

1-бөлім

Раздел 1

Section 1

Математика

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Mathematics

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ON POTENTIAL THEORY FOR THE GENERALIZED BI-AXIALLY SYMMETRIC ELLIPTIC EQUATION IN THE PLANE

Fundamental solutions of the generalized biaxially symmetric elliptic equation are expressed in terms of the well-known Appel hypergeometric function in two variables, the properties of which are necessary for studying boundary value problems for the above equation. In this paper, using some properties of the Appel hypergeometric function, we prove limit theorems and derive integral equations for the double- and simple-layer potentials and apply the results of the constructed potential theory to the study of the Dirichlet problem for a two-dimensional elliptic equation with two singular coefficients in a domain bounded in the first quarter of the plane.

Key words: Appell hypergeometric function, generalized bi-axially symmetric elliptic equation, potential theory, Green's function, Dirichlet problem.

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Жазықтықтағы жалпыланған екі өске симметриялық эллиптикалық теңдеудің потенциалдық теориясы жайында

Жалпыланған екі өске симметриялық эллиптикалық теңдеудің іргелі шешімдері екі айнымалысы бар Аппелдің гипергеометриялық функциясы арқылы өрнектеледі, олардың қасиеттері жоғарыда келтірілген теңдеу үшін шекті есептерді зерттеу үшін қажет. Бұл жұмыста Аппелдің гипергеометриялық функциясының кейбір қасиеттерін қолдана отырып, біз қос қабатты және жай қабатты потенциалдардың тығыздығы үшін шекті теоремаларды дәлелдейміз және интегралдық теңдеулер аламыз. Құрылған потенциалдар теориясының нәтижелерін жазықтықтың бірінші ширегінде шектелген облыста екі сингулярлы коэффициенті бар екі өлшемді эллиптикалық теңдеу үшін Дирихле есебін зерттеуге қолданамыз.

Түйін сөздер: Аппелдің екі айнымалы гипергеометриялық функциясы, жай қабатты және қос қабатты потенциалдар, Грин функциясы, іргелі шешім, Дирихле есебі.

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О теории потенциала для обобщенного двусесимметричного эллиптического уравнения на плоскости

Фундаментальные решения обобщенного двусесимметричного эллиптического уравнения выражаются через известную гипергеометрическую функцию Аппеля с двумя переменными, свойства которой необходимы для изучения краевых задач для указанного выше уравнения. В данной работе, используя некоторые свойства гипергеометрической функции Аппеля, доказываем предельные теоремы и выводим интегральные уравнения, касающиеся плотности потенциалов двойного и простого слоев. Применим результаты построенной теории потенциала к исследованию задачи Дирихле для двумерного эллиптического уравнения с двумя сингулярными коэффициентами в области, ограниченной в первой четверти плоскости.

Ключевые слова: гипергеометрическая функция Аппеля двух переменных, потенциалы двойного и простого слоев, функция Грина, фундаментальное решение, задача Дирихле.

1 Introduction

Numerous applications of simple- and double-layer potentials, as well as volumetric potentials, occur in fluid mechanics, elastodynamics, electromagnetism, and acoustics [3]; therefore, the theory of potentials plays an important role in solving boundary value problems for elliptic equations. This, in particular, allows one to reduce the solution of boundary value problems to the solution of integral equations [1, 2].

For the first time S. Gellerstedt [4] constructed a potential theory and applied it to the solution of basic boundary value problems for the model Tricomi equation, i.e. for a two-dimensional elliptic equation with one singular coefficient of the form

$$u_{xx} + u_{yy} + \frac{2\alpha}{x}u_x = 0, \quad 0 < 2\alpha < 1,$$

which, later, was developed in the works of F.I. Frankl [5], S.P. Pulkin [6], M.M. Smirnov [7]. This line of research adjoin works [8–10].

The papers [11] and [12] are devoted to investigation of the double- and simple-layer potentials for a three-dimensional singular elliptic equation of the form

$$u_{xx} + u_{yy} + u_{zz} + \frac{2\alpha}{x}u_x = 0, \quad 0 < 2\alpha < 1 \tag{1}$$

and solving the mixed problem and the Dirichlet problem for the equation (1) in a domain bounded in the half-space $x > 0$, respectively.

The authors of the papers [13, 14] constructed a potential theory for multidimensional elliptic equation with one singular coefficient

$$\sum_{k=1}^m u_{x_k x_k} + \frac{2\alpha}{x_1} u_{x_1} = 0, \quad 0 < 2\alpha < 1, \quad m \geq 2$$

in the domain bounded in a half-space $x_1 > 0$ and with the help of this theory, the solutions of the Dirichlet [13] and Holmgren problems [14] are obtained in forms convenient for further research.

On potential theory for an elliptic equation with two singular coefficients

$$E(u) \equiv u_{xx} + u_{yy} + \frac{2\alpha}{x}u_x + \frac{2\beta}{y}u_y = 0, \quad 0 < 2\alpha, 2\beta < 1 \quad (2)$$

are devoted to relatively few works. In the works [15–18] the authors studied only the properties of the double-layer potentials for generalized biaxially symmetric elliptic equation (2).

In this paper, for the equation (2), we construct the theory potential and apply it to the solution of the Dirichlet problem in the domain bounded in the first quarter $R_2^{2+} := \{(x, y) : x > 0, y > 0\}$ of the xOy -plane.

2 Preliminaries

The Pochhammer symbol $(p)_n$ is defined by the equality

$$(p)_n = p(p+1)\dots(p+n-1), \quad n = 1, 2, \dots; \quad (p)_0 \equiv 1. \quad (3)$$

The Gaussian hypergeometric function is defined inside the circle $|z| < 1$ as the sum of the hypergeometric series [19, Ch.2, eq. 2.1(2)]

$$F(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{k! (c)_k} z^k, \quad (4)$$

and for $|z| \geq 1$ is obtained by an analytic continuation of (4).

For the Gaussian hypergeometric function the summation formula [19, Ch.2, eq. 2.1(14)]

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad \operatorname{Re}(c-a-b) > 0, \quad (5)$$

and Bolts's formula [19, Ch.2, eq. 2.1(22)]

$$F(a, b; c; z) = (1-z)^{-b} F\left(c-a, b; c; \frac{z}{z-1}\right) \quad (6)$$

are valid.

The Appel hypergeometric function of two variables has a form [19, Ch.5, eq. 5.7(7)]

$$\begin{aligned} F_2(a; b_1, b_2; c_1, c_2; x, y) &\equiv \\ &\equiv F_2 \left[\begin{matrix} a, b_1, b_2; \\ c_1, c_2; \end{matrix} x, y \right] = \sum_{m, n=0}^{\infty} \frac{(a)_{m+n} (b_1)_m (b_2)_n}{m! n! (c_1)_m (c_2)_n} x^m y^n, \quad |x| + |y| < 1, \end{aligned}$$

where the parameters a, b_1, b_2, c_1, c_2 and variables x, y are arbitrary complex numbers and $c_1, c_2 \neq 0, -1, -2, \dots$

We give some elementary relations for F_2 necessary in this study:

$$\begin{aligned} \frac{\partial^{m+n}}{\partial x^m \partial y^n} F_2(a; b_1, b_2; c_1, c_2; x, y) &= \\ &= \frac{(a)_{m+n} (b_1)_m (b_2)_n}{m! n! (c_1)_m (c_2)_n} F_2 \left[\begin{matrix} a + m + n, b_1 + m, b_2 + n; \\ c_1 + m, c_2 + n; \end{matrix} x, y \right], \end{aligned} \quad (7)$$

$$\begin{aligned} \frac{b_1}{c_1} x F_2 \left[\begin{matrix} a + 1, b_1 + 1, b_2; \\ c_1 + 1, c_2; \end{matrix} x, y \right] + \frac{b_2}{c_2} y F_2 \left[\begin{matrix} a + 1, b_1, b_2 + 1; \\ c_1, c_2 + 1; \end{matrix} x, y \right] &= \\ = F_2(a + 1; b_1, b_2; c_1, c_2; x, y) - F_2(a; b_1, b_2; c_1, c_2; x, y), \end{aligned} \quad (8)$$

$$\begin{aligned} F_2(a, b_1, b_2; c_1, c_2; x, y) &= \\ = (1 - x - y)^{-a} F_2 \left(a, c_1 - b_1, c_2 - b_2; c_1, c_2; \frac{x}{x + y - 1}, \frac{y}{x + y - 1} \right). \end{aligned} \quad (9)$$

We note, that every point of the line $x + y = 1$ is a logarithmic singularity of the function F_2 .

