IRSTI 27.31.17 DOI: https://doi.org/10.26577/JMMCS.2021.v111.i3.06

M.I. Ramazanov, N.K. Gulmanov*
E. A. Academician E.A. Buketov Karaganda University, Kazakhstan, Karaganda
*e-mail: gulmanov.nurtay@gmail.com

SOLUTION OF A TWO-DIMENSIONAL BOUNDARY VALUE PROBLEM OF HEAT CONDUCTION IN A DEGENERATING DOMAIN

In the paper we consider the boundary value problem of heat conduction outside the cone, i.e. in the domain degenerating into a point at the initial moment of time. In this case, the boundary condition contain a derivative with respect to the time variable. The peculiarity of the problem under consideration consists precisely in the presence of a moving boundary and the degeneration of the solution domain into a point at the initial moment of time. The well-known classical methods are generally not applicable to this type of problems. By the method of heat potentials, such boundary value problems of heat conduction are reduced to the solution of singular Volterra type integral equations of the second kind A singular Volterra type equation is understood as an equation whose kernel has the following property: the integral of the kernel of the equation does not tend to zero as the upper limit tends to the lower one. Such integral equations cannot be solved by the method of successive approximations, and in most cases the corresponding homogeneous integral equations have nonzero solutions. We prove a theorem on the solvability of the considered boundary value problem in weighted spaces of essentially bounded functions. The issues of solvability of the singular Volterra integral equation of the second kind, to which the original problem is reduced, are studied. We found a nonzero solution of this singular integral equation.

Key words: noncylindrical domain, cone, boundary value problem of heat conduction, singular Volterra integral equation, regularization.

М.И. Рамазанов, Н.К. Гульманов*
Академик Е.А. Бөкетов атындағы Қарағанды мемлекеттік университеті, Қазақстан, Қарағанды қ.
*e-mail: gulmanov.nurtay@gmail.com

ЖОЙЫЛАТЫН ОБЛЫСТА ЖЫЛУ ӨТКІЗГІШТІКТІҢ ЕКІ ӨЛШЕМДІ ШЕКАРАЛЫҚ ЕСЕБІН ШЕШУ

Мақалада конустан тыс, яғни бастапқы уақытта нүктеге айналатын облыста жылу өткізгіштіктің шекаралық есебі қарастырылады. Мұндағы шекаралық шарт уақыт бойынша алынған туындыны қамтиды. Қарастырылып отырған есеп жылжымалы шекараның бар болуымен және бастапқы уақытта шешу облысының нүктеге айналуымен ерекшеленеді. Мұндай түрдегі есептерді жалпы жағдайда белгілі классикалық әдістермен шешуге болмайды. Бұл шекаралық есептер жылу потенциалдарының әдісімен екінші текті Вольтерра типіндегі сингулярлық интегралдық теңдеулерді шешуге келтіріледі. Вольтерра типіндегі сингулярлық теңдеуді ядросы келесі қасиетке ие болатын теңдеу деп түсіну керек: жоғарғы шек төменгі шекке ұмтылғанда ядродан алынған интеграл нөлге ұмтылмайды. Елеулі шенелген функциялар кеңістігінде қарастырылатын шекаралық есептің шешімділігі туралы теорема дәлелденді. Берілген есеп түрленетін екінші текті Вольтерраның сингуялрлық интегралдық теңдеуінің шешімділігі туралы маселелер зерттелді. Осы сингулярлык интегралдык теңдеудің нөлдік емес шешімі табылды.

Түйін сөздер: цилиндрлік емес облыс, конус, жылу өткізгіштіктің шекаралық есебі, Вольтерраның сингулярлық интегралдық теңдеуі, регуляризация.

М.И. Рамазанов, Н.К. Гульманов*
Карагандинский университет имени академика Е.А. Букетова, Казахстан, г. Караганда
*e-mail: gulmanov.nurtay@gmail.com

РЕШЕНИЕ ДВУМЕРНОЙ ГРАНИЧНОЙ ЗАДАЧИ ТЕПЛОПРОВОДНОСТИ В ВЫРОЖДАЮЩЕЙСЯ ОБЛАСТИ

В работе рассматривается краевая задача теплопроводности вне конуса, то есть в области вырождающейся в точку в начальный момент времени. При этом граничное условие содержит производную по временной переменной. Особенность рассматриваемой задачи состоит именно в наличии подвижной границы и вырождения области решения в начальный момент времени в точку. К этому типу задач в общем случае не применимы известные классические методы. Методом тепловых потенциалов подобные краевые задачи теплопроводности редуцируются к решению сингулярных интегральных уравнений типа Вольтерра второго рода. Под сингулярным уравнением типа Вольтерра подразумевается уравнение, ядро которого обладает следующим свойством: интеграл от ядра уравнения при стремлении верхнего предела к нижнему не стремится к нулю. Такого рода интегральные уравнения нельзя решить методом последовательных приближений и в большинстве случаев соответствующие однородные интегральные уравнения имеют ненулевые решения. Доказана теорема о разрешимости рассматриваемой краевой задачи в весовых пространствах существенно ограниченных функций. Исследованы вопросы разрешимости сингулярного интегрального уравнения Вольтерра второго рода, к которому редуцирована исходная задача. Найдено ненулевое решение этого сингулярного интегрального уравнения.

