-n}, & n \geq 3,\ |x-y| = \sqrt{\sum\limits_{k=1}^{n} (x_k - y_k)^2}. \end{cases}$$

Along with the Dirichlet problem, the Neumann problem for the Poisson equation is a classical and well-studied one

$$-\Delta u(x) = f(x),\ x \in \Omega,\ \frac{\partial u(x)}{\partial n} = \psi(x),\ x \in \partial\Omega. \tag{4}$$

It is known that the solution of the Neumann problem (4) from class $C^2(\Omega) \cap C^1(\overline{\Omega})$ is not unique up to the constant term. For the existence of a solution to the problem, it is necessary and sufficient to fulfill the condition

$$\int_\Omega f(y) dy + \int_{\partial\Omega} \psi(y) dS_y = 0. \tag{5}$$

If a solution to problem (4) exists, then this solution can be represented in integral form using the Green function of the Neumann problem $G_N(x,y)$ according to the formula [1]

$$u(x) = \int_\Omega G_N(x,y) f(y) dy + \int_{\partial\Omega} G_N(x,y) \psi(y) dS_y + \text{const}. \tag{6}$$

The Green function of the Neumann problem (4) is understood as [1] a function that has the representation

$$G_N(x,y) = \frac{1}{\omega_n}\left[\varepsilon_n(x-y) + g(x,y)\right],$$

where $g(x,y)-$ the harmonic function in region Ω.

In this case, the boundary condition must be met

$$\frac{\partial G_N}{\partial n_y}(x,y) = -\frac{1}{\omega_n},\ y \in \partial\Omega. \tag{7}$$

If such a Green's function $G_N(x,y)$ exists, then it follows from (5) and (7) that function (6) satisfies all conditions of problem (4).

For a unit ball, the Green function of the Neumann problem is presented explicitly for cases $n=2$ and $n=3$

$$G_N(x,y) = \frac{1}{2\pi}\left[\ln\frac{1}{|x-y|} + \ln\frac{1}{\left|x|y| - \frac{y}{|y|}\right|}\right],\ n=2,$$

$$G_N(x,y) = \frac{1}{4\pi}\left[\frac{1}{|x-y|} + \frac{1}{\left|x|y| - \frac{y}{|y|}\right|} - \ln\left|1 + (x,y) + \left|x|y| - \frac{y}{|y|}\right|\right|\right],\ n=3,$$

where $(x,y) = x_1y_1 + ... + x_ny_n-$ the scalar product in $\mathbb{R}^n$ of vectors x and y.

The Green function of the Neumann problem (4) has the following representation [5]

$$G_N(x,y) = \frac{1}{\omega_n}\left[\varepsilon_n(x-y) + \varepsilon_n\left(x|y| - \frac{y}{|y|}\right) + \widetilde{\varepsilon}(x,y)\right] + const,$$

where $\widetilde{\varepsilon}(x,y)$ expressed by the identity

$$\widetilde{\varepsilon}(x,y) = \int_0^1 \left[(n-2)s|x|y| - \frac{y}{|y|}| - 1\right]\frac{ds}{s} \equiv \int_0^1 \left[s\left|x|y| - \frac{y}{|y|}\right|^{2-n} - 1\right]\frac{ds}{s}, n \geq 3,$$

and they are written through elementary functions

$$\widetilde{\varepsilon}(x,y) = \ln\frac{2}{\left|1 - (x,y) + \left|x|y| - \frac{y}{|y|}\right|\right|},\ n=3; \tag{i}$$

$$\widetilde{\varepsilon}(x,y) = \ln\frac{(x,y)}{\sqrt{|x|^2|y|^2 - (x,y)^2}}\arctan\frac{\sqrt{|x|^2|y|^2-(x,y)^2}}{1-(x,y)} - \ln\left|x|y| - \frac{y}{|y|}\right|,\ n=4; \tag{ii}$$

$$\widetilde{\varepsilon}(x,y) = \ln\frac{2}{\left|1 - (x,y) + \left|x|y| - \frac{y}{|y|}\right|\right|} + \sum_{k=1}^{m-1}\frac{1}{(2k-1)}\left\{\left|x|y| - \frac{y}{|y|}\right|^{1-2k} - 1\right\} +$$

$$\sum_{k=1}^{m-1}\sum_{i=0}^{m-k-1}\frac{2^i(k+i-1)(2k-3)!!(x,y)|x|^{2i}|y|^{2i}}{(k-1)!(2k+2i-1)!!(|x|^2|y|^2-(x,y)^2)^{i+1}}\left[\frac{|x|^2|y|^2-(x,y)}{\left|x|y| - \frac{y}{|y|}\right|^{2k-1}} + (x,y)\right],$$

$$n \geq 5,\ n = 2m+1,\ m \geq 2; \qquad (iii)$$

$$\widetilde{\varepsilon}(x,y) = -\ln\left|x|y| - \frac{y}{|y|}\right| + \sum_{k=1}^{m-1} \frac{1}{2k}\left\{\left|x|y| - \frac{y}{|y|}\right|^{-2k} - 1\right\} +$$

$$(x,y)\arctan\frac{\sqrt{|x|^2|y|^2-(x,y)^2}}{1-(x,y)} \sum_{k=0}^{m-1} \frac{(2k-1)!!}{2^k k!} \frac{|x|^{2k}|y|^{2k}}{(|x|^2|y|^2-(x,y)^2)^{k-1/2}} +$$

$$\sum_{k=1}^{m-1}\sum_{i=0}^{m-k-1} \frac{(2k+2i-1)!!(k+1)!\,(x,y)|x|^{2i}|y|^{2i}}{2^{i+1}(2k-1)!!(k+i)!(|x|^2|y|^2-(x,y)^2)^{i+1}} \left[\frac{|x|^2|y|^2-(x,y)}{\left|x|y|-\frac{y}{|y|}\right|^{2k}} - (x,y)\right],$$

$$n \geq 6,\ n = 2m+2,\ m \geq 2. \qquad (iv)$$

Along with the Dirichlet and Neumann problems, the Robin problem (the third boundary value problem) for the Poisson equation is a classical and well-studied one

$$-\Delta u(x) = f(x),\ x \in \Omega,\ \frac{\partial u(x)}{\partial n} + au(x) = \psi(x),\ x \in \partial\Omega. \qquad (8)$$