Lemma 1 [20]. *If x and y are positive and $\alpha > 0$, $\beta > 0$, then*

$$F_2(\alpha + \beta, \alpha, \beta; 2\alpha, 2\beta; x, y) \sim -\frac{\Gamma(2\alpha)\Gamma(2\beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha + \beta)} x^{-\alpha} y^{-\beta} \ln(1 - x - y) \quad (10)$$

as $x + y \rightarrow 1 - 0$.

Let $c_1 > b_1$, $c_2 > b_2$ u $a + b_1 + b_2 = c_1 + c_2$. If $x > 0$, $y > 0$, then

$$F_2(a, b_1, b_2; c_1, c_2; x, y) \sim -\frac{\Gamma(c_1)\Gamma(c_2)}{\Gamma(a)\Gamma(b_1)\Gamma(b_2)} x^{b_1 - c_1} y^{b_2 - c_2} \ln(1 - x - y) \quad (11)$$

as $x + y \rightarrow 1 - 0$.

If $c_1 + c_2 < a + b_1 + b_2$, then

$$\begin{aligned} F_2(a, b_1, b_2; c_1, c_2; x, y) &\sim \frac{\Gamma(c_1)\Gamma(c_2)\Gamma(a + b_1 + b_2 - c_1 - c_2)}{\Gamma(a)\Gamma(b_1)\Gamma(b_2)} \times \\ &\times x^{b_1 - c_1} y^{b_2 - c_2} (1 - x - y)^{c_1 + c_2 - a - b_1 - b_2}. \end{aligned} \quad (12)$$

In addition, the fundamental solutions of the equation (2) are expressed in terms of the Appell hypergeometric function F_2 , one of which has the form [21]:

$$\begin{aligned} q(x, y; \xi, \eta) &= \kappa r^{2\alpha + 2\beta - 4} x^{1 - 2\alpha} y^{1 - 2\beta} \xi^{1 - 2\alpha} \eta^{1 - 2\beta} \times \\ &\times F_2(2 - \alpha - \beta, 1 - \alpha, 1 - \beta; 2 - 2\alpha, 2 - 2\beta; \sigma_1, \sigma_2); \end{aligned} \quad (13)$$

where

$$\sigma_1 = 1 - \frac{r_1^2}{r^2}, \quad \sigma_2 = 1 - \frac{r_2^2}{r^2}; \quad r^2 = (x - \xi)^2 + (y - \eta)^2,$$

$$r_1^2 = (x + \xi)^2 + (y - \eta)^2, \quad r_2^2 = (x - \xi)^2 + (y + \eta)^2,$$

$$\kappa = \frac{2^{4-2\alpha-2\beta} \Gamma(1-\alpha)\Gamma(1-\beta)\Gamma(2-\alpha-\beta)}{4\pi \Gamma(2-2\alpha)\Gamma(2-2\beta)}.$$

The function $q(x, y; \xi, \eta)$ satisfies the equation by the variables (x, y) , and by virtue of the formula (10), it has a logarithmic singularity at $r \rightarrow 0$ ($x > 0, y > 0$) and, therefore, the function $q(x, y; \xi, \eta)$ is a fundamental solution to the equation (2).

The fundamental solution given by (13) possesses the following potentially useful property:

$$q(x, y; \xi, \eta)|_{x=0} = q(x, y; \xi, \eta)|_{y=0} = 0. \quad (14)$$

3 Green's formula

We consider the following identity:

$$x^{2\alpha}y^{2\beta} [uE(v) - vE(u)] =$$

$$= y^{2\beta} \frac{\partial}{\partial x} \left[x^{2\alpha} \left(u \frac{\partial v}{\partial x} - v \frac{\partial u}{\partial x} \right) \right] + x^{2\alpha} \frac{\partial}{\partial y} \left[y^{2\beta} \left(u \frac{\partial v}{\partial y} - v \frac{\partial u}{\partial y} \right) \right]. \quad (15)$$

Integrating both sides of this identity in a domain D , which is located and bounded in the quarter-plane $x > 0, y > 0$, and using the Ostrogradsky formula, we obtain

$$\int \int_D x^{2\alpha}y^{2\beta} [uE(v) - vE(u)] dx dy =$$

$$= \int_{\gamma} x^{2\alpha}y^{2\beta} \left[- \left(u \frac{\partial v}{\partial y} - v \frac{\partial u}{\partial y} \right) dx + \left(u \frac{\partial v}{\partial x} - v \frac{\partial u}{\partial x} \right) dy \right], \quad (16)$$

where γ is a contour of D .

The Green's formula (16) is derived under the following assumptions: (a) The functions $u(x, y)$ and $v(x, y)$, and their first-order derivatives, are continuous in the closed domain \bar{D} ; (b) The second-order partial derivatives are continuous inside the domain D .

The integrals over D , consisting of $E(u)$ and $E(v)$, have a meaning. If $E(u)$ and $E(v)$ are not continuous up to S , then they are improper integrals obtained as limits on any sequence of domains D_n contained inside D when these domains D_n tend to D , so that any point in this D_n will be inside of D , starting with some number n .

If u and v are solutions of equation (2), then we find from formula (16) that

$$\int_{\gamma} (uA_n^{\alpha,\beta}[v] - vA_n^{\alpha,\beta}[u]) ds = 0, \quad (17)$$

where $A_n^{\alpha,\beta} [\]$ is the conormal derivative with respect to (x, y) :

$$A_n^{\alpha,\beta} [\] \equiv x^{2\alpha}y^{2\beta} \left(\frac{dy}{ds} \frac{\partial}{\partial x} - \frac{dx}{ds} \frac{\partial}{\partial y} \right).$$

Here $\frac{dy}{ds} = \cos(n, x)$, $\frac{dx}{ds} = -\cos(n, y)$, n is the outer normal to the curve γ .

Assuming that $v \equiv 1$ in (16) and replacing u by u^2 , we obtain

$$\int_D x^{2\alpha} y^{2\beta} \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right] dx dy = \int_\gamma u A_n^{\alpha, \beta} [u] ds, \quad (18)$$

where $u(x, y)$ is the solution of equation (2).

The special case of (17) when $v \equiv 1$ reduces to the following form:

$$\int_\gamma A_n^{\alpha, \beta} [u] ds = 0. \quad (19)$$

We note from (19) that the integral of the conormal derivative of the solution of equation (2) along the boundary γ of the domain is equal to zero.

4 A double-layer potential

Let D be a domain bounded by two segments $[0, a]$ of the axes x and y , and a curve Γ with the ends at the points $A(a, 0)$ and $B(0, a)$ lying in the quarter-plane $x > 0$, $y > 0$.

Let the parametric equation of the curve Γ be $x = x(s)$, $y = y(s)$, where s is the length of the arc measured from the point A . With respect to the curve Γ , we will assume that:

(i) the functions $x(s)$ and $y(s)$ have the continuous derivatives $x'(s)$ and $y'(s)$ on the segment $[0, l]$, which do not vanish simultaneously; the derivatives $x''(s)$ and $y''(s)$ satisfy the Holder condition on $[0, l]$, where l is the length of the curve Γ ;

(ii) in a neighborhoods of the points A and B on the curve Γ the following conditions are satisfied

$$\left| \frac{dx}{ds} \right| \leq C_1 y(s), \quad \left| \frac{dy}{ds} \right| \leq C_2 x(s), \quad (20)$$

respectively.

The coordinates of a variable point on the curve Γ will be denoted by (ξ, η) .

