1-бөлім **Раздел 1** **Section 1**

Математика **Математика** **Mathematics**

IRSTI 27.31.15 DOI: https://doi.org/10.26577/JMMCS.2021.v111.i3.01

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ON STABILIZATION PROBLEM FOR A LOADED HEAT EQUATION: THE TWO-DIMENSIONAL CASE

One of the important properties that characterize the behavior of solutions of boundary value problems for differential equations is stabilization, which has a direct relationship with the problems of controllability. The problems of solvability of stabilization problems of two-dimensional loaded equations of parabolic type with the help of feedback control given on the boundary of the region are investigated in the article. These equations have numerous applications in the study of inverse problems for differential equations. The problem consists in the choice of boundary conditions (controls), so that the solution of the boundary value problem tends to a given stationary solution at a certain speed at $t \to \infty$. This requires that the control is feedback, i.e. that it responds to unintended fluctuations in the system, suppressing the results of their impact on the stabilized solution. The spectral properties of the loaded two-dimensional Laplace operator, which are used to solve the initial stabilization problem, are also studied. The paper presents an algorithm for solving the stabilization problem, which consists of constructively implemented stages. The idea of reducing the stabilization problem for a parabolic equation by means of boundary controls to the solution of an auxiliary boundary value problem in the extended domain of independent variables belongs to A.V. Fursikov. At the same time, recently, the so-called loaded differential equations are actively used in problems of mathematical modeling and control of nonlocal dynamical systems.

Key words: boundary stabilization, heat equation, spectrum, eigenfunction, loaded Laplace operator.

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Жүктелген жылуөткізгіштік теңдеуі үшін стабилизациялау есебі: екі өлшемді жағдай

Дифференциалдық теңдеулер үшін шекаралық есептер шешімдерінің табиғатын сипаттайтын маңызды қасиеттердің бірі – стабилизация, бұл басқарылатын есептермен тікелей байланысты. Мақалада екі өлшемді жүктелген параболалық теңдеулер үшін стабилизация есептерінің шешімділігі мәселесі облыс шекарасында орнатылған кері байланысты басқаруды қолдана отырып зерттелген. Бұл теңдеулердің дифференциалдық теңдеулерге арналған кері есептерді зерттеуде көптеген қосымшалары бар. Шектік есептерді белгілі жылдамдықпен шешу берілген стационар шешімге $t \to \infty$ кезінде жуықтайтындай етіп, шекаралық шарттарды (басқаруларды) таңдау болып табылады. Бұл басқарудың кері байланысты болуын, яғни жүйенің күтпеген өзгерістеріне жауап беруі мен олардың стабилизацияланған шешімге әсерінің нәтижелерін болдырмауын талап етеді. Жүктелген екі өлшемді Лаплас операторының спектрлік қасиеттері де зерттелген, олар бастапқы стабилизация мәселесін шешу үшін қолданылады. Мақалада конструктивті түрде іске асырылған кезеңдерден тұратын стабилизация есебін шешудің алгоритмі келтірілген.

Параболалық теңдеу үшін стабилизация есебін тәуелсіз айнымалылардың кеңейтілген облысында көмекші шекаралық есепті шешуге шекаралық басқаруды қолдану арқылы шешу идеясы А.В. Фурсиковке тиесілі. Сонымен қатар, кейінгі кездерде жүктелген дифференциалдық теңдеулер математикалық модельдеу және локальды емес динамикалық жүйелерді басқару есептерінде белсенді қолданылады.

Түйін сөздер: шекаралық стабилизация, жылуөткізгіштік теңдеуі, спектр, меншікті функциялар, жүктелген Лаплас операторы.

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Задача стабилизации для нагруженного уравнения теплопроводности: двумерный случай

Одним из важных свойств, характеризующих поведение решений краевых задач для дифференциальных уравнений, является стабилизация, имеющая прямое отношение к задачам управляемости. В статье исследуются вопросы разрешимости задач стабилизации двумерных нагруженных уравнений параболического типа с помощью управления с обратной связью, заданного на границе области. Эти уравнения имеют многочисленные приложения при исследовании обратных задач для дифференциальных уравнений. Задача состоит в выборе граничных условий (управлений) так, чтобы решение краевой задачи с определенной скоростью приближалось к заданному стационарному решению при $t \to \infty$. Для этого требуется, чтобы управление было обратной связью, то есть чтобы оно реагировало на непредвиденные возмущения в системе, подавляя результаты их воздействия на стабилизированное решение. Также исследуются спектральные свойства нагруженного двумерного оператора Лапласа, которые используются для решения начальной задачи стабилизации. В статье представлен алгоритм решения задачи стабилизации, состоящий из конструктивно реализованных этапов. Идея сведения задачи стабилизации параболического уравнения с помощью граничных управлений к решению вспомогательной краевой задачи в расширенной области независимых переменных принадлежит А.В. Фурсикову. Кроме того, в последнее время так называемые нагруженные дифференциальные уравнения активно используются в задачах математического моделирования и управления нелокальными динамическими системами.