The solution of the problem Robin (8) from class $C^2(\Omega) \cap C^1(\overline{\Omega})$ is represented as follows

$$u(x) = \int_\Omega G_a(x,y)f(y)dy - \int_{\partial\Omega} \frac{\partial G_a(x,y)}{\partial n_y}\varphi(y)dS_y. \qquad (9)$$

The Green function of the Robin problem (8) has the form [6]

a) if $a > 0$, then

$$G_a(x,y) = G_D(x,y) + \frac{1}{2\pi}\int_0^1 s^{a-1}P(r\rho s,\gamma)ds =$$

$$\varepsilon(x-y) - \varepsilon\left(x|y| - \frac{y}{|y|}\right) + \frac{n-2-2a}{\omega_n}\int_0^1 s^{a-1}\varepsilon\left(sx|y| - \frac{y}{|y|}\right)ds,$$

where $\gamma = \theta - \varphi$ и $P(r\rho s,\gamma) = \dfrac{1-t^2}{1-2tcos\gamma+t^2}-$ the Poisson kernel;

b) if $a < 0$ and $a-$ non-integer, then

$$G_a(x,y) = G_D(x,y) + \frac{1}{2\pi}\left[\frac{2}{a} + 2\sum_{k=1}^{m}\frac{1}{k+a}(r\rho)^s\cos k\gamma + \right.$$

$$\left.\int_0^1 s^{a-1}\Big(P(r\rho s,\gamma) + 1 - 2\sum_{k=0}^{m}(r\rho s)^k cosk\gamma\Big)ds\right],$$

where $m = -[a] + 1$.

4 Green's function of the Dirichlet problem for a polyharmonic equation in a multidimensional ball

Let m be a natural number and in the $n-$ dimensional ball $\Omega = \{x \in \mathbb{R}^n : |x| < r\}$ consider Dirichlet problem for a polyharmonic equation

$$\Delta^m u(x) = f(x), \quad x \in \Omega, \tag{10}$$

$$\frac{\partial^j u(x)}{\partial n_x^j} = \varphi_j(x), \;\; 0 \le j \le m-1, \;\; x \in \partial\Omega. \tag{11}$$

The classical solution $u(x) \in C^{2m}(\Omega) \cap C^{m-1}(\overline{\Omega})$ to the Dirichlet problem (10), (11) exists, is unique, and it is represented by the Green's function $G_{2m,n}(x,y)$ in the following form [3]

$$u(x) = \int_\Omega G_{2m,n}(x,y) f(y) dy + \sum_{j=0}^{m-1} \int_{\partial\Omega} \left[\frac{\partial}{\partial n_y} \Delta_y^j G_{2m,n}(x,y) \cdot \Delta_y^{m-1-j} \varphi(y) - \right.$$

$$\left. - \Delta_y^j G_{2m,n}(x,y) \cdot \frac{\partial}{\partial n_y} \Delta_y^{m-1-j} \varphi(y) \right] dS_y, \tag{12}$$

where $\frac{\partial}{\partial n_y}-$ external normal $\partial\Omega$.

The Green function of the Dirichlet problem (10), (11) is defined as follows

$$\Delta^m G_{2m,n}(x,y) = \delta(x-y), \;\; x, y \in \Omega, \tag{13}$$

$$\frac{\partial^j G_{2m,n}(x,y)}{\partial n_x^j} = 0, \; x \in \partial\Omega, \; y \in \Omega, \; 0 \le j \le m-1, \tag{14}$$

where $\delta(x-y)-$ the Dirac delta function.

Theorem 1 *[7–9] a) In the case of odd n, as well as for even n, if $2m < n$ the Green's function of the Dirichlet problem (13), (14) can be represented in the form*

$$G_{2m,n}(x,y) = \varepsilon_{2m,n}(x,y) - g^0_{2m,n}(x,y) - \sum_{k=1}^{m-1} g^k_{2m,n}(x,y), \tag{15}$$

where

$$\varepsilon_{2m,n}(x,y) = d_{2m,n}|x-y|^{2m-n},$$

$$g^0_{2m,n}(x,y) = d_{2m,n} \left[\left| \frac{y}{r} \right| \cdot \left| x - \frac{y}{|y|^2} r^2 \right| \right]^{2m-n},$$

$$g^k_{2m,n}(x,y) = d_{2m,n}(2m-n)...(2(m-k+1)-n) \cdot \left[\left| \frac{y}{r} \right| \cdot \left| x - \frac{y}{|y|^2} r^2 \right| \right]^{2m-n-2k} \cdot$$

$$\cdot \left(1 - \left| \frac{y}{r} \right|^2 \right)^k \left(1 - \left| \frac{x}{r} \right|^2 \right)^k \left(\frac{r^2}{-2} \right)^k \frac{1}{k!}, \quad k = 1, 2, ..., m-1,$$

$$d_{2m,n} = \frac{1}{(m-1)!\,(2m-n)(2(m-1)-n)...(4-n)(2-n)} \cdot \frac{\Gamma(n/2)}{2^m \pi^{n/2}},$$

$\Gamma(\cdot)-$ *gamma function;*

b) In the case of even n and $2m \geq n$, the Green's function of the Dirichlet problem (13), (14) can be represented in the form (15), where

$$\varepsilon_{2m,n}(x,y) = d_{2m,n}|x-y|^{2m-n}\ln|x-y|,$$

$$g^0_{2m,n}(x,y) = d_{2m,n}\left[\left|\frac{y}{r}\right| \cdot \left|x - \frac{y}{|y|^2}\,r^2\right|\right]^{2m-n} \ln\left[\left|\frac{y}{r}\right|\left|x - \frac{y}{|y|^2}r^2\right|\right],$$

$$g^k_{2m,n}(x,y) = d_{2m,n}\left[\left|\frac{y}{r}\right| \cdot \left|x - \frac{y}{|y|^2}r^2\right|\right]^{2m-n-2k}\left(1-\left|\frac{y}{r}\right|^2\right)^k\left(1-\left|\frac{x}{r}\right|^2\right)^k r^{2k}\cdot$$

$$\left[\frac{(-2)^k}{k!}(2m-n)(2(m-1)-n)...(2(m-k+1)-n)\ln\left[\left|\frac{y}{r}\right|\left|x - \frac{y}{|y|^2}r^2\right|\right] - \right.$$

$$\left.\frac{2^{2k}}{2}\left(\frac{1}{k} + \sum_{i=1}^{k-1}\frac{1}{i}\frac{(-1)^{k-i}}{(k-i)!}\frac{(2m-n)}{2}...\frac{(2m-n-2(k-i)+2)}{2}\right)\right],\ \ k=1,2,...,m-1,$$

$$d_{2m,n} = \frac{(-1)^{n/2-1}}{\Gamma(m)\Gamma(m-n/2+1)\cdot 2^{2m-1}\pi^{n/2}}.$$