We now consider the following integral:

$$w(x, y) = \int_0^l \mu(s) A_\nu^{\alpha, \beta} [q(\xi, \eta; x, y)] ds, \quad (21)$$

where $\mu(s) \in C(\bar{\Gamma})$ and $q(\xi, \eta; x, y)$ is a fundamental solution of the equation (2) defined by (13). Here

$$A_\nu^{\alpha, \beta} [] = \xi^{2\alpha} \eta^{2\beta} \left[\cos(\nu, \xi) \cdot \frac{\partial []}{\partial \xi} + \cos(\nu, \eta) \cdot \frac{\partial []}{\partial \eta} \right], \quad (22)$$

is the conormal derivative with respect to (ξ, η) , ν is outer normal to the curve Γ .

Definition 1 . We call the integral (21) a double-layer potential with density $\mu(s)$.

In the study of the double layer potential (21), the conormal derivative of the fundamental solution $q(\xi, \eta; x, y)$ plays an important role. Applying successively the formula for the derivative of the Appel hypergeometric function (7) and the adjacent relation (8), taking into account (22), we obtain (for details, see [18]):

$$\begin{aligned}
A_{\nu}^{\alpha, \beta} [q(\xi, \eta; x, y)] = & -(2 - \alpha - \beta) \kappa \frac{1}{r^{4-2\alpha-2\beta}} x^{1-2\alpha} y^{1-2\beta} \xi^{1-2\alpha} \eta^{1-2\beta} \times \\
& \times F_2 \left[\begin{matrix} 3 - \alpha - \beta, 1 - \alpha, 1 - \beta; \\ 2 - 2\alpha, 2 - 2\beta; \end{matrix} \sigma_1, \sigma_2 \right] A_{\nu}^{\alpha, \beta} [\ln r^2] - \\
& - 2(2 - \alpha - \beta) \kappa \frac{x^{2-2\alpha} y^{1-2\beta} \xi \eta}{r^{6-2\alpha-2\beta}} F_2 \left[\begin{matrix} 3 - \alpha - \beta, 2 - \alpha, 1 - \beta; \\ 3 - 2\alpha, 2 - 2\beta; \end{matrix} \sigma_1, \sigma_2 \right] \frac{d\eta(s)}{ds} + \\
& + 2(2 - \alpha - \beta) \kappa \frac{x^{1-2\alpha} y^{2-2\beta} \xi \eta}{r^{6-2\alpha-2\beta}} F_2 \left[\begin{matrix} 3 - \alpha - \beta, 1 - \alpha, 2 - \beta; \\ 2 - 2\alpha, 3 - 2\beta; \end{matrix} \sigma_1, \sigma_2 \right] \frac{d\xi(s)}{ds} + \\
& + (1 - 2\alpha) \kappa \frac{x^{1-2\alpha} y^{1-2\beta} \eta}{r^{4-2\alpha-2\beta}} F_2 \left[\begin{matrix} 2 - \alpha - \beta, 1 - \alpha, 1 - \beta; \\ 2 - 2\alpha, 2 - 2\beta; \end{matrix} \sigma_1, \sigma_2 \right] \frac{d\eta(s)}{ds} - \\
& - (1 - 2\beta) \kappa \frac{x^{1-2\alpha} y^{1-2\beta} \xi}{r^{4-2\alpha-2\beta}} F_2 \left[\begin{matrix} 2 - \alpha - \beta, 1 - \alpha, 1 - \beta; \\ 2 - 2\alpha, 2 - 2\beta; \end{matrix} \sigma_1, \sigma_2 \right] \frac{d\xi(s)}{ds}.
\end{aligned} \tag{23}$$

We introduce the notation:

$$w_1(x, y) \equiv \int_0^l A_{\nu}^{\alpha, \beta} [q(\xi, \eta; x, y)] ds,$$

Lemma 2 . *The following formula holds true:*

$$w_1(x, y) = \begin{cases} i(x, y) - 1, & (x, y) \in D, \\ i(x, y) - \frac{1}{2}, & (x, y) \in \Gamma, \\ i(x, y), & (x, y) \notin D \cup \Gamma, \end{cases} \tag{24}$$

where

$$\begin{aligned}
i(x, y) \equiv & (1 - 2\beta) \kappa x^{1-2\alpha} y^{1-2\beta} \int_0^a \frac{\xi F(2 - \alpha - \beta, 1 - \alpha; 2 - 2\alpha; \sigma_{10})}{[(x - \xi)^2 + y^2]^{2-\alpha-\beta}} d\xi + \\
& + (1 - 2\alpha) \kappa x^{1-2\alpha} y^{1-2\beta} \int_0^a \frac{\eta F(2 - \alpha - \beta, 1 - \beta; 2 - 2\beta; \sigma_{20})}{[x^2 + (y - \eta)^2]^{2-\alpha-\beta}} d\eta, \\
\sigma_{10} = & -\frac{4x\xi}{(x - \xi)^2 + y^2}, \quad \sigma_{20} = -\frac{4y\eta}{x^2 + (y - \eta)^2}.
\end{aligned}$$

Proof. Lemma 2 was proved in [18].

Lemma 3 . *If $(x, y) \in \Gamma$, then*

$$|A_{\nu}^{\alpha, \beta} [q(\xi, \eta; x, y)]| \leq \frac{B_1}{r_1^{2\alpha} r_2^{2\beta}} \left(\ln \frac{r_1 r_2}{r_{12} r} + 1 \right). \tag{25}$$

where B_1 is a constant.

Proof. The estimate (25) follows from the formula (23) and Lemma 1.

Lemma 4 . *If a curve Γ satisfies the conditions (i) and (ii), then the following inequality holds true:*

$$\int_0^l |A_\nu^{\alpha,\beta}[q(\xi, \eta; x, y)]| ds \leq \frac{B_2}{x^\alpha y^\beta},$$

whrer B_2 is a constant.

Proof. Using the formula transformations (9), the conormal derivative $A_\nu^{\alpha,\beta}[q(\xi, \eta; x, y)]$, defined by the formula (23), can be represented as

$$A_\nu^{\alpha,\beta}[q(\xi, \eta; x, y)] = \sum_{i=0}^4 P_i(s; x, y),$$

where

$$P_0(s; x, y) = -\kappa \frac{(2 - \alpha - \beta)r^2}{r_{12}^{6-2\alpha-2\beta}} x^{1-2\alpha} y^{1-2\beta} \xi^{1-2\alpha} \eta^{1-2\beta} \times \\ \times F_2 \left[\begin{matrix} 3 - \alpha - \beta, 1 - \alpha, 1 - \beta; \\ 2 - 2\alpha, 2 - 2\beta; \end{matrix} \bar{\sigma}_1, \bar{\sigma}_2 \right] [\ln r^2],$$

$$P_1(s; x, y) = -2(2 - \alpha - \beta)\kappa \times \\ \times \frac{x^{2-2\alpha} y^{1-2\beta} \xi \eta}{r_{12}^{6-2\alpha-2\beta}} F_2 \left[\begin{matrix} 3 - \alpha - \beta, 2 - \alpha, 1 - \beta; \\ 3 - 2\alpha, 2 - 2\beta; \end{matrix} \bar{\sigma}_1, \bar{\sigma}_2 \right] \frac{d\eta(s)}{ds},$$

$$P_2(s; x, y) = 2(2 - \alpha - \beta)\kappa \times \\ \times \frac{x^{1-2\alpha} y^{2-2\beta} \xi \eta}{r_{12}^{6-2\alpha-2\beta}} F_2 \left[\begin{matrix} 3 - \alpha - \beta, 1 - \alpha, 2 - \beta; \\ 2 - 2\alpha, 3 - 2\beta; \end{matrix} \bar{\sigma}_1, \bar{\sigma}_2 \right] \frac{d\xi(s)}{ds},$$