Ключевые слова: граничная стабилизация, уравнение теплопроводности, спектр, собственные функций, нагруженный оператор Лапласа.

1 Introduction

The idea of reducing the stabilization problem for a parabolic equation by means of boundary controls to the solution of an auxiliary boundary value problem in the extended domain of independent variables belongs to A.V. Fursikov [1]. In [1] the stabilization problem from the boundary $\partial\Omega$ for a parabolic equation given in a bounded domain $\Omega \subset R^n$, consists in choosing a boundary condition (a control) such that the solution of the resulting mixed boundary value problem tends as $t \to \infty$ to a given steady-state solution at a prescribed rate $\exp(-\sigma t)$. It was proposed in his work [1] and developed further in the works [2–4]. Note that in the works [1–4] stabilization problems for differential equations without load were considered, i.e. there was no phenomenon of nonlocality. Last thing complicates the implementation of the idea proposed by A.V. Fursikov, especially in parts of constructing eigenfunctions and associated functions for loaded differential operators. At the same time, recently, the so-called loaded differential equations [5–10] are actively used in problems of mathematical modelling and control of nonlocal dynamical systems. We have previously studied stabilization problems for a loaded one-dimensional heat equation [11], [12]. In this paper, we investigate stabilization problems for the loaded two-dimensional thermal conductivity equation.

Our work consists of eight sections. Section 1 is an introduction. Section 2 discusses the statement on the problem. In Section 3, an auxiliary boundary value problem is introduced. In Section 4, the spectral problem for the loaded two-dimensional Laplace operator is investigated. In Section 5 gives main results. In Sections 6 and 7, a biorthogonal system of functions is constructed on the base of eigenfunctions and associated functions. In Section 8, the algorithm for solving the above stabilization problem is described. Finally, Section 9 gives conclusions and discusses possible applications of the results.

2 Statement of the problem

Let $\Omega = \{x, y : -\pi/2 < x, y < \pi/2\}$ be a domain with a boundary $\partial\Omega$. In the cylinder $Q = \Omega \times \{t > 0\}$ with lateral surface $\Sigma = \partial\Omega \times \{t > 0\}$ we consider the boundary value problem for the loaded heat equation

$$u_t - \Delta u + \alpha u(0, y, t) + \beta u(x, 0, t) = 0, \ \{x, y, t\} \in Q, \tag{1}$$

$$u(x, y, 0) = u_0(x, y), \ \{x, y\} \in \Omega, \tag{2}$$

$$u(x, y, t) = p(x, y, t), \ \{x, y, t\} \in \Sigma, \tag{3}$$

where α, $\beta \in \mathbb{C}$ are given (in general case are complex) bounded constants, $u_0(x, y)$ is given function. The aim is to find a function $p(x, y, t)$ such that a solution of the boundary value problem (1)–(3) satisfies the inequality

$$\|u(x, y, t)\|_{L_2(\Omega)} \le C_0 e^{-\sigma t}, \ \sigma > 0, \ t > 0. \tag{4}$$

Note that here σ is a given constant and C_0 is an arbitrary bounded constant.

Remark 1. *In section 8 it will be shown that the solution of the stabilization problem (1)–(4) significantly depends on the values of the coefficients α and β, including the sign of their real parts.*

Equation (1) is called a loaded equation [5,6]. We note that problem (1)–(4) with a single load point was studied in [12].

3 Auxiliary boundary value problem (BVP)

Let $\Omega_1 = \{x, y : -\pi < x, y < \pi\}$ and $Q_1 = \Omega_1 \times \{t > 0\}$.

$$z_t - \Delta z + \alpha z(0, y, t) + \beta z(x, 0, t) = 0, \ \{x, y, t\} \in Q_1, \tag{5}$$

$$z(x, y, 0) = z_0(x, y), \ \{x, y\} \in \Omega_1, \tag{6}$$

$$\frac{\partial^j z(-\pi, y, t)}{\partial x^j} = \frac{\partial^j z(\pi, y, t)}{\partial x^j}, \ \{y, t\} \in (-\pi, \pi) \times \{t > 0\},$$

$$\frac{\partial^j z(x, -\pi, t)}{\partial y^j} = \frac{\partial^j z(x, \pi, t)}{\partial y^j}, \ \{x, t\} \in (-\pi, \pi) \times \{t > 0\}, \ j = 0, 1. \tag{7}$$

The problem is to find an initial function $z_0(x,y)$ such that a solution of the BVP (5)–(7) satisfies the inequality

$$\|z(x,y,t)\|_{L_2(\Omega_1)} \le C_0 e^{-\sigma t},\ \sigma > 0,\ t > 0. \tag{8}$$

We recall, as we indicated above, that here σ is a given constant and C_0 is an arbitrary bounded constant.