Lemma 1 *a) It is known [3] that in the case of odd n and even n, when $2m \leq n$, the function*

$$\varepsilon_{2m,n}(x,y) = d_{2m,n}|x-y|^{2m-n},$$

and in the case of even n, when $2m \geq n$, the function

$$\varepsilon_{2m,n}(x,y) = d_{2m,n}|x-y|^{2m-n}\ln|x-y|$$

is a fundamental solution to equation (10);

b) for all $0 \leq k \leq m-1$ functions

$$g^k_{2m,n}(x,y) = d_k\left[\left|\frac{y}{r}\right| \cdot \left|x - \frac{y}{|y|^2}r^2\right|\right]^{2m-n-2k} \cdot \left(1-\left|\frac{y}{r}\right|^2\right)^k\left(1-\left|\frac{x}{r}\right|^2\right)^k r^{2k},$$

where

$$d_k = \frac{1}{(-2)^k k!}\,d_{2m,n}(2m-n)(2(m-1)-n)...(2(m-k+1)-n)$$

are solutions to the homogeneous polyharmonic equation

$$\Delta^m g^k_{2m,n}(x,y) = 0, \ \ x,y \in \Omega. \tag{16}$$

Proof 1 *Indeed, the function* $g^k_{2m,n}(x,y)$ *can be represented in the form* $g^k_{2m,n}(x,y) = g(x,y)f_{2k}(|x|,|y|)$, *where* $f_{2k}(|x|,|y|)-$ *polynomial of degree* $2k$ *in* $|x|$ *for fixed* $|y|$, *and* $g(x,y)$ *satisfy the equation* $\Delta^{m-k}g(x,y)=0$.

By Almanzi's theorem [3], the function $g(x,y)$ *can be represented as*

$$g(x,y) = \sum_{j=0}^{m-k-1} |x|^{2j}\Psi_j(x,y),$$

where $\Psi_j(x,y)-$ *harmonic functions, i.e.* $\Delta_x\Psi_j(x,y) = 0$. *Then the function* $g^k_{2m,n}(x,y)$ *satisfies the representation*

$$g^k_{2m,n}(x,y) = \sum_{j=0}^{m-k-1} |x|^{2j}\Psi_j(x,y)f_{2k}(|x|,|y|) = \sum_{j=0}^{m-k-1} |x|^{2j}\widetilde{\Psi}_j(x,y),$$

where $\widetilde{\Psi}_j(x,y)-$ *some harmonic functions.*

Therefore, according to Almanzi's theorem, the function $g^k_{2m,n}(x,y)$ *for all* $0 \le k \le m-1$ *satisfies homogeneous polyharmonic equation (16).*

It is easy to show that in the following notation

$$|x-y|^2 = X^2(x,y) = X^2,\ \left|\frac{y}{r}\right|^2 \left|x - \frac{y}{|y|^2}r^2\right|^2 = Y^2(x,y) = Y^2,$$

$$\left(1 - \left|\frac{y}{r}\right|^2\right)\left(1 - \left|\frac{x}{r}\right|^2\right) r^2 = Z^2(x,y) = Z^2, \tag{17}$$

we have the identity

$$X^2 - Y^2 = -Z^2,\ \forall x,y \in \Omega. \tag{18}$$

Proof 2 *a) Using equality (18) and the expansion of functions* $f(x) = (1-x)^\alpha$, $0 < x \le 1$ *[4], we represent the fundamental solution of equation (10) as a series*

$$\varepsilon_{2m,n}(x,y) = X^{2m-n} = Y^{2m-n}\Big(1 - \frac{Z^2}{Y^2}\Big)^{\frac{2m-n}{2}} =$$

$$Y^{2m-n} + \sum_{k=1}^{m-1} \frac{(-1)^k}{k!}(m-\frac{n}{2})(m-\frac{n}{2}-1)...\Big(m-\frac{n}{2}-k+1\Big)Y^{2m-n-2k}Z^{2k} +$$

$$\sum_{k=m}^{\infty} \frac{(-1)^k}{k!}(m-\frac{n}{2})(m-\frac{n}{2}-1)...\Big(m-\frac{n}{2}-k+1\Big)Y^{2m-n-2k}Z^{2k}.$$

Moving the m *terms to the left, we get the required Green's function in the following form:*

$$G_{2m,n}(x,y) = \mathfrak{G}^m_{2m,n}(x,y) = \mathfrak{G}^\infty_{2m,n}(x,y),$$

where

$$\mathfrak{G}^m_{2m,n}(x,y) = d_{2m,n}\Big[X^{2m-n} - Y^{2m-n} - \sum_{k=1}^{m-1} \frac{(-1)^k}{k!}(m-\frac{n}{2})...\Big(m-\frac{n}{2}-k+1\Big)Y^{2m-n-2k}Z^{2k}\Big],$$

$$\mathfrak{G}^{\infty}_{2m,n}(x,y) = d_{2m,n}\sum_{k=m}^{\infty}\frac{(-1)^k}{k!}(m-\frac{n}{2})(m-\frac{n}{2}-1)...\Big(m-\frac{n}{2}-k+1\Big)Y^{2m-n-2k}Z^{2k}.$$