$$P_3(s; x, y) = (1 - 2\alpha)\kappa \times \\ \times \frac{x^{-2\alpha} y^{1-2\beta} \xi \eta}{r_{12}^{4-2\alpha-2\beta}} F_2 \left[\begin{matrix} 2 - \alpha - \beta, 1 - \alpha, 1 - \beta; \\ 2 - 2\alpha, 2 - 2\beta; \end{matrix} \bar{\sigma}_1, \bar{\sigma}_2 \right] \frac{d\eta(s)}{ds},$$

$$P_4(s; x, y) = -(1 - 2\beta)\kappa \times \\ \times \frac{x^{1-2\alpha} y^{-2\beta} \xi \eta}{r_{12}^{4-2\alpha-2\beta}} F_2 \left[\begin{matrix} 2 - \alpha - \beta, 1 - \alpha, 1 - \beta; \\ 2 - 2\alpha, 2 - 2\beta; \end{matrix} \bar{\sigma}_1, \bar{\sigma}_2 \right] \frac{d\xi(s)}{ds},$$

$$r_{12}^2 = (x + \xi)^2 + (y + \eta)^2, \quad \bar{\sigma}_1 = \frac{4x\xi}{r_{12}^2}, \quad \bar{\sigma}_2 = \frac{4y\eta}{r_{12}^2}, \quad 0 \leq \bar{\sigma}_1 + \bar{\sigma}_2 \leq 1.$$

By virtue of (12), we obtain

$$\begin{aligned} \int_0^l |P_0(s; x, y)| ds &\leq C_2 \int_0^l \frac{x^{1-2\alpha} y^{1-2\beta} \xi \eta r^2}{r_{12}^{6-2\alpha-2\beta}} \times \\ &\times \left(\frac{x\xi}{r_{12}^2} \right)^{\alpha-1} \left(\frac{y\eta}{r_{12}^2} \right)^{\beta-1} \left(\frac{r^2}{r_{12}^2} \right)^{-1} \left| \frac{\partial}{\partial \nu} \left(\ln \frac{1}{r} \right) \right| ds \leq \\ &\leq \frac{C_2}{x^\alpha y^\beta} \int_0^l \xi^\alpha \eta^\beta \left| \frac{\partial}{\partial \nu} \left(\ln \frac{1}{r} \right) \right| ds \leq \frac{C_3}{x^\alpha y^\beta} \int_0^l \frac{|\cos \vartheta|}{r} ds, \end{aligned} \quad (26)$$

ϑ is an angle between r and outer normal ν to the curve Γ .

From the theory of the logarithmic potential we have

$$\int_0^l \frac{|\cos \vartheta|}{r} ds < C_4. \quad (27)$$

Similarly we estimate $P_1(s; x, y)$ and $P_2(s; x, y)$:

$$\int_0^l |P_k(s; x, y)| ds \leq \frac{D_k}{x^\alpha y^\beta} \quad (k = 1, 2). \quad (28)$$

Now we will estimate $P_3(s; x, y)$ and $P_4(s; x, y)$. It is easy to see that

$$\int_{\varepsilon_k}^{l-\varepsilon_k} |P_k(s; x, y)| ds \leq \frac{D_k}{x^\alpha y^\beta} \quad (\varepsilon_k > 0, \quad k = 3, 4), \quad (29)$$

where D_3 and D_4 are independent of (x, y) .

Integrals $\int_0^{\varepsilon_k} |P_k(s; x, y)| ds$ and $\int_{l-\varepsilon_k}^l |P_k(s; x, y)| ds$ are estimated similarly. Let us estimate the first of them for $k = 3$. Using the estimate (11), taking into account the first of the conditions (20), we get

$$\int_0^{\varepsilon_3} |P_3(s; x, y)| ds \leq \frac{E_1}{x^\alpha y^\beta} \int_0^{\varepsilon_3} \ln \left[\frac{r}{r_{12}} \right] ds \leq \frac{E_2}{x^\alpha y^\beta}. \quad (30)$$

Thus, the obtained estimates (26) - (30) imply the validity of the Lemma 4.

Theorem 1 . *The following limit formulas hold true for a double-layer potential (21):*

$$\begin{aligned} w_i(s) &= -\frac{1}{2}\mu(s) + \int_0^l \mu(t)K(s, t)dt, \\ w_e(s) &= \frac{1}{2}\mu(s) + \int_0^l \mu(t)K(s, t)dt, \end{aligned} \quad (31)$$

where

$$K(s, t) = A_v^{\alpha, \beta} [q(\xi(t), \eta(t); x(s), y(s))].$$

$A_n^{\alpha, \beta} [w(x, y)]_i$ and $A_n^{\alpha, \beta} [w(x, y)]_e$ are limiting values of the double-layer potential (21) at the point $t \in \Gamma$ from the inside and the outside, respectively.

Proof. Theorem 1 follows from the Lemmas 2 and 4.

5 The simple-layer potential

In this section, we consider the following integral:

$$v(x, y) = \int_0^l \rho(t)q(\xi, \eta; x, y)dt, \quad (32)$$

where the density $\rho(t) \in C(\bar{\Gamma})$ and $q(\xi, \eta; x, y)$ is given in (13). We call the integral (32) a simple-layer potential with density $\rho(t)$.

The simple-layer potential (32) is defined throughout the quarter-plane $x > 0, y > 0$ and is a continuous function when passing through the curve Γ . Obviously, a simple-layer potential is a regular solution of equation (2) in any domain lying in the quarter-plane $x > 0, y > 0$. It is easy to see that, as the point (x, y) tends to ∞ , a simple-layer potential $v(x, y)$ tends to 0. Indeed, we let the point (x, y) be on the quarter-circle given by $C_R: x^2 + y^2 = R^2$ ($x > 0, y > 0$). Then, by virtue of (13), we have

$$|v(x, y)| \leq \int_0^l |\rho(t)| |q(\xi, \eta; x, y)| dt \leq \frac{M}{R^2}, \quad (33)$$

where M is a constant. ($R \geq R_0$).

We take an arbitrary point $N(x(x), y(s))$ on the curve Γ and draw a normal at this point. By considering on this normal any point $M(x, y)$, not lying on the curve Γ , we find the conormal derivative of the simple-layer potential (32):

$$A_n^{\alpha, \beta} [v(x, y)] = \int_0^l \rho(t) A_n^{\alpha, \beta} [q(\xi, \eta; x, y)] dt, \quad (34)$$

where

$$A_n^{\alpha, \beta} [] = x^{2\alpha} y^{2\beta} \left(\cos(n, x) \cdot \frac{\partial}{\partial x} + \cos(n, y) \cdot \frac{\partial}{\partial y} \right).$$

The integral in (34) exists also in the case when the point $M(x, y)$ coincides with the point N , which we mentioned above.

Theorem 2 . *The following limit formulas hold true for a simple-layer potential (32):*

$$\begin{aligned} A_n^{\alpha, \beta} [v(x, y)]_i &= \frac{1}{2} \rho(s) + \int_0^l \rho(t) K(t, s) dt, \\ A_n^{\alpha, \beta} [v(x, y)]_e &= -\frac{1}{2} \rho(s) + \int_0^l \rho(t) K(t, s) dt, \end{aligned} \quad (35)$$

where

$$K(t, s) = A_n^{\alpha, \beta} [q(\xi(t), \eta(t); x(s), y(s))].$$

$A_n^{\alpha, \beta} [v(x, y)]_i$ and $A_n^{\alpha, \beta} [v(x, y)]_e$ are limiting values of the normal derivative of simple-layer potential (32) at the point $t \in \Gamma$ from the inside and the outside, respectively.

Proof. Theorem 2 is proved in the same way as theorem 1.

Making use of these formulas, the jump in the normal derivative of the simple-layer potential follows immediately:

$$A_n^{\alpha,\beta} [v(x, y)]_i - A_n^{\alpha,\beta} [v(x, y)]_e = \rho(x, y). \quad (36)$$

For future researches on the subject of the present investigation, it will be useful to note that when the point (x, y) tends to ∞ , the following inequality

$$|A_n^{\alpha,\beta} [v(x, y)]| \leq \frac{M}{R^{4-2\alpha-2\beta}}, \quad (37)$$

is valid, M is a constant ($R \geq R_0$).