We will define the function $z_0(x,y)$ as a continuation of the function $u_0(x,y)$, which was given in the original domain Ω. Thus in the auxiliary boundary value problem (5)–(7) it is needed to find the function $z_0(x,y)$ on the square Ω_1, so that the requirement (8) is satisfied for a solution $z(x,y,t)$ of the problem (5)–(7). In this case the condition (4) holds for restriction $u(x,y,t)$ of $z(x,y,t)$ too and a required boundary control $p(x,y,t)$, $\{x,y\} \in \Sigma$ is defined as trace of function $z(x,y,t)$ for $\{x,y,t\} \in \Sigma$.

4 Spectral problem for the loaded twodimensional Laplace operator

Let us search a solution of the problem (5)–(7) in the form

$$z(x,y,t) = \sum_{k,l\in\mathbb{Z}} Z_{kl}(t)\psi_{kl}(x,y), \tag{9}$$

where $\{\psi_{kl}(x,y), k,l \in \mathbb{Z}\}$ is a biorthogonal basis of the space $L_2(\Omega_1)$ and $\mathbb{Z} = \{0,\pm1,\pm2,...\}$. The following two spectral problems are considered for the construction of the biorthogonal basis $\{\psi_{kl}(x,y), k,l \in \mathbb{Z}\}$ in the domain $\Omega_1 = \{x,y : -\pi < x < \pi, -\pi < y < \pi\}$:

$$\begin{cases} -\Delta\varphi(x,y) + \alpha\varphi(0,y) = \lambda\varphi(x,y), \\ \dfrac{\partial^j\varphi(-\pi,y)}{\partial x^j} = \dfrac{\partial^j\varphi(\pi,y)}{\partial x^j},\ \dfrac{\partial^j\varphi(x,-\pi)}{\partial y^j} = \dfrac{\partial^j\varphi(x,\pi)}{\partial y^j}, \end{cases} \tag{10}$$

$$\begin{cases} -\Delta\varphi(x,y) + \alpha\varphi(0,y) + \beta\varphi(x,0) = \lambda\varphi(x,y), \\ \dfrac{\partial^j\varphi(-\pi,y)}{\partial x^j} = \dfrac{\partial^j\varphi(\pi,y)}{\partial x^j},\ \dfrac{\partial^j\varphi(x,-\pi)}{\partial y^j} = \dfrac{\partial^j\varphi(x,\pi)}{\partial y^j}, \end{cases} \tag{11}$$

where $j = 0,1$, Δ is the Laplace operator, α, $\beta \in \mathbb{C}$ are given complex numbers, $\lambda \in \mathbb{C}$ is a spectral parameter. A one-dimensional analogue of the problems (10) and (11) is studied in [12].

5 Main results

Let $\mathbb{Z} = \{0,\pm1,\pm2,\dots\}$. The following propositions are valid [13].

Theorem 1. *(a). Let $\forall\, l \in \mathbb{Z} : \alpha \ne l^2$. Then a system of eigenfunctions and eigenvalues of the problem (10) is defined in the form:*

$$\left\{\varphi_{kl}(x,y) = \left(e^{ilx} + \frac{\alpha}{l^2-\alpha}\right)e^{iky},\ \lambda_{kl} = l^2 + k^2,\ l \in \mathbb{Z}' \equiv \mathbb{Z}\backslash\{0\};\right.$$

$$\left.\varphi_{k0}(x,y)=e^{iky},\ \lambda_{k0}=\alpha+k^2\ (l=0),\ k\in\mathbb{Z}\right\}. \tag{12}$$

(b). Let $\exists\ l_0\in\mathbb{Z}:\ \alpha=l_0^2$. Then a system of eigenfunctions, associated functions (marked with $\sim$) and eigenvalues of the problem (10) is defined in the form:

$$\left\{\varphi_{kl}(x,y)=\left(e^{ilx}+\frac{\alpha}{l^2-\alpha}\right)e^{iky},\ \lambda_{kl}=l^2+k^2,\ l\in\mathbb{Z}_1'\equiv\mathbb{Z}'\backslash\{\pm l_0\};\right.$$

$$\left.\varphi_{kl_0}(x,y)=e^{iky},\ \tilde{\varphi}_{kl_0}^{\pm}(x,y)=e^{\pm il_0x+iky},\ \lambda_{kl_0}=\alpha+k^2\ (\alpha=l_0^2),\ k\in\mathbb{Z}\right\}. \tag{13}$$

Theorem 2. *(a). Let $\forall\ k,l\in\mathbb{Z}:\ \beta\neq k^2,\ \alpha\neq l^2$. Then a system of eigenfunctions and eigenvalues for the problem (11) is defined in the form:*

$$\left\{\varphi_{kl}(x,y)=\left(e^{ilx}+\frac{\alpha}{l^2-\alpha}\right)\left(e^{iky}+\frac{\beta}{k^2-\beta}\right),\lambda_{kl}=k^2+l^2,\ k,\ l\in\mathbb{Z}';\right.$$

$$\varphi_{0l}(x,y)=e^{ilx}+\frac{\alpha}{l^2-\alpha},\ \lambda_{0l}=\beta+l^2,\ l\in\mathbb{Z}';$$

$$\left.\varphi_{k0}(x,y)=e^{iky}+\frac{\beta}{k^2-\beta},\ \lambda_{k0}=k^2+\alpha,\ k\in\mathbb{Z}';\ \varphi_{00}(x,y)=1,\ \lambda_{00}=\alpha+\beta\right\}. \tag{14}$$