Because

$$\Big(X^2-Y^2\Big)\Big|_{x\in\partial\Omega,y\in\Omega} = -Z^2\Big|_{x\in\partial\Omega,y\in\Omega} = -r^2\left(1-\left|\frac{y}{r}\right|^2\right)\left(1-\left|\frac{x}{r}\right|^2\right)\Big|_{x\in\partial\Omega,y\in\Omega} = 0,$$

then using the equalities

$$\frac{\partial^j}{\partial n_x^j}Z^{2m}\Big|_{x\in\partial\Omega,y\in\Omega} = 0,\ j=\overline{0,m-1},$$

it is easy to show that the function

$$\mathfrak{G}^{\infty}_{2m,n}(x,y) =$$

$$Z^{2m}\Big[d_{2m,n}\sum_{k=m}^{\infty}\frac{(-1)^k}{k!}(m-\frac{n}{2})(m-\frac{n}{2}-1)...\Big(m-\frac{n}{2}-k+m+1\Big)Y^{2m-n-2k}Z^{2k-2m}\Big] =$$

$$\left(r^2\left(1-\left|\frac{y}{r}\right|^2\right)\left(1-\left|\frac{x}{r}\right|^2\right)\right)^m\times$$

$$\Big[d_{2m,n}\sum_{k=m}^{\infty}\frac{(-1)^k}{k!}(m-\frac{n}{2})(m-\frac{n}{2}-1)...\Big(m-\frac{n}{2}-k+m+1\Big)Y^{2m-n-2k}Z^{2k-2m}\Big]$$

satisfies the boundary condition (14).

According to Lemma 1 and the last equality, we have

$$(-\Delta_x)^m G_{2m,n}(x,y) = (-\Delta_x)^m\mathfrak{G}^{m}_{2m,n}(x,y) = \delta(x-y),\ x,y\in\Omega,$$

$$\frac{\partial^j}{\partial n_x^j}G_{2m,n}(x,y)\Big|_{x\in\partial\Omega} = \frac{\partial^j}{\partial n_x^j}\mathfrak{G}^{\infty}_{2m,n}(x,y)\Big|_{x\in\partial\Omega} = 0,\ j=\overline{0,m-1}.$$

By virtue of the uniqueness of the solution to the Dirichlet problem for the polyharmonic equation, the Green's function of problem (13), (14) is

$$G_{2m,n}(x,y) = d_{2m,n}\Big[X^{2m-n}-Y^{2m-n}-\sum_{k=1}^{m-1}\frac{(-1)^k}{k!}(m-\frac{n}{2})...\Big(m-\frac{n}{2}-k+1\Big)Y^{2m-n-2k}Z^{2k}\Big].$$

b) Using Lemma 1 and the expansion of functions $f(x)=\ln(1-x)$, $0<x\le 1$ [4], we represent the fundamental solution of equation (10) as a series

$$\varepsilon_{2m,n}(x,y) = |x-y|^{2m-n}\ln|x-y| = X^{2m-n}\ln X =$$

$$\Big[Y^{2m-n}\ln Y+\sum_{\mu=1}^{m-\frac{n}{2}}\frac{1}{2\mu}\Big]\cdot\Big(1-\frac{Z^2}{Y^2}\Big)^{m-\frac{n}{2}}+\frac{1}{2}\Big[\ln\Big(1-\frac{Z^2}{Y^2}\Big)-\sum_{\mu=1}^{m-\frac{n}{2}}\frac{1}{2\mu}\Big]\cdot\Big(1-\frac{Z^2}{Y^2}\Big)^{m-\frac{n}{2}} =$$

$$Y^{2m-n}\ln Y+\sum_{\nu=1}^{m-\frac{n}{2}}C_{\nu}^{m-\frac{n}{2}}Z^{2\nu}Y^{2(m-\nu)-n}\ln Y+\sum_{\nu=1}^{m-\frac{n}{2}}(-1)^{\nu}C_{\nu}^{m-\frac{n}{2}}\sum_{\mu=m-\nu+1-\frac{n}{2}}^{m-\frac{n}{2}}\frac{1}{2\mu}Z^{2\nu}Y^{2(m-\nu)-n}+$$

$$(-1)^{m+1-\frac{n}{2}}\sum_{\nu=1}^{\infty}\frac{1}{2\nu C_{\nu}^{\nu+m-\frac{n}{2}}}Z^{2(m+\nu)-n}Y^{-2\nu}.$$

Moving the $m-1$ terms to the left, we get the equality

$$G_{2m,n}(x,y)=\mathfrak{F}_{2m,n}^{m}(x,y)=\mathfrak{F}_{2m,n}^{\infty}(x,y),$$

where

$$\mathfrak{F}_{2m,n}^{m}(x,y)=d_{2m,n}\Big[X^{2m-n}\ln X-Y^{2m-n}\ln Y-$$

$$\sum_{\nu=1}^{m-n/2}(-1)^{\nu}C_{\nu}^{m-n/2}\Big[\ln Y+\widetilde{C}\Big]Z^{2\nu}Y^{2m-2\nu-n}+(-1)^{m-n/2}\sum_{\nu=1}^{n/2-1}\frac{2^{2m+2\nu-n}}{2\nu C_{\nu+n/2}^{m+\nu}}Z^{2(m+\nu)}Y^{-2\nu-n}\Big],$$

$$\mathfrak{F}_{2m,n}^{\infty}(x,y)=-d_{2m,n}\sum_{\nu=0}^{\infty}\frac{2^{2(m+\nu)}}{(2\nu+n)C_{\nu+n/2}^{m+\nu}}Y^{-n-2\nu}Z^{2(m+\nu)},\ \widetilde{C}=\sum_{\mu=m-n/2+1-\nu}^{m-n/2}\frac{1}{2\mu}.$$

Using this representation, just as in the proof of assertion a), we make sure that $G_{2m,n}(x,y)$ is the required Green's function for even n for $2m\geq n$. The theorem is proved.

5 Correct constrictions and extensions of differential operators

In the early 80s of the last century, M.O. Otelbaev and his students [21-23] constructed an abstract theory that allows us to describe all correct constrictions of a certain maximum operator and separately - all correct extensions of a certain minimum operator, independently of each other, in terms of the inverse operator. Moreover, this theory was extended to the case of Banach spaces and it was possible to partially abandon the linearity of operators. We give a brief summary of this theory in the case of Hilbert spaces.