In exactly the same way as in the derivation of (18), it is not difficult to show that Green's formulas are applicable to the simple-layer potential (32) as follows:

$$\int \int_D x^{2\alpha} y^{2\beta} \left[\left(\frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 \right] dx dy = \int_{\Gamma} v A_n^{\alpha,\beta} [v]_i ds, \quad (38)$$

$$\int \int_{D'} x^{2\alpha} y^{2\beta} \left[\left(\frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 \right] dx dy = - \int_{\Gamma} v A_n^{\alpha,\beta} [v]_e ds. \quad (39)$$

Hereinafter $D' = R_2^{2+} \setminus \bar{D}$ is the unbounded domain at $x > 0, y > 0$.

6 Integral Equations For Denseness

Formulas (31) and (35) can be written as the following integral equations for densities:

$$\mu(s) - \lambda \int_0^l K(s, t) \mu(t) dt = f(s), \quad (40)$$

$$\rho(s) - \lambda \int_0^l K(t, s) \rho(t) dt = g(s), \quad (41)$$

where

$$\lambda = 2, \quad f(s) = -2w_i(s), \quad g(s) = -2A_n^{\alpha,\beta} [v]_e,$$

$$\lambda = -2, \quad f(s) = 2w_e(s), \quad g(s) = 2A_n^{\alpha,\beta} [v]_i.$$

Equations (40) and (41) are mutually conjugated and, by Lemma 3, Fredholm theory is applicable to them. We show that $\lambda = 2$ is not an eigenvalue of the kernel $K(s, t)$. This assertion is equivalent to the fact that the homogeneous integral equation

$$\rho(s) - 2 \int_0^l K(t, s)\rho(t)dt = 0, \quad (42)$$

has no non-trivial solutions.

Let $\tilde{\rho}(t)$ be a continuous non-trivial solution of the equation (42). The simple-layer potential with density $\tilde{\rho}(t)$ gives us a function $\tilde{v}(x, y)$, which is a solution of the equation (2) in the domains D and D' . By virtue of the equation (42), the limiting values of the normal derivative of $A_n^{\alpha, \beta}[\tilde{v}]_e$ are zero. The formula (39) is applicable to the simple-layer potential $\tilde{v}(x, y)$, from which it follows that $\tilde{v}(x, y) = \text{const}$ in domain D' . At infinity, a simple layer potential is zero, and consequently $\tilde{v}(x, y) \equiv 0$ in D' , and also on the curve Γ . Applying now (38), we find that $\tilde{v}(x, y) \equiv 0$ is valid also inside the domain D . But then $A_n^{\alpha, \beta}[\tilde{v}]_i = 0$, and by virtue of formula (36) we obtain $\tilde{\rho}(t) \equiv 0$. Thus, clearly, the homogeneous equation (42) has only the trivial solution; consequently, $\lambda = 2$ is not an eigenvalue of the kernel $K(s; t)$.

7 The Uniqueness of the Solution of Dirichlet Problem

We apply the obtained results of potential theory to the solving the boundary value problem for the equation (2) in the domain D .

We consider the Dirichlet problem for equation (2) in the domain D defined in Section 4. We assume that the curve Γ satisfies conditions (i) and (ii) in Section 4.

Dirichlet problem. Find a regular solution $u(x, y)$ of equation (2) in the domain D that is continuous in the closed domain \bar{D} and satisfies the following boundary conditions:

$$u|_{\Gamma} = \varphi(s) \quad (0 \leq s \leq l), \quad (43)$$

$$\lim_{x \rightarrow 0} u(x, y) = \tau_1(y), \quad \lim_{y \rightarrow 0} u(x, y) = \tau_2(x) \quad (0 \leq x, y \leq a), \quad (44)$$

where $\varphi(s)$ is given continuous function in $0 \leq s \leq l$; $\tau_1(y)$ and $\tau_2(x)$ are continuous functions at $0 \leq x, y \leq a$; $\tau_1(0) = \tau_2(0)$, $\tau_1(a) = \varphi(l)$, $\tau_2(a) = \varphi(0)$.

Theorem 3 . *If the Dirichlet problem has a regular solution, then it is unique.*

Proof. Consider the domain $D_{\varepsilon, \delta_1, \delta_2} \subset D$, bounded by the curve Γ_{ε} , parallel to the curve Γ , and line segments $x = \delta_1 > \varepsilon$ and $y = \delta_2 > \varepsilon$.

Integrating both sides of the identity (15) along the domain D_{ε} and using the Gauss-Ostrogradsky formula, we obtain

$$\begin{aligned} \int \int_{D_{\varepsilon, \delta_1, \delta_2}} x^{2\alpha} y^{2\beta} [uE(v) - vE(u)] dx dy = \\ = \int_{S_{\varepsilon, \delta_1, \delta_2}} (uA_n^{\alpha, \beta}[v] - vA_n^{\alpha, \beta}[u]) dS_{\varepsilon, \delta_1, \delta_2}, \end{aligned}$$

where $S_{\varepsilon, \delta_1, \delta_2}$ is a contour of the domain $D_{\varepsilon, \delta_1, \delta_2}$.

One can easily check that the following equality holds:

$$\begin{aligned} \int \int_{D_{\varepsilon, \delta_1, \delta_2}} x^{2\alpha} y^{2\beta} u E(u) dx dy &= \int \int_{D_{\varepsilon, \delta_1, \delta_2}} x^{2\alpha} y^{2\beta} [u_x^2 + u_y^2] dx dy - \\ &- \int \int_{D_{\varepsilon, \delta_1, \delta_2}} \left[y^{2\beta} \frac{\partial}{\partial x} (x^{2\alpha} u u_x) + x^{2\alpha} \frac{\partial}{\partial y} (y^{2\beta} u u_y) \right] dx dy. \end{aligned}$$

Application of the Ostrogradsky formula to this equality after $\delta_1 \rightarrow 0$, $\delta_2 \rightarrow 0$ and $\varepsilon \rightarrow 0$ yields

$$\begin{aligned} \int \int_D x^{2\alpha} y^{2\beta} [u_x^2 + u_y^2] dx dy &= - \int_{\Gamma} \varphi(s) A_n^{\alpha, \beta} [u] ds + \\ &+ \int_0^a x^{2\alpha} \frac{\partial u}{\partial x} \Big|_{x=0} \cdot y^{2\beta} \tau_1(y) dy + \int_0^a y^{2\beta} \frac{\partial u}{\partial y} \Big|_{y=0} \cdot x^{2\alpha} \tau_2(x) dx. \end{aligned} \quad (45)$$

If we consider the homogeneous Dirichlet problem, then we find from (45):

$$\int \int_D x^{2\alpha} y^{2\beta} [u_x^2 + u_y^2] dx dy = 0.$$

Hence, it follows that $u(x, y) = 0$ in \bar{D} .

8 Green's Function Revisited

To solve this problem, we use the Green's function method. First, we construct the Green's function for solving the Dirichlet problem for an equation in a domain which is bounded by an arbitrary curve and two mutually perpendicular line segments. We then show that, in view of the Green's function, the solution of the Dirichlet problem in a quadrant takes a simpler form as described below.

Definition 2 . We refer to $G(x, y; x_0, y_0)$ as Green's function of the Dirichlet problem, if it satisfies following conditions:

1) The function $G(x, y; x_0, y_0)$ is a regular solution of equation (2) in the domain D , except at the point (x_0, y_0) , which is any fixed point of D .

2) The function $G(x, y; x_0, y_0)$ satisfies the boundary conditions given by

$$G(x, y; x_0, y_0)|_{\Gamma} = 0, \quad G(x, y; x_0, y_0)|_{x=0} = 0, \quad G(x, y; x_0, y_0)|_{y=0} = 0; \quad (46)$$

3) The function $G(x, y; x_0, y_0)$ can be represented as follows:

$$G(x, y; x_0, y_0) = q(x, y; x_0, y_0) + v(x, y; x_0, y_0) \quad (47)$$

where $q(x, y; x_0, y_0)$ is a fundamental solution of the equation (2), defined in the domain D , and the function $v(x, y; x_0, y_0)$ is a regular solution of the equation (2) in the domain D .