(b). Let $\forall\ k\in\mathbb{Z}:\ \beta\neq k^2$ and $\exists\ l_0\in\mathbb{Z}:\ \alpha=l_0^2$. Then a system of eigenfunctions, associated functions (marked with $\sim$) and eigenvalues for the problem (11) is defined in the form (where $\mathbb{Z}_1'=\mathbb{Z}'\backslash\{\pm l_0\}$):

$$\left\{\varphi_{kl}(x,y)=\left(e^{ilx}+\frac{\alpha}{l^2-\alpha}\right)\left(e^{iky}+\frac{\beta}{k^2-\beta}\right),\lambda_{kl}=k^2+l^2,\ k\in\mathbb{Z}',\ l\in\mathbb{Z}_1';\right.$$

$$\varphi_{kl_0}(x,y)=e^{iky}+\frac{\beta}{k^2-\beta},\ \tilde{\varphi}_{kl_0}^{\pm}(x,y)=e^{\pm il_0x}\left(e^{iky}+\frac{\beta}{k^2-\beta}\right),$$

$$\left.\lambda_{kl_0}=k^2+\alpha,\ \alpha=l_0^2,\ k\in\mathbb{Z}';\ \varphi_{0l_0}(x,y)=1,\ \tilde{\varphi}_{0l_0}^{\pm}(x,y)=e^{\pm il_0x},\ \lambda_{0l_0}=\alpha+\beta\right\}. \tag{15}$$

(c). Let $\forall\ l\in\mathbb{Z}:\ \alpha\neq l^2$ and $\exists\ k_0\in\mathbb{Z}:\ \beta=k_0^2$. Then a system of eigenfunctions, associated (marked with $\sim$) functions and eigenvalues for the problem (11) is defined in the form (where $\mathbb{Z}_2'=\mathbb{Z}'\backslash\{\pm k_0\}$):

$$\left\{\varphi_{kl}(x,y)=\left(e^{ilx}+\frac{\alpha}{l^2-\alpha}\right)\left(e^{iky}+\frac{\beta}{k^2-\beta}\right),\ \lambda_{kl}=k^2+l^2,\ k\in\mathbb{Z}_2',\ l\in\mathbb{Z}';\right.$$

$$\varphi_{k_0l}(x,y)=e^{ilx}+\frac{\alpha}{l^2-\alpha},\ \tilde{\varphi}_{k_0l}^{\pm}(x,y)=e^{\pm ik_0y}\left(e^{ilx}+\frac{\alpha}{l^2-\alpha}\right),$$

$$\left.\lambda_{k_0 l} = \beta + l^2,\ \beta = k_0^2,\ l \in \mathbb{Z}';\ \varphi_{k_0 0}(x,y) = 1,\ \tilde{\varphi}_{k_0 0}(x,y) = e^{\pm i k_0 y},\ \lambda_{k_0 0} = \alpha + \beta \right\}. \quad (16)$$

(d). Let $\exists\ k_0,\ l_0 \in \mathbb{Z}:\ \beta = k_0^2,\ \alpha = l_0^2$. Then a system of eigenfunctions, associated functions (marked with $\sim$) and eigenvalues for the problem (11) is defined in the form:

$$\left\{\varphi_{kl}(x,y) = \left(e^{ilx} + \frac{\alpha}{l^2-\alpha}\right)\left(e^{iky} + \frac{\beta}{k^2-\beta}\right),\right.$$

$$\lambda_{kl} = k^2 + l^2,\ k \in \mathbb{Z}_2',\ l \in \mathbb{Z}_1';\ \varphi_{k_0 l}(x,y) = e^{ilx} + \frac{\alpha}{l^2-\alpha},$$

$$\tilde{\varphi}_{k_0 l}^{\pm}(x,y) = e^{\pm i k_0 y}\left(e^{ilx} + \frac{\alpha}{l^2-\alpha}\right),\ \lambda_{k_0 l} = \beta + l^2,\ \beta = k_0^2,\ l \in \mathbb{Z}_1';$$

$$\varphi_{k l_0}(x,y) = e^{iky} + \frac{\beta}{k^2-\beta},\ \alpha + k^2,\ k \in \mathbb{Z}_2';$$

$$\varphi_{k_0 l_0}(x,y) = 1,\ \tilde{\varphi}_{k_0 l_0}(x,y) = e^{\pm i k_0 y},\ \lambda_{k_0 l_0} = \alpha + \beta;$$

$$\tilde{\varphi}_{k l_0}(x,y) = e^{\pm i l_0 x}\left(e^{iky} + \frac{\beta}{k^2-\beta}\right),\ \alpha + k^2,\ k \in \mathbb{Z}_2';$$

$$\left.\tilde{\varphi}_{k_0 l_0}(x,y) = e^{\pm i l_0 x},\ \tilde{\varphi}_{k_0 l_0}(x,y) = e^{\pm i l_0 x \pm i k_0 y},\ \lambda_{k_0 l_0} = \alpha + \beta\right\}. \quad (17)$$

6 Construction of biorthogonal system of functions for the cases of Theorem 1

Let us find a biorthogonal sequences for (12) (case (a) from Theorem 1) and for (13) (case (b) of Theorem 1) [13].