Let the Hilbert space H be a linear operator L with a domain of definition $D(L)$ and a domain of value $R(L)$. The kernel of operator L is the set

$$KerL=\{f\in D(L):Lf=0\}.$$

Definition 1 *A linear closed operator $\widehat{L}$ in a Hilbert space H is called* maximal, *if $R(\widehat{L})=H$ and $Ker\widehat{L}\neq\{0\}$.*

Definition 2 *A linear closed operator L_0 in a Hilbert space H is called* is called, *if $\overline{R(L_0)}\neq H$ and there is a bounded inverse operator L_0^{-1} by $R(L_0)$.*

Definition 3 *A linear closed operator L in a Hilbert space H is called* correct, *if there is a bounded inverse operator L^{-1} defined on all H.*

Definition 4 *Operator L is called* a contraction *of operator L_1, and operator L_1 is called* an extension *of operator L, and briefly write $L\subset L_1$, if*

1) $D(L)\subset D(L_1)$,

2) $Lf=L_1f,\ \forall f\in D(L)$.

Definition 5 *The correct operator L in the Hilbert space H is called* the correct contraction of the maximum operator $\widehat{L}$ *(*the correct extension of the minimum operator L_0*), if $L \subset \widehat{L}$ ($L_0 \subset L$).*

Definition 6 *A correct operator L in a Hilbert space H is called* a boundary-correct extension, *if L is both a correct contraction of the maximum operator $\widehat{L}$ and a correct extension of the minimum operator L_0, i.e. $L_0 \subset L \subset \widehat{L}$.*

Theorem 2 *[21, 22] Let $\widehat{L}$ be a maximal linear operator in a Hilbert space H, $L-$ a known correct narrowing of operator $\widehat{L}$ and $K-$ an arbitrary linear operator bounded in H that satisfies the following condition*

$$R(K) \subset Ker\widehat{L}. \tag{19}$$

Then the operator L_K^{-1}, defined by the formula

$$L_K^{-1} f = L^{-1} f + K f, \ \forall f \in H, \tag{20}$$

is the inverse of some correct narrowing of L_K of the maximal operator $\widehat{L}$, i.e. $L_K \subset \widehat{L}$. Conversely, if L_1 is some correct narrowing of the maximal operator $\widehat{L}$, then there exists a linear operator K_1 bounded in H that satisfies condition (19), such that the equality holds

$$L_1^{-1} f = L^{-1} f + K_1 f, \ \forall f \in H.$$

As a rule, it is difficult to describe the kernel of the maximal operator. Therefore, often the following Theorem 3 is more effective than Theorem 2.

Theorem 3 *[23] Let $\widehat{L}$ be the maximal operator, L_ϕ be the known correct constriction of $\widehat{L}$, and K be the continuous operator acting from H to $D(\widehat{L})$ be the domain of the definition of operator $\widehat{L}$. Then operator L_K^{-1}, defined by the formula*

$$L_K^{-1} f = L_\phi^{-1} f + (E - L_\phi^{-1}\widehat{L}) K f \tag{21}$$

is the inverse of some correct narrowing $\widehat{L}$, i.e. $L_K \subset \widehat{L}$.
Conversely, any correct narrowing of operator $\widehat{L}$ is represented as (21) with some operator K.

In what follows, this theorem will be applied to the polyharmonic operator.

6 Correct boundary value problems for a polyharmonic operator in a multidimensional ball

In this section $\Omega = \{x \in \mathbb{R}^n : |x| < r\}$. On $D(\widehat{L}) = W_2^{2m}(\Omega)$ we define the maximal operator $\widehat{L}$ by the formula

$$\widehat{L} u = \Delta^m u, \ \forall u \in D(\widehat{L}).$$

By definition $R(\widehat{L}) = L_2(\Omega)$, and $Ker\widehat{L}$ is not trivial.

In the previous section, it was proved that the Dirichlet boundary value problem for the polyharmonic equation

$$L_\phi u := \begin{cases} \Delta_x^m u(x) = f(x), \quad x \in \Omega = \{x : |x| < r\ \}, \\ \dfrac{\partial^j u(x)}{\partial n_x^j} = 0, \ \ 0 \le j \le m-1, \ \ x \in \partial\Omega \end{cases}$$

has a unique solution $u(x)$ for any $f \in L_2(\Omega)$, which has an integral representation

$$L_\phi^{-1} f = u(x) = \int_\Omega G_{2m,n}^D(x,y) f(y) dy, \tag{22}$$

where $G_{2m,n}^D(x,y) \equiv G_{2m,n}(x,y)-$ Green's function of the Dirichlet problem from (15).

Note that the zero Dirichlet boundary conditions for a polyharmonic equation are equivalent to the following boundary conditions for the same equation.

Theorem 4 *a) For any $f \in L_2(\Omega)$, the function $u(x)$ given by formula (22) with $m = 2p$ is a solution to the boundary value problem*

$$\Delta_x^m u(x) = f(x), \quad x \in \Omega, \tag{23}$$

$$u(x)\,|_{\partial\Omega} = 0, \ \frac{\partial}{\partial n_x} u(x)\bigg|_{\partial\Omega} = 0, \ \Delta_x u(x)\,|_{\partial\Omega} = 0, \ \frac{\partial}{\partial n_x}\Delta_x u(x)\bigg|_{\partial\Omega} = 0,$$

$$\dots\dots\dots\ \Delta_x^{p-1} u(x)\,\big|_{\partial\Omega} = 0, \ \frac{\partial}{\partial n_x}\Delta_x^{p-1} u(x)\bigg|_{\partial\Omega} = 0. \tag{24}$$

b) For any $f \in L_2(\Omega)$, the function $u(x)$ given by formula (22) with $m = 2p+1$ is a solution to the boundary value problem

$$\Delta_x^m u(x) = f(x), \quad x \in \Omega,$$

$$u(x)\,|_{\partial\Omega} = 0, \ \frac{\partial}{\partial n_x} u(x)\bigg|_{\partial\Omega} = 0, \ \Delta_x u(x)\,|_{\partial\Omega} = 0, \ \frac{\partial}{\partial n_x}\Delta_x u(x)\bigg|_{\partial\Omega} = 0,$$

$$\dots\dots\dots\ \frac{\partial}{\partial n_x}\Delta_x^{p-1} u(x)\bigg|_{\partial\Omega} = 0, \ \Delta_x^p u(x)\,|_{\partial\Omega} = 0. \tag{25}$$