The construction of the Green's function $G(x, y; x_0, y_0)$ reduces to finding its regular part $v(x, y; x_0, y_0)$ which, by virtue of (14), (46) and (47), must satisfy the following boundary conditions:

$$v(x, y; x_0, y_0)|_{\Gamma} = -q(x, y; x_0, y_0)|_{\Gamma}, \quad (48)$$

$$v(x, y; x_0, y_0)|_{x=0} = 0, \quad v(x, y; x_0, y_0)|_{y=0} = 0.$$

We now look for the function $v(x, y; x_0, y_0)$ in the form of a double-layer potential given by

$$v(x, y; x_0, y_0) = \int_0^l \mu(t; x_0, y_0) A_{\nu}^{\alpha, \beta}[q(\xi, \eta; x, y)] dt. \quad (49)$$

By taking into account the equality (31) and the boundary condition (48), we obtain the integral equation for the density $\mu(t; x_0, y_0)$ as follows:

$$\mu(s; x_0, y_0) - 2 \int_0^l K(s, t) \mu(t; x_0, y_0) dt = 2q(x(s), y(s); x_0, y_0). \quad (50)$$

The right-hand side of (50) is a continuous function of s (the point (x_0, y_0) lies inside D). In Section 6, it was proved that $\lambda = 2$ is not an eigenvalue of the kernel $K(s, t)$ and, consequently, the Equation (50) is solvable and its continuous solution can be written in the following form:

$$\mu(s; x_0, y_0) = 2q(x(s), y(s); x_0, y_0) + 4 \int_0^l R(s, t; 2) q(\xi, \eta; x_0, y_0) dt, \quad (51)$$

where $R(s, t; 2)$ is the resolvent of kernel $K(s, t)$; $(x(s), y(s)) \in \Gamma$. Thus, upon substituting from (51) into (49), we obtain

$$\begin{aligned} v(x, y; x_0, y_0) &= 2 \int_0^l q(\xi, \eta; x_0, y_0) A_{\nu}^{\alpha, \beta}[q(\xi, \eta; x, y)] dt + \\ &+ 4 \int_0^l \int_0^l A_{\nu}^{\alpha, \beta}[q(\xi, \eta; x, y)] R_0(t, s; 2) q(x(s), y(s); x_0, y_0) dt ds. \end{aligned} \quad (52)$$

We now define the function $g(x, y)$ as follows:

$$g(x, y) = \begin{cases} v(x, y; x_0, y_0), & (x, y) \in D, \\ -q(x, y; x_0, y_0), & (x, y) \in D'. \end{cases} \quad (53)$$

The function $g(x, y)$ is a regular solution of equation (2) both inside the domain D and inside D' and equal to zero at infinity. Because the point (x_0, y_0) lies inside D , therefore, in D' , the function $g(x, y)$ has derivatives of any order in all variables that are continuous up to Γ . We can consider $g(x, y)$ in D' as a solution of Equation (2) satisfying the boundary conditions given by

$$A_n^{\alpha, \beta}[g(x, y)]|_{\Gamma} = -A_n^{\alpha, \beta}[q(x(s), y(s); x_0, y_0)],$$

$$g(x, y)|_{x=0} = 0, \quad g(x, y)|_{y=0} = 0.$$

We represent this solution in the form of a simple-layer potential as follows:

$$g(x, y) = \int_0^l \rho(t; x_0, y_0) q(\xi, \eta; x, y) dt, \quad (x, y) \in D' \quad (54)$$

with an unknown density $\rho(t; x_0, y_0)$.

Using the formula (35), we obtain the following integral equation for the density $\rho(s; x_0, y_0)$:

$$\rho(s; x_0, y_0) - 2 \int_0^l K(t, s) \rho(t; x_0, y_0) dt = 2A_n^{\alpha, \beta} [q(x(s), y(s); x_0, y_0)]. \quad (55)$$

Equation (55) is conjugated with the equation (50). Its right-hand side is a continuous function of s . Thus, clearly, the equation (55) has the following continuous solution:

$$\begin{aligned} \rho(s; x_0, y_0) = & 2A_n^{\alpha, \beta} [q(x(s), y(s); x_0, y_0)] + \\ & + 4 \int_0^l R(t, s; 2) A_\nu^{\alpha, \beta} [q(\xi, \eta; x_0, y_0)] dt. \end{aligned} \quad (56)$$

The values of a simple-layer potential $g(x, y)$ on the curve Γ are equal to $-q(x, y; x_0, y_0)$, that is, just as the functions $v(x, y; x_0, y_0)$ and on the axes x and y their partial derivatives with respect to y and x multiplied, respectively, by $y^{2\beta}$ and $x^{2\alpha}$ are equal to zero. Hence, by virtue of the uniqueness theorem for the Dirichlet problem, it follows that the formula (54) for the function $g(x, y)$ defined by (53) holds throughout in the quarter-plane $x \geq 0, y \geq 0$, that is,

$$v(x, y; x_0, y_0) = \int_0^l \rho(t; x_0, y_0) q(\xi, \eta; x, y) dt, \quad (x, y) \in D. \quad (57)$$

Thus, the regular part $v(x, y; x_0, y_0)$ of Green's function is representable in the form of a simple-layer potential.

Applying the formula (35) to (57), we obtain

$$2A_n^{\alpha, \beta} [v(x(s), y(s); x_0, y_0)]_i = \rho(s; x_0, y_0) + 2 \int_0^l K(t, s) \rho(t; x_0, y_0) dt,$$

But, according to (55), we have

$$2A_n^{\alpha, \beta} [q(x(s), y(s); x_0, y_0)]_i = \rho(s; x_0, y_0) - 2 \int_0^l K(t, s) \rho(t; x_0, y_0) dt.$$

Summing the last two equalities by term-by-term and taking equation (47) into account, we find that

$$A_n^{\alpha, \beta} [G(x(s), y(s); x_0, y_0)] = \rho(s; x_0, y_0). \quad (58)$$

Consequently, formula (57) can be written in the following form:

$$v(x, y; x_0, y_0) = \int_0^l A_\nu^{\alpha, \beta} [G(\xi, \eta; x_0, y_0)] q(\xi, \eta; x, y) dt.$$

Multiplying both sides of (56) by $q(x(s), y(s); x, y)$, integrating by s over the curve Γ from 0 to l and, by virtue of (51) and (49), we obtain

$$v(x_0, y_0; x, y) = \int_0^l \rho(t; x_0, y_0) q(\xi, \eta; x, y) dt.$$

Comparing this last equation with the formula (57), we have

$$v(x, y; x_0, y_0) = v(x_0, y_0; x, y). \quad (59)$$

if the points (x, y) and (x_0, y_0) are inside the domain D .

Lemma 5 . *If points (x, y) and (x_0, y_0) are inside domain D , then Green's function $G(x, y; x_0, y_0)$ is symmetric about those points.*

Proof. The proof of Lemma 5 follows from the representation (47) of Green's function and the equality (59).

For a quarter circle D_0 bounded by two segments $[0, a]$ of the axes x and y and a quarter circle given by $x^2 + y^2 = a^2$ ($x \geq 0, y \geq 0$), the Green's function of the Dirichlet problem has the following form

$$G_0(x, y; x_0, y_0) = q(x, y; x_0, y_0) - \left(\frac{a}{R}\right)^{2\alpha+2\beta} q(x, y; \bar{x}_0, \bar{y}_0), \quad (60)$$

where

$$R^2 = x_0^2 + y_0^2, \quad \bar{x}_0 = \frac{a^2}{R^2} x_0, \quad \bar{y}_0 = \frac{a^2}{R^2} y_0.$$

We now show that the function given by

$$v_0(x, y; x_0, y_0) = - \left(\frac{a}{R}\right)^{2\alpha+2\beta} q(x, y; \bar{x}_0, \bar{y}_0)$$

can be represented in the following form:

$$v_0(x, y; x_0, y_0) = - \int_0^l \rho(s; x, y) v_0(x(s), y(s); x_0, y_0) ds, \quad (61)$$

where $\rho(s; x, y)$ is a solution of equation (57).