Theorem 3. *(a). Let $\forall\ l \in \mathbb{Z}: \alpha \neq l^2$. Then a biorthogonal sequence for the basis (12) (case (a) of Theorem 1) is*

$$\{\psi_{kl}(x,y),\ k,l \in \mathbb{Z}\} = \left\{e^{i(lx+ky)},\ -\frac{1}{2\pi}\sum_{n\in\mathbb{Z}} \frac{\alpha}{n^2-\alpha}\cdot e^{i(nx+ky)},\ l \in \mathbb{Z}',\ k \in \mathbb{Z}\right\}, \quad (18)$$

which defines an biorthogonal basis in $L_2(\Omega_1)$.

(b). Let $\exists\ l_0 \in \mathbb{Z}:\ \alpha = l_0^2$. Then a biorthogonal sequence for the basis (13) (case (b) of Theorem 1) is

$$\{\psi_{kl}(x),\ k,\ l \in \mathbb{Z}\} = \left\{e^{i(lx+ky)},\ -\frac{1}{2\pi}\sum_{n\in\mathbb{Z}\setminus\{\pm l_0\}} \frac{\alpha}{n^2-\alpha}\cdot e^{i(nx+ky)},\ k \in \mathbb{Z}, l \in \mathbb{Z}'\right\}, \quad (19)$$

which defines an biorthogonal basis in $L_2(\Omega_1)$.

Where we applying the Paley-Wieners theorem ([14], p.206–207).

7 Construction of biorthogonal systems for the cases of Theorem 2

Let us find a biorthogonal sequences for (14), (15), (16) and (17) (cases (a), (b), (c) and (d) from Theorem 2, respectively) [13].

Theorem 4. *(a). Let $\forall\ k, l \in \mathbb{Z}:\ \beta \neq k^2,\ \alpha \neq l^2$. Then a biorthogonal sequence for the basis (14) (case (a) from Theorem 2) is*

$$\{\psi_{kl}(x,y),\ k,l \in \mathbb{Z}\} = \Bigg\{ e^{i(lx+ky)},\ -\frac{1}{2\pi}\sum_{n\in\mathbb{Z}} \frac{\beta}{n^2-\beta}\cdot e^{i(ny+lx)},$$

$$-\frac{1}{2\pi}\sum_{n\in\mathbb{Z}} \frac{\alpha}{n^2-\alpha}\cdot e^{i(nx+ky)},\ k,\ l \in \mathbb{Z}',$$

$$\frac{1}{4\pi^2}\sum_{m,n\in\mathbb{Z}} \frac{\alpha\beta}{(n^2-\alpha)(m^2-\beta)}\cdot e^{i(nx+my)} \Bigg\}, \tag{20}$$

which defines a biorthogonal basis in $L_2(\Omega_1)$.

(b). Let $\forall\ k \in \mathbb{Z}:\ \beta \neq k^2$ and $\exists\ l_0 \in \mathbb{Z}:\ \alpha = l_0^2$. Then a biorthogonal sequence for the basis (15) (case (b) from Theorem 2) is

$$\{\psi_{kl}(x,y),\ k,\ l \in \mathbb{Z}\} = \Bigg\{ e^{i(lx+ky)},\ -\frac{1}{2\pi}\sum_{n\in\mathbb{Z}} \frac{\beta}{n^2-\beta}\cdot e^{i(ny+lx)},$$

$$-\frac{1}{2\pi}\sum_{n\in\mathbb{Z}\setminus\{\pm l_0\}} \frac{\alpha}{n^2-\alpha}\cdot e^{i(nx+ky)},\ k,\ l \in \mathbb{Z}',$$

$$\frac{1}{4\pi^2}\sum_{n\in\mathbb{Z},\, m\in\mathbb{Z}\setminus\{\pm l_0\}} \frac{\alpha\beta}{(m^2-\alpha)(n^2-\beta)}\cdot e^{i(mx+ny)} \Bigg\}, \tag{21}$$

which defines a biorthogonal basis in $L_2(\Omega_1)$.