Proof 3 *Let us show that* $\Delta u|_{\partial\Omega} = \dfrac{\partial^2}{\partial n^2} u|_{\partial\Omega} = 0$, *if* $u|_{\partial\Omega} = 0$ *и* $\dfrac{\partial}{\partial n} u|_{\partial\Omega} = 0$.
This fact follows from the following identity

$$\Delta u = \frac{1}{r^{n-1}} \frac{\partial}{\partial r} r^{n-1} \frac{\partial}{\partial r} u + \frac{1}{r^2}\Delta_\theta u, \ x = r\cdot\theta \in \mathbb{R}^n.$$

In the case of the ball Ω, the direction of the outward normal to the boundary $\partial\Omega$ coincides with the direction of the radius of the vector $\overrightarrow{r}$, therefore the derivative with respect to the outward normal at the boundary $\partial\Omega$ coincides with the derivative in the direction of the radius. From here we get

$$\Delta u|_{\partial\Omega} = \frac{\partial^2}{\partial r^2} u|_{\partial\Omega} + \frac{n-1}{r}\frac{\partial}{\partial r} u|_{\partial\Omega} + \frac{1}{r^2}\Delta_\theta u|_{\partial\Omega} = 0,$$

because $\Delta u|_{\partial\Omega} = 0, \ \dfrac{\partial}{\partial n} u|_{\partial\Omega} = 0, \ u|_{\partial\Omega} = 0.$

In this section, based on the representation of the solution (12) of the Dirichlet problem, we present other well-posed boundary value problems for an inhomogeneous polyharmonic equation. For this we apply Theorem 3 to describe correct restrictions of the maximal operator $\widehat{L}$.

Lemma 2 *For any $h \in W_2^{2m}(\Omega)$ fair representation*

$$(E - L_\phi^{-1}\widehat{L})h(x) =$$

$$\sum_{j=0}^{m-1} \int_{\partial\Omega} \left[\frac{\partial}{\partial n_y} \Delta_y^j G_{2m,n}^D(x,y) \cdot \Delta_y^{m-1-j} h(y) - \Delta_y^j G_{2m,n}^D(x,y) \cdot \frac{\partial}{\partial n_y} \Delta_y^{m-1-j} h(y) \right] dS_y.$$

Proof 4 *For this purpose, we introduce into consideration the integral*

$$I(x) = L_\phi^{-1}\widehat{L}h = \int_\Omega G_{2m,n}^D(x,y)\Delta_y^m h(y)dy, \tag{26}$$

where $h(y)$ is sufficiently smooth, for example, from the class $W_2^{2m}(\Omega)$, and the rest is an arbitrary function.

Taking into account the second Green's formula for the polyharmonic equation, the integral (26) can be written in the form

$$I(x) = \int_\Omega h(y)\Delta_y^m G_{2m,n}^D(x,y)dy -$$

$$\sum_{j=0}^{m-1} \int_{\partial\Omega} \left[\frac{\partial}{\partial n_y} \Delta_y^j G_{2m,n}^D(x,y) \cdot \Delta_y^{m-1-j} h(y) - \Delta_y^j G_{2m,n}^D(x,y) \cdot \frac{\partial}{\partial n_y} \Delta_y^{m-1-j} h(y) \right] dS_y =$$

$$h(x) - \sum_{j=0}^{m-1} \int_{\partial\Omega} \left[\frac{\partial}{\partial n_y} \Delta_y^j G_{2m,n}^D(x,y) \cdot \Delta_y^{m-1-j} h(y) - \Delta_y^j G_{2m,n}^D(x,y) \cdot \frac{\partial}{\partial n_y} \Delta_y^{m-1-j} h(y) \right] dS_y.$$

From here, on the one hand, we get

$$h - L_\phi^{-1}\widehat{L}h =$$

$$\sum_{j=0}^{m-1} \int_{\partial\Omega} \left[\frac{\partial}{\partial n_y} \Delta_y^j G_{2m,n}^D(x,y) \cdot \Delta_y^{m-1-j} h(y) - \Delta_y^j G_{2m,n}^D(x,y) \cdot \frac{\partial}{\partial n_y} \Delta_y^{m-1-j} h(y) \right] dS_y.$$

Lemma 2 is proved.

Lemma 3 *The Green function of the Dirichlet problem $G_{2m,n}(x,y)$ on the boundary $\partial\Omega$ has the following properties*

$$\Delta_y^j G_{2m,n}^D(x,y)|_{x\in\partial\Omega} = 0,\ j = 0, 1, ..., m-1,\ \forall y \in \partial\Omega, \tag{27.1}$$

$$\frac{\partial}{\partial n_y} \Delta_y^j G_{2m,n}^D(x,y)|_{x\in\partial\Omega} = 0,\ j = 0, 1, ..., m-2,\ \forall y \in \partial\Omega, \tag{27.2}$$

$$\frac{\partial}{\partial n_y} \Delta_y^{m-1} G_{2m,n}^D(x,y)|_{x\in\partial\Omega} = \delta(x-y)|_{x\in\partial\Omega},\ \forall y \in \partial\Omega \tag{27.3}$$

and $(28.i) - (30.i)$ with $m = 2p$; $(28.i) - (31.i)$ with $m = 2p+1$, $i = 1, 2, 3$.

Proof 5 *It follows from representation (26) that* $I(x)$ *satisfies the Dirichlet boundary conditions. Therefore, for* $x \in \partial\Omega$*, taking into account (24) or (25), we obtain the relation*

$$0 \equiv I(x)|_{x\in\partial\Omega} = h(x)|_{x\in\partial\Omega} - \sum_{j=0}^{m-1} \int_{\partial\Omega} \left[\frac{\partial}{\partial n_y}\Delta_y^j G_{2m,n}^D(x,y)|_{x\in\partial\Omega} \cdot \Delta_y^{m-1-j} h(y) - \right.$$

$$\left. \Delta_y^j G_{2m,n}^D(x,y)|_{x\in\partial\Omega} \cdot \frac{\partial}{\partial n_y}\Delta_y^{m-1-j} h(y)\right] dS_y,$$

i.e. $I(x) \equiv 0$, $\forall x \in \partial\Omega$*, which is valid for all sufficiently smooth* $h(x)$*. Since the function* $h(y)$ *is arbitrary, we conclude from this that the relations* $(27.i)$, $i = 1, 2, 3$.