Indeed, by letting an arbitrary point (x_0, y_0) be inside the domain D , we consider the function given by

$$u(x, y; x_0, y_0) = - \int_0^l \rho(s; x, y) v_0(x(s), y(s); x_0, y_0) ds.$$

As a function of (x, y) , the function $u(x, y; x_0, y_0)$ satisfies equation (2), because this equation is satisfied by the function $\rho(s; x, y)$. Substituting the expression (56) for $\rho(s; x, y)$, we obtain

$$u(x, y; x_0, y_0) = - \int_0^l \psi(s; x_0, y_0) A_n^{\alpha, \beta} [q(x(s), y(s); x, y)] ds, \quad (62)$$

where

$$\psi(s; x_0, y_0) = 2v_0(x(s), y(s); x_0, y_0) + 4 \int_0^l R(s, t; 2)v_0(\xi, \eta; x_0, y_0) dt,$$

that is, $\psi(s; x_0, y_0)$ is a solution of the integral equation

$$\psi(s; x_0, y_0) - 2 \int_0^l K(s, t)\psi(t; x_0, y_0)dt = 2v_0(x(s), y(s); x_0, y_0). \quad (63)$$

Applying formula (31) to the double-layer potential (62), we obtain

$$u_i(x(s), y(s); x_0, y_0) = \frac{1}{2}\psi(s; x_0, y_0) - \int_0^l K(s, t)\psi(t; x_0, y_0)dt,$$

whence, by virtue of (63) we get

$$u_i(x(s), y(s); x_0, y_0) = v_0(x(s), y(s); x_0, y_0), (x(s), y(s)) \in \Gamma.$$

It is easy to see that

$$\begin{aligned} u(x, y; x_0, y_0)|_{x=0} &= 0, & v_0(x, y; x_0, y_0)|_{x=0} &= 0, \\ u(x, y; x_0, y_0)|_{y=0} &= 0, & v_0(x, y; x_0, y_0)|_{y=0} &= 0. \end{aligned}$$

Thus, clearly, the functions $u(x, y; x_0, y_0)$ and $v_0(x, y; x_0, y_0)$ satisfy the same equation (2) and the same boundary conditions. Also, by virtue of the uniqueness of the solution of the Dirichlet problem, the equality

$$u(x, y; x_0, y_0) \equiv v_0(x, y; x_0, y_0).$$

is satisfied.

Now, subtracting the expression (60) from (47), we obtain

$$\begin{aligned} H(x, y; x_0, y_0) &= G(x, y; x_0, y_0) - G_0(x, y; x_0, y_0) = \\ &= v(x, y; x_0, y_0) - v_0(x, y; x_0, y_0) \end{aligned}$$

or, by virtue of (57), (59), (60) and (61), we obtain

$$H(x, y; x_0, y_0) = \int_0^l \rho(t; x, y)G_0(\xi, \eta; x_0, y_0)dt. \quad (64)$$

Solving the Dirichlet Problem for Equation (2)

Theorem 4 . *The following function*

$$\begin{aligned}
u(x_0, y_0) &= \int_0^a y^{2\beta} \left(x^{2\alpha} \frac{\partial G(x, y; x_0, y_0)}{\partial x} \right) \Big|_{x=0} \tau_1(y) dy + \\
&+ \int_0^a x^{2\alpha} \left(y^{2\beta} \frac{\partial G(x, y; x_0, y_0)}{\partial y} \right) \Big|_{y=0} \tau_2(x) dx - \\
&- \int_0^l A_\nu^{\alpha, \beta} [G(\xi, \eta; x_0, y_0)] \varphi(s) ds \\
&= I_1(x_0, y_0) + I_2(x_0, y_0) + I_3(x_0, y_0),
\end{aligned} \tag{65}$$

where $\varphi(s)$ is given continuous function in $0 \leq s \leq l$; $\tau_1(y)$ and $\tau_2(x)$ are given continuous functions in $0 \leq x, y \leq a$ with $\tau_1(0) = \tau_2(0)$, $\tau_1(a) = \varphi(l)$, $\tau_2(a) = \varphi(0)$, is the solution of the Dirichlet problem for equation (2) in the domain D .

Proof. Let (x_0, y_0) be a point inside the domain D . Consider the domain $D_{\varepsilon, \delta_1, \delta_2} \subset D$ bounded by the curve Γ_ε , which is parallel to the curve Γ , and the line segments $x = \delta_1 > \varepsilon$ and $y = \delta_2 > \varepsilon$.

We choose ε , δ_1 and δ_2 to be so small that the point (x_0, y_0) is inside $D_{\varepsilon, \delta_1, \delta_2}$. We cut out from the domain $D_{\varepsilon, \delta_1, \delta_2}$ a circle of small radius ρ with center at the point (x_0, y_0) , and we denote the remainder part of $D_{\varepsilon, \delta_1, \delta_2}$ by $D_{\varepsilon, \delta}^\rho$, in which the Green's function $G(x, y; x_0, y_0)$ is a regular solution of equation (2).

Let $u(x, y)$ be a regular solution of the equation (2) in the domain D that satisfies the boundary conditions (43) and (44). Applying the formula (17), we obtain

$$\begin{aligned}
\int_{C_\rho} (GA_n^{\alpha, \beta}[u] - uA_n^{\alpha, \beta}[G]) ds &= \int_{\delta_2}^{y_1} x^{2\alpha} y^{2\beta} \left(u \frac{\partial G}{\partial x} - G \frac{\partial u}{\partial x} \right) \Big|_{x=\delta_1} dy + \\
+ \int_{\delta_1}^{x_1} x^{2\alpha} y^{2\beta} \left(u \frac{\partial G}{\partial y} - G \frac{\partial u}{\partial y} \right) \Big|_{y=\delta_2} dx &+ \int_{\Gamma_\varepsilon} (GA_n^{\alpha, \beta}[u] - uA_n^{\alpha, \beta}[G]) ds,
\end{aligned}$$

x_1 and y_1 are an abscissa and ordinate of the intersection points of the curve Γ_ε with the straight lines $y = \delta_2$ and $x = \delta_1$, respectively, and C_ρ is a circumference of the cut circle.

Proceeding to the limit as $\rho \rightarrow 0$ and then as $\varepsilon \rightarrow 0$, $\delta_1 \rightarrow 0$ and $\delta_2 \rightarrow 0$, we obtain the formula (65).

We show that the formula (65) gives a solution of the Holmgren problem.

It is easy to see that the first integral $I_1(x_0, y_0)$ in the formula (65) is a solution of the equation (2) and is regular in the domain D , continuous in \bar{D} .