(c). Let $\forall\ l \in \mathbb{Z}:\ \alpha \neq l^2$ and $\exists\ k_0 \in \mathbb{Z}:\ \beta = k_0^2$. Then a biorthogonal sequence for the basis (16) (case (c) from Theorem 2) is

$$\{\psi_{kl}(x,y),\ k,\ l \in \mathbb{Z}\} = \Bigg\{ e^{i(lx+ky)},\ -\frac{1}{2\pi}\sum_{n\in\mathbb{Z}\setminus\{\pm k_0\}} \frac{\beta}{n^2-\beta}\cdot e^{i(ny+lx)},$$

$$-\frac{1}{2\pi}\sum_{n\in\mathbb{Z}} \frac{\alpha}{n^2-\alpha}\cdot e^{i(nx+ky)},\ k,\ l \in \mathbb{Z}',$$

$$\frac{1}{4\pi^2}\sum_{n\in\mathbb{Z}\setminus\{\pm k_0\},\, m\in\mathbb{Z}} \frac{\alpha\beta}{(m^2-\alpha)(n^2-\beta)}\cdot e^{i(mx+ny)} \Bigg\}, \tag{22}$$

which defines a biorthogonal basis in $L_2(\Omega_1)$.

(d). Let $\exists\ k_0,\, l_0 \in \mathbb{Z}:\ \beta = k_0^2,\ \alpha = l_0^2$. Then a biorthogonal sequence for the basis (17) (case (d) from Theorem 2) is

$$\{\psi_{kl}(x,y),\ k,\, l \in \mathbb{Z}\} = \Bigg\{ e^{i(lx+ky)},\ -\frac{1}{2\pi}\sum_{n\in\mathbb{Z}\setminus\{\pm k_0\}} \frac{\beta}{n^2-\beta}\cdot e^{i(ny+lx)},$$

$$-\frac{1}{2\pi}\sum_{n\in\mathbb{Z}\setminus\{\pm l_0\}} \frac{\alpha}{n^2-\alpha}\cdot e^{i(nx+ky)},\ k,\, l \in \mathbb{Z}',$$

$$\frac{1}{4\pi^2}\sum_{n\in\mathbb{Z}\setminus\{\pm k_0\},\, m\in\mathbb{Z}\setminus\{\pm l_0\}} \frac{\alpha\beta}{(m^2-\alpha)(n^2-\beta)}\cdot e^{i(mx+ny)} \Bigg\}, \tag{23}$$

which defines a biorthogonal basis in $L_2(\Omega_1)$.

8 Algorithm for solving stabilization problem

We propose the following algorithm for solving the stabilization problem for the heat equation with a loaded two-dimensional Laplace operator. It consists of the following constructively implemented steps.

Step 1. We define the function $z_0(x,y)$ as a continuation of the given function $u_0(x,y)$. Thus in the auxiliary boundary value problem (5)–(7) it is needed to continue the function $z_0(x,y)$ on the square Ω_1, so that the requirement (8) is satisfied for a solution $z(x,y,t)$ of the problem (5)–(7). In this case the condition (4) holds as well for its restriction $u(x,y,t)$ and a required boundary control $p(x,y,t)$, $\{x,y\} \in \Sigma$ is defined as trace of function $z(x,y,t)$ for $\{x,y,t\} \in \Sigma$.

Step 2. We construct complete biorthogonal system of functions on the square Ω_1 by solving appropriate spectral problems.

Step 3. Find the coefficients of the decomposition for the desired function $z_0(x,y)$ on the square Ω_1 from constructed at the previous step complete biorthogonal system so that the condition (8) holds.

We will show estimates of values C_0 and σ from inequality (8) for the case (a) of Theorem 1. And for case (b) of Theorem 1 and for cases (a)–(d) of Theorem 2 required estimates of values C_0 and σ can be obtained similarly. For this purpose the solution of initial-boundary value problem (5)–(7) can be written in form (9):

$$z(x,y,t) = \sum_{k\in\mathbb{Z}'} z_{0k0}e^{-(\alpha+k^2)t}\psi_{k0}(x,y) + \sum_{l\in\mathbb{Z}'} z_{00l}e^{-l^2t}\psi_{0l}(x,y) +$$

$$+ z_{000}e^{-\alpha t}\psi_{00}(x,y) + \sum_{k,l\in\mathbb{Z}'} z_{0kl}e^{-(k^2+l^2)t}\psi_{kl}(x,y), \tag{24}$$

where

$$z_{0kl} = \int\limits_{\Omega_1} \overline{\varphi_{kl}(x,y)}\, z_0(x,y)\,dx\,dy,\quad k,l \in \mathbb{Z},$$

are Fourier coefficients $z_0(x,y)$, where $\{\varphi_{kl}(x,y),\ k,l\in\mathbb{Z}\}$ and $\{\psi_{kl}(x,y),\ k,l\in\mathbb{Z}\}$ are defined respectively by the formulas (12) and (18).