Using the second boundary condition of Dirichlet, arguing similarly, we obtain the relation

$$0 \equiv \frac{\partial}{\partial n_x} I(x)|_{x\in\partial\Omega} = \frac{\partial}{\partial n_x} h(x)|_{x\in\partial\Omega} - \sum_{j=0}^{m-1} \int_{\partial\Omega} \left[\frac{\partial}{\partial n_x}\frac{\partial}{\partial n_y}\Delta_y^j G_{2m,n}^D(x,y)|_{x\in\partial\Omega} \cdot \Delta_y^{m-1-j} h(y) - \right.$$

$$\left. \frac{\partial}{\partial n_x}\Delta_y^j G_{2m,n}^D(x,y)|_{x\in\partial\Omega} \cdot \frac{\partial}{\partial n_y}\Delta_y^{m-1-j} h(y)\right] dS_y,$$

for an arbitrary sufficiently smooth function $h(y)$*. Since the values* $\{\Delta_y^{m-1-j} h(y), \frac{\partial}{\partial n_y}\Delta_y^{m-1-j} h(y), j = 0, 1, ..., m-1\}$ *are linearly independent from each other, therefore*

$$\frac{\partial}{\partial n_x}\Delta_y^j G_{2m,n}^D(x,y)|_{x\in\partial\Omega} = 0,\ j = 0, 1, ..., m-2,\ \forall y \in \partial\Omega, \tag{28.1}$$

$$\frac{\partial}{\partial n_x}\frac{\partial}{\partial n_y}\Delta_y^j G_{2m,n}^D(x,y)|_{x\in\partial\Omega} = 0,\ j = 0, 1, ..., m-1,\ \forall y \in \partial\Omega, \tag{28.2}$$

$$\frac{\partial}{\partial n_x}\Delta_y^{m-1} G_{2m,n}^D(x,y)|_{x\in\partial\Omega} = \delta(x-y)|_{x\in\partial\Omega},\ \forall y \in \partial\Omega. \tag{28.3}$$

Taking into account the above statement, the third Dirichlet boundary condition allows us to write out the relation

$$0 \equiv \frac{\partial^2}{\partial n_x^2} I(x)\Big|_{x\in\partial\Omega} = \Delta_x I(x)\Big|_{x\in\partial\Omega} = \Delta_x h(x)\Big|_{x\in\partial\Omega} -$$

$$\sum_{j=0}^{m-1} \int_{\partial\Omega} \left[\Delta_x \frac{\partial}{\partial n_y}\Delta_y^j G_{2m,n}^D(x,y)|_{x\in\partial\Omega} \cdot \Delta_y^{m-1-j} h(y) - \right.$$

$$\left. \Delta_x \Delta_y^j G_{2m,n}^D(x,y)|_{x\in\partial\Omega} \cdot \frac{\partial}{\partial n_y}\Delta_y^{m-1-j} h(y)\right] dS_y,$$

for an arbitrary sufficiently smooth function $h(y)$*. Therefore*

$$\Delta_x \frac{\partial}{\partial n_y}\Delta_y^j G_{2m,n}^D(x,y)|_{x\in\partial\Omega} = 0,\ j = 0, 1, ..., m-1,\ j \neq m-2,\ \forall y \in \partial\Omega, \tag{29.1}$$

$$\Delta_x \Delta_y^j G_{2m,n}^D(x,y)|_{x\in\partial\Omega} = 0,\ j=0,1,...,m-1,\ \forall y \in \partial\Omega, \tag{29.2}$$

$$\Delta_x \frac{\partial}{\partial n_y} \Delta_y^{m-2} G_{2m,n}^D(x,y)|_{x\in\partial\Omega} = \delta(x-y)|_{x\in\partial\Omega},\ \forall y \in \partial\Omega. \tag{29.3}$$

Similarly, we write out other conditions that the Green's function $G_{2m,n}^D(x,y)$ on the boundary $\partial\Omega$ satisfies

for $m = 2p$

$$0 \equiv \frac{\partial^{2p-1}}{\partial n_x^{2p-1}} I(x)\Big|_{x\in\partial\Omega} = \frac{\partial}{\partial n_x}\Delta_x^{p-1} I(x)\Big|_{x\in\partial\Omega} = \frac{\partial}{\partial n_x}\Delta_x^{p-1} h(x)\Big|_{x\in\partial\Omega} -$$

$$\sum_{j=0}^{2p-1}\int_{\partial\Omega}\left[\frac{\partial}{\partial n_x}\Delta_x^{p-1}\frac{\partial}{\partial n_y}\Delta_y^j G_{2m,n}^D(x,y)|_{x\in\partial\Omega}\cdot \Delta_y^{2p-1-j}h(y) - \right.$$

$$\left.\frac{\partial}{\partial n_x}\Delta_x^{p-1}\Delta_y^j G_{2m,n}^D(x,y)|_{x\in\partial\Omega}\cdot \frac{\partial}{\partial n_y}\Delta_y^{2p-1-j}h(y)\right] dS_y,$$

for $m = 2p+1$

$$0 \equiv \frac{\partial^{2p}}{\partial n_x^{2p}} I(x)\Big|_{x\in\partial\Omega} = \Delta_x^p I(x)\Big|_{x\in\partial\Omega} = \Delta_x^p h(x)\Big|_{x\in\partial\Omega} -$$

$$\sum_{j=0}^{2p}\int_{\partial\Omega}\left[\Delta_x^p \frac{\partial}{\partial n_y}\Delta_y^j G_{2m,n}^D(x,y)|_{x\in\partial\Omega}\cdot \Delta_y^{2p-j}h(y) - \right.$$

$$\left.\Delta_x^p \Delta_y^j G_{2m,n}^D(x,y)|_{x\in\partial\Omega}\cdot \frac{\partial}{\partial n_y}\Delta_y^{2p-j}h(y)\right] dS_y,$$