We use the following notation:

$$\begin{aligned}
\vartheta(x_0, y_0) &= \int_0^a y^{2\beta} \left(x^{2\alpha} \frac{\partial q(x, y; x_0, y_0)}{\partial x} \right) \Big|_{x=0} \tau_1(y) dy = (1 - 2\alpha)\kappa \times \\
&\times x_0^{1-2\alpha} y_0^{1-2\beta} \int_0^a \frac{yF\left(\beta - \alpha, 1 - \beta; 2 - 2\beta; \frac{4yy_0}{x_0^2 + (y + y_0)^2}\right)}{[x_0^2 + (y - y_0)^2]^{1-\alpha} [x_0^2 + (y + y_0)^2]^{1-\beta}} \tau_1(y) dy.
\end{aligned} \tag{66}$$

Here, $\vartheta(x_0, y_0)$ is a continuous function in \bar{D} . In view of (66) and (52) and the symmetry of the function $v(x, y; x_0, y_0)$, the integral $I_1(x_0, y_0)$ can be represented in the following form:

$$\begin{aligned}
I_1(x_0, y_0) = & \vartheta(x_0, y_0) + 2 \int_0^l \vartheta(\xi, \eta) A_{\nu}^{\alpha, \beta}[q(\xi, \eta; x_0, y_0)] dt + \\
& + 4 \int_0^l \int_0^l R(t, s; 2) \vartheta(x(s), y(s)) A_{\nu}^{\alpha, \beta}[q(\xi, \eta; x_0, y_0)] dt ds.
\end{aligned} \tag{67}$$

The last two integrals in the formula (67) are double-layer potentials. Taking into account the formula (31) and the integral equation for the resolvent $R(s, t; 2)$ from formula (67), we obtain

$$I_1(x_0, y_0)|_{\Gamma} = 0,$$

It is easy to see that

$$\lim_{x_0 \rightarrow 0} u(x, y) = \tau_1(y_0) \quad (0 \leq y_0 \leq a).$$

In fact, by virtue of (57) and the symmetry of the function $v(x, y; x_0, y_0)$, the above integral can also be written in the following form:

$$\begin{aligned}
I_1(x_0, y_0) = & \int_0^a \tau_1(y) q(0, y; x_0, y_0) dy + \\
& + \int_0^a \tau_1(y) dy \int_0^l \rho(t; 0, y) q(\xi, \eta; x_0, y_0) dt.
\end{aligned}$$

Following the work [7], it is easy to show that

$$\lim_{x_0 \rightarrow 0} \int_0^a \tau_1(y) q(0, y; x_0, y_0) dy = \tau(y_0) \quad (0 \leq y_0 \leq a)$$

and

$$\lim_{x_0 \rightarrow 0} \int_0^a \tau_1(y) dy \int_0^l \rho(t; 0, y) q(\xi, \eta; x_0, y_0) dt = 0 \quad (0 \leq y_0 \leq a),$$

because

$$q(\xi, \eta; x_0, y_0) = 0$$

when $x_0 = 0$, $0 \leq y_0 \leq a$.

By virtue of the last from the conditions (46), we have

$$\lim_{y_0 \rightarrow 0} u(x, y) = 0 \quad (0 \leq x_0 \leq a).$$

Similarly, we get

$$I_2(x_0, y_0)|_{\Gamma} = 0; \quad \lim_{x_0 \rightarrow 0} I_2(x_0, y_0) = 0, \quad \lim_{y_0 \rightarrow 0} I_2(x_0, y_0) = \tau_2(x_0).$$

We consider the third integral $I_3(x_0, y_0)$ in the formula (65), which, by virtue of (58) and (56), can be written in the following form:

$$I_3(x_0, y_0) = - \int_0^l \varphi(s) \rho(s; x_0, y_0) ds = - \int_0^l \theta(t) A_\nu^{\alpha, \beta} [q(\xi, \eta; x_0, y_0)] dt,$$

where

$$\theta(t) = 2\varphi(t) + 4 \int_0^l R(t, s; 2) \varphi(s) ds,$$

that is, the function $\theta(s)$ is a solution of the integral equation

$$\theta(s) - 2 \int_0^l K(s, t) \theta(t) dt = 2\varphi(s). \quad (68)$$

Because $\theta(s)$ is a continuous function, $I_3(x_0, y_0)$ is a solution of Equation (2), regular in the domain D , that is continuous in \bar{D} , which, by virtue of (31) and (68), satisfies following condition:

$$I_3(x_0, y_0)|_\Gamma = \varphi(s).$$

It is now easy to see that

$$\lim_{x_0 \rightarrow 0} I_3(x_0, y_0) = 0 \quad (0 \leq y_0 \leq a), \quad \lim_{y_0 \rightarrow 0} I_3(x_0, y_0) = 0 \quad (0 \leq x_0 \leq a).$$

Theorem 4 is proved.

By using formulas (64) and (60), solution (65) of the Dirichlet problem given by (43) and (44) for Equation (2) can be written in the following form:

$$\begin{aligned} u(x_0, y_0) &= \\ &= \int_0^a \tau_1(y) y^{2\beta} \cdot x^{2\alpha} \frac{\partial}{\partial x} [G_0(x, y; x_0, y_0) + H(x, y; x_0, y_0)] \Big|_{x=0} dy + \\ &+ \int_0^a \tau_2(x) x^{2\alpha} \cdot y^{2\beta} \frac{\partial}{\partial y} [G_0(x, y; x_0, y_0) + H(x, y; x_0, y_0)] \Big|_{y=0} dx - \\ &- \int_0^l \varphi(s) \{ A_\nu^{\alpha, \beta} [G_0(\xi, \eta; x_0, y_0)] + A_\nu^{\alpha, \beta} [H(\xi, \eta; x_0, y_0)] \} ds, \end{aligned} \quad (69)$$

where

$$H(x, y; x_0, y_0) = \int_0^l \rho_0(t; x_0, y_0) G_0(\xi, \eta; x, y) dt.$$

We remark that solution (69) of the Dirichlet problem is more convenient for further investigations.

In the case of a quarter circle D_0 , the function $H(x, y; x_0, y_0) \equiv 0$ and solution (69) assumes a simpler form as follows:

$$\begin{aligned}
u(x_0, y_0) &= \\
&= (1 - 2\alpha)\kappa x_0^{1-2\alpha} y_0^{1-2\beta} \int_0^a \tau_1(y) y \left[\frac{\tilde{F}_1\left(-\frac{4yy_0}{X_1^2}\right)}{X_1^{4-2\alpha-2\beta}} - \frac{\tilde{F}_1\left(-\frac{4yy_0}{Y_1^2}\right)}{Y_1^{4-2\alpha-2\beta}} \right] dy + \\
&+ (1 - 2\beta)\kappa x_0^{1-2\alpha} y_0^{1-2\beta} \int_0^a \tau_2(x) x \left[\frac{\tilde{F}_2\left(-\frac{4xx_0}{X_2^2}\right)}{X_2^{4-2\alpha-2\beta}} - \frac{\tilde{F}_2\left(-\frac{4xx_0}{Y_2^2}\right)}{Y_2^{4-2\alpha-2\beta}} \right] dx - \\
&+ 2(2 - \alpha - \beta)\kappa x_0^{1-2\alpha} y_0^{1-2\beta} \int_0^l \varphi(s)\xi(s)\eta(s) \frac{R^2 - a^2}{r_{12}^{6-2\alpha-2\beta}} \times \\
&\times F_2\left(3 - \alpha - \beta, 1 - \alpha, 1 - \beta; 2 - 2\alpha, 2 - 2\beta; \frac{r_1^2 - r^2}{r_{12}^2}, \frac{r_2^2 - r^2}{r_{12}^2}\right) ds,
\end{aligned} \tag{70}$$

where

$$\tilde{F}_1(z) = F(2 - \alpha - \beta, 1 - \beta; 2 - 2\beta; z), \quad \tilde{F}_2(z) = F(2 - \alpha - \beta, 1 - \alpha; 2 - 2\alpha; z);$$

$$R^2 = x_0^2 + y_0^2, \quad a^2 = \xi^2 + \eta^2; \quad r^2 = (\xi - x_0)^2 + (\eta - y_0)^2,$$

$$r_1^2 = (\xi + x_0)^2 + (\eta - y_0)^2, \quad r_2^2 = (\xi - x_0)^2 + (\eta + y_0)^2;$$

$$X_1^2 = x_0^2 + (y - y_0)^2, \quad Y_1^2 = \left(a - \frac{yy_0}{a}\right)^2 + \frac{y^2}{a^2} x_0^2;$$

$$X_2^2 = (x - x_0)^2 + y_0^2, \quad Y_2^2 = \left(a - \frac{xx_0}{a}\right)^2 + \frac{x^2}{a^2} y_0^2.$$

The resulting explicit integral representations (69) and (70) play an important role in the study of problems for equation of the mixed type (that is, elliptic-hyperbolic or elliptic-parabolic types): they make it easy to derive the basic functional relationship between the traces of the sought solution and of its derivative on the line of degeneration from the elliptic part of the mixed domain.

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