We introduce the following sets of indices

$$\mathbb{W}=\mathbb{Z}\times\mathbb{Z}=\mathbb{W}_k\cup\mathbb{W}_l\cup\mathbb{W}',\ \mathbb{W}'=\mathbb{Z}'\times\mathbb{Z}',\ \mathbb{W}_0=\bigcup_{j=1}^{3}\overline{\mathbb{W}}_{0j}\cup\mathbb{W}_{04}, \tag{25}$$

$$\overline{\mathbb{W}}_{01}=\mathbb{W}_k\setminus\mathbb{W}_{01},\ \overline{\mathbb{W}}_{02}=\mathbb{W}_l\setminus\mathbb{W}_{02},\ \overline{\mathbb{W}}_{03}=\mathbb{W}'\setminus\mathbb{W}_{03}. \tag{26}$$

where

$$\mathbb{W}_k=\mathbb{Z}'\times\{0\},\ \mathbb{W}_l=\{0\}\times\mathbb{Z}',\ \mathbb{Z}'=\mathbb{Z}\setminus\{0\},$$

$$\mathbb{W}_{01}=\Big\{\{k,0\}:\ \mathrm{Re}\{\alpha\}+k^2\geq\sigma\Big\}\subset\mathbb{W}_k,$$

$$\mathbb{W}_{02}=\Big\{\{0,l\}:\ l^2\geq\sigma\Big\}\subset\mathbb{W}_l,$$

$$\mathbb{W}_{03}=\Big\{\{k,l\}:\ k^2+l^2\geq\sigma\Big\}\subset\mathbb{W}',$$

$$\mathbb{W}_{04}=\{\{0,0\}:\ \mathrm{Re}\{\alpha\}\geq\sigma\}.$$

Remark 2. *Sets $\mathbb{W}_{01}$, $\mathbb{W}_{02}$, $\mathbb{W}_{03}$ (25) and $\mathbb{W}_0$ (26) are finite.*

Thus, let the conditions of case (a) of Theorem 1 hold. Then the following assertion is true.

Theorem 5. *Let conditions*

$$z_{0kl}=0\ \ at\ \{k,l\}\in\mathbb{W}_0, \tag{27}$$

be satisfied for solution (24), then the stabilized solution $z_{stab}(x,y,t)$ of problem (5)–(7) takes the form

$$z_{stab}(x,y,t)=\sum_{\{k,0\}\in\mathbb{W}_{01}} z_{0k0}e^{-(\alpha+k^2)t}\psi_{k0}(x,y)+\sum_{\{0,l\}\in\mathbb{W}_{02}} z_{00l}e^{-l^2t}\psi_{0l}(x,y)+$$

$$+A(\alpha)e^{-\alpha t}\psi_{00}(x,y)+\sum_{\{k,l\}\in\mathbb{W}_{03}} z_{0kl}e^{-(k^2+l^2)t}\psi_{kl}(x,y), \tag{28}$$

which will satisfy the inequality (8), where in conformity with (25)–(27):

$$A(\alpha)=\begin{cases} z_{000}, & if\ \ \mathbb{W}_{04}\neq\emptyset,\\ 0, & if\ \ \mathbb{W}_{04}=\emptyset.\end{cases}$$

Remark 3. *If $\mathrm{Re}\,\alpha\geq\sigma$, then $z_{000}\neq 0$. In addition, we note that additional restrictions on the constant α, which were indicated in Remark 1 (section 2), are contained in conditions (27). Similar restrictions also takes place for the constant β too in cases (a)–(d) of Theorem 2.*

The proof of Theorem 5 directly follows from our further reasoning. Thus, each of the sets $\mathbb{W}_{01}$, $\mathbb{W}_{02}$, $\mathbb{W}_{03}$ contains a set of indices $\{k,l\}$ that not satisfy conditions (27). From (28) we obtain that for constant C_0 the following equality is true:

$$C_0^2=\int\limits_{\Omega_1}|z_0(x,y)|^2\,dx\,dy=\int\limits_{\Omega_1}\Big|\sum_{\substack{\{k,0\}\in\mathbb{W}_{01}\\ \{0,l\}\in\mathbb{W}_{02}\\ \{k,l\}\in\mathbb{W}_{03}}} z_{0kl}\psi_{kl}(x,y)+A(\alpha)\psi_{00}(x,y)\Big|^2\,dx\,dy<\infty, \tag{29}$$

where

$$z_0(x,y) = \begin{cases} u_0(x,y), & \text{at } \{x,y\} \in \Omega, \\ z_1(x,y), & \text{at } \{x,y\} \in \Omega_1 \setminus \Omega, \end{cases} \tag{30}$$

and here the function $z_1(x,y)$ and its Fourier coefficients $\{z_{0kl},\ \{k,l\} \in \mathbb{W}_0\}$ are unknown, and there is a need to find them. And for this we will use equalities (27), from which we obtain:

$$\int\limits_{\Omega_1 \setminus \Omega} \overline{\varphi_{kl}(x,y)} z_1(x,y)\, d\,x\, d\,y = -\hat{u}_0(k,l),\ \{k,l\} \in \mathbb{W}_0, \tag{31}$$

where

$$\hat{u}_0(k,l) = \int\limits_{\Omega} \overline{\varphi_{kl}(x,y)} u_0(x,y)\, d\,x\, d\,y.$$