for an arbitrary sufficiently smooth function $h(y)$. Therefore, for $m = 2p$

$$\frac{\partial}{\partial n_x}\Delta_x^{p-1}\frac{\partial}{\partial n_y}\Delta_y^j G_{2m,n}^D(x,y)|_{x\in\partial\Omega} = 0,\ j=0,1,...,2p-1,\ \forall y\in\partial\Omega, \tag{30.1}$$

$$\frac{\partial}{\partial n_x}\Delta_x^{p-1}\Delta_y^j G_{2m,n}^D(x,y)|_{x\in\partial\Omega} = 0,\ j=0,1,...,2p-1,\ j\neq p, \forall y\in\partial\Omega, \tag{30.2}$$

$$\frac{\partial}{\partial n_x}\Delta_x^{p-1}\Delta_y^p G_{2m,n}^D(x,y)|_{x\in\partial\Omega} = -\delta(x-y)|_{x\in\partial\Omega},\ \forall y\in\partial\Omega; \tag{30.3}$$

for $m = 2p+1$

$$\Delta_x^p \frac{\partial}{\partial n_y}\Delta_y^j G_{2m,n}^D(x,y)|_{x\in\partial\Omega} = 0,\ j=0,1,...,2p,\ j\neq p\,\forall y\in\partial\Omega, \tag{31.1}$$

$$\Delta_x^p \Delta_y^j G_{2m,n}^D(x,y)|_{x\in\partial\Omega} = 0,\ j=0,1,...,2p,\ \forall y\in\partial\Omega, \tag{31.2}$$

$$\Delta_x^p \Delta_y^p G_{2m,n}^D(x,y)|_{x\in\partial\Omega} = \delta(x-y)|_{x\in\partial\Omega},\ \forall y\in\partial\Omega. \tag{31.3}$$

Lemma 3 is proved.

Now we can describe the domain of the maximum operator $\widehat{L}$ in terms of the Green's function $G_{2m,n}$.

Lemma 4 *The domain of the maximum operator $\widehat{L}$ has the representation*

$$D(\widehat{L})=\{w:\, w(x)=\int_{\Omega}G_{2m,n}(x,y)f(y)dy+\sum_{j=0}^{m-1}\int_{\partial\Omega}\left[\frac{\partial}{\partial n_y}\Delta_y^{j}G_{2m,n}(x,y)\cdot\Delta_y^{m-1-j}h(y)-\right.$$

$$\left.-\Delta_y^{j}G_{2m,n}(x,y)\cdot\frac{\partial}{\partial n_y}\Delta_y^{m-1-j}h(y)\right]dS_y,\ \forall f\in L_2(\Omega),\ \forall h\in W_2^{2m}(\Omega)\}. \tag{32}$$

In particular, if

$$\Delta_y^{m-1-j}h(y)|_{y\in\partial\Omega}=0,\ \frac{\partial}{\partial n_y}\Delta_y^{m-1-j}h(y)|_{y\in\partial\Omega}=0,\ j=0,...,m-1,$$

then we get $D(L_\phi)$ domain of the operator L_ϕ.

Now the question arises: how to describe the domains of definition of other possible correct restrictions of the maximal operator $\widehat{L}$?

Let K be an operator that puts each function $f(x)\in L_2(\Omega)$ in there is a unique function $h(x)\in W_2^{2m}(\Omega)$, such that $\|Kf\|_{L_2(\Omega)}\le C\|f\|_{L_2(\Omega)}$. Using the chosen operator K, construct the set

$$D_K=\{\,w(x)\in D(\widehat{L}):\ h=Kf\}.$$

On the set D_K we define the operator

$$\widehat{L}\Big|_{D_K}=L_K.$$

It follows from Theorem 3 that L_K is a correct restriction of the maximal operator $\widehat{L}$. In conclusion, we give another description of the operator L_K in terms of boundary conditions.

Theorem 5 *Let K be an arbitrary continuous operator acting from $L_2(\Omega)$ to $D(\widehat{L})$. Then the inhomogeneous operator equation $L_Kw=f$ is equivalent to the following boundary value problem*

a) for $m=2p$

$$\Delta_x^m w(x)=f(x),\quad x\in\Omega,$$

$$w\,|_{\partial\Omega}=K(\Delta_x^m w\,)|_{\partial\Omega},\ \frac{\partial}{\partial n_x}w\Big|_{\partial\Omega}=\frac{\partial}{\partial n_x}K(\Delta_x^m w\,)\Big|_{\partial\Omega},\ldots,$$

$$\Delta_x^{p-1}w\,|_{\partial\Omega}=\Delta_x^{p-1}K(\Delta_x^m w\,)\big|_{\partial\Omega},\ \frac{\partial}{\partial n_x}\Delta_x^{p-1}w\Big|_{\partial\Omega}=\frac{\partial}{\partial n_x}\Delta_x^{p-1}K(\Delta_x^m w\,)\Big|_{\partial\Omega};$$

b) for $m=2p+1$

$$w\,|_{\partial\Omega}=K(\Delta_x^m w\,)|_{\partial\Omega},\ \frac{\partial}{\partial n_x}w\Big|_{\partial\Omega}=\frac{\partial}{\partial n_x}K(\Delta_x^m w\,)\Big|_{\partial\Omega},\ldots,$$

$$\frac{\partial}{\partial n_x}\Delta_x^{p-1}w\Big|_{\partial\Omega}=\frac{\partial}{\partial n_x}\Delta_x^{p-1}(K\Delta_x^m w\,)\Big|_{\partial\Omega},\ \Delta_x^p w\,|_{\partial\Omega}=\Delta_x^p(K\Delta_x^m w\,)|_{\partial\Omega}.$$

Other applications of M.Otelbaev's results in various branches of the theory of differential equations can be found in the works [26–29].

7 Conclusion

The studies carried out in this article are of significant importance in the theory of boundary value problems of linear and nonlinear partial differential equations, spectral theory, and the theory of numerical methods for approximate solutions of certain classes of boundary value problems for differential equations.

8 Acknowledgments

This work was done by grant AP 09559378 of the Ministry of Education and Science of the Republic of Kazakhstan.

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