Now we will look for the unknown function $z_1(x,y)$ in the form of next linear combination:

$$z_1(x,y) = \sum_{\{m,n\} \in \mathbb{W}_0} \hat{z}_1(m,n) \varphi_{mn}(x,y). \tag{32}$$

As a result, substituting $z_1(x,y)$ (32) into the relation (31), we obtain a system of algebraic equations relatively unknown constant matrix $\{\hat{z}_1(m,n),\ \ \{m,n\} \in \mathbb{W}_0\}$:

$$\sum_{\{m,n\} \in \mathbb{W}_0} a_{klmn} \hat{z}_1(m,n) = -\hat{u}_0(k,l),\ \{k,l\} \in \mathbb{W}_0, \tag{33}$$

where

$$a_{klmn} = \int\limits_{\Omega_1 \setminus \Omega} \overline{\varphi_{kl}(x,y)} \varphi_{mn}(x,y)\, d\,x\, d\,y,\ \{k,l\}, \{m,n\} \in \mathbb{W}_0. \tag{34}$$

We fix the indices k_0 and m_0. Then the system of equations (33) with a known matrix (34) we will represent as next family of independent systems of linear equations

$$\sum_{\{m_0,n\} \in \mathbb{W}_0} a_{k_0 l m_0 n} \hat{z}_1(m_0,n) = -\hat{u}_0(k_0,l),\ \{k_0,l\} \in \mathbb{W}_0, \tag{35}$$

with an unknown vector $\hat{z}_1(m_0,n)$, and with known vector right parts of $-\hat{u}_0(k_0,l)$, and with well-known matrix

$$a_{k_0 l m_0 n} = \int\limits_{\Omega_1 \setminus \Omega} \overline{\varphi_{k_0 l}(x,y)} \varphi_{m_0 n}(x,y)\, d\,x\, d\,y,\ \{k_0,l\}, \{m_0,n\} \in \mathbb{W}_0. \tag{36}$$

Since matrices (36) are built on elements $\{\varphi_{k,l}(x,y),\ \{k,l\} \in \mathbb{W}_0\}$, which are the finite subsystems of the basis $\{\varphi_{kl}(x,y),\ k,l \in \mathbb{Z}\}$ (12), then for each fixed pair of indices k_0 and

m_0 they are Gram matrices. As it is known, the determinants of Gram matrices are different from zero ([15], p.219). Therefore, we will have the unique solvability for the equation (35), and as corollary of it and for the equation (33) too.

Next, by finding the unknown matrix $\{\hat{z}_1(m,n),\ \{m,n\} \in \mathbb{W}_0\}$ according to the formula (32) we find the function $\{z_1(x,y),\ \{x,y\} \in \Omega_1 \backslash \Omega\}$, and together with it, according to (30) we find the function $\{z_0(x,y),\ \{x,y\} \in \Omega_1\}$ as a continuation of the function $\{u_0(x,y), \{x,y\} \in \Omega\}$.

Further, analyzing the formula (24) and taking into account the definitions of sets $\mathbb{W}_{0j}, j = \overline{1,3}$ in (26), we obtain the estimate real constant σ_r, which determines the decay order in the exponent of (8):

$$\sigma_r \triangleq \min\left\{ \min_{\{k,0\}\in\mathbb{W}_{01}} \{\mathrm{Re}\{\alpha\} + k^2\};\ \min_{\{0,l\}\in\mathbb{W}_{02}} \{l^2\};\ B(\alpha);\ \min_{\{k,l\}\in\mathbb{W}_{03}} \{k^2 + l^2\} \right\} \geq \sigma,$$

where in conformity with (25)–(27) and Remark 1:

$$B(\alpha) = \begin{cases} \mathrm{Re}\{\alpha\}, & \text{if } \ \mathbb{W}_{04} \neq \emptyset, \\ 0, & \text{if } \ \mathbb{W}_{04} = \emptyset. \end{cases}$$

Now, according to formulas (28)–(30), we can find the value of the bounded constant C_0 from (8).

Step 4. By the solution found $z(x,y,t)$ of the auxiliary boundary value problem (5)–(7) as restriction of it to the cylinder Q we find a solution $u(x,y,t)$ to the given boundary value problem (1)–(3), satisfying the required condition (4). A boundary control $p(x,y,t)$, $\{x,y\} \in \Sigma$ is found as trace of the solution $z(x,y,t)$, i.e.

$$p(x,y,t) = z(x,y,t)_{|\{x,y,t\}\in\Sigma}.$$

9 Conclusion

The results of the work can be useful in solving stabilization problems for a loaded parabolic equation with the help of boundary control actions that can be used in problems of mathematical modeling by controlled loaded differential equations.

Acknowledgements

This research has is funded by the Science Committee of the Ministry of Education and Science of the RK (Grant № AP08855372, 2020-2022).

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