Ключевые слова: нецилиндрическая область, конус, краевая задача теплопроводности, сингулярное интегральное уравнение Вольтерра, регуляризация.

1 Introduction

In the literature, a domain is usually called non-cylindrical if at least one of the parts of its boundary moves with time. The domain, the boundary of which does not change its shape with time, is cylindrical, for such domains, the theory of boundary value problems of heat conduction is well developed.

In most papers, the domain in which the solution of the boundary value problem is sought does not degenerate into a point at the initial moment of time. In [1–6] authors for solving such problems used a technique which consists in reducing a non-cylindrical domain to a cylindrical one. There are a number of works devoted to numerical methods for solving such problems [7–9].

Of particular interest are the boundary value problems of heat conduction in domains that degenerate into a point at the initial moment of time. For example, in the study of thermophysical processes in an electric arc of high-current disconnecting devices, the effect of contracting the axial section of the arc into a contact spot in the cathode field is observed [10–14]. The solution domain changes over time according to the law determined by the conditions for opening the contacts. At the initial moment of time, the contacts are in a closed state and there is no domain for solving the problem. One-dimensional with respect to the spatial variable boundary value problems in degenerate domains were studied in [15–20].

In this paper, we consider a two-dimensional boundary value problem of heat conduction outside the cone with boundary conditions containing the time derivative. By the method of heat potentials, boundary value problem of heat conduction is reduced to the solution of singular Volterra type integral equation of the second kind. In order to solve the obtained singular Volterra integral equation we use the method of equivalent regularization by solving the characteristic equation.

2 Statement of the boundary value problem

We consider in the domain $G = \{(r,t) : r > t > 0\}$ the following boundary value problem:

$$\frac{\partial u}{\partial t} - a^2 \cdot \frac{1}{r^{2\nu-1}} \frac{\partial}{\partial r}\left(r^{2\nu-1}\frac{\partial u}{\partial r}\right) = 0, \tag{1}$$

$$\left.\left(2 \cdot \frac{\partial u}{\partial r} + \frac{\partial u}{\partial t}\right)\right|_{r=t} = g\left(t\right), \tag{2}$$

$$\left. r^{2\nu-1}\frac{\partial u\left(r,t\right)}{\partial r}\right|_{r=0} = 0, \tag{3}$$

where $0 < \nu < 1$.

Remark 1 *Solution of the problem (1)–(3) for $g(t) = 0$, i.e. solution of a complete homogeneous problem, can be only a constant.*

3 Main result

For the problem (1)–(3), we proved the following theorem.

Theorem 1 *If the condition $t^{\nu-\frac{1}{2}}g\left(t\right) \in L_\infty\left(0,\infty\right)$ is satisfied, then the boundary value problem (1)–(3) has a solution $u\left(r,t\right) = \tilde{u}\left(r,t\right) + C, \;\; \tilde{u}\left(r,t\right) \in L_\infty\left(G\right), \;\; C = const.$*

4 Boundary value problem transformation

We make some transformations of the problem (1)–(3) by introducing a new unknown function::

$$\omega\left(r,t\right) = r^{2\nu-1}\frac{\partial u}{\partial r}. \tag{4}$$

Then, taking into account (4), problem (1)–(3) is reduced to the following one:

$$\frac{\partial \omega}{\partial t} = a^2\frac{\partial^2 \omega}{\partial r^2} - a^2\frac{2\nu-1}{r}\frac{\partial \omega}{\partial r}, \tag{5}$$

$$\left.\frac{a^2}{r^{2\nu-1}}\left(\frac{\partial \omega}{\partial r} + \frac{2}{a^2}\omega\right)\right|_{r=t} = g\left(t\right), \tag{6}$$

$$\left.\omega\left(r,t\right)\right|_{r=0} = q\left(t\right). \tag{7}$$

5 Integral representation of the solution of the problem (5)–(7) using heat potentials

We seek the solution of problem (5)–(7) as a single layer heat potential:

$$\omega(r,t) = \int_0^t G(r,\xi,t-\tau)\big|_{\xi=\tau} \cdot \varphi(\tau)\, d\tau, \tag{8}$$

where function

$$G(r,\xi,t-\tau) = \frac{1}{2a^2} \cdot \frac{r^\nu \cdot \xi^{1-\nu}}{t-\tau} \cdot \exp\left[-\frac{r^2+\xi^2}{4a^2(t-\tau)}\right] \cdot I_\nu\left(\frac{r\xi}{2a^2(t-\tau)}\right)$$

is the fundamental solution of the equation (5), ξ is a parameter. Hereinafter, $I_\nu(z)$ is the modified Bessel function of order ν. Function (8) satisfies our equation (5) for any potential density $\varphi(t)$ from the class

$$t^{\frac{1}{2}-\nu} e^{\frac{t}{4a^2}} \varphi(t) \in L_\infty(0,\infty). \tag{9}$$

6 Reduction of boundary value problem (5)–(7) to a singular Volterra type integral equation

Using the value of the derivative:

$$\frac{\partial \omega(r,t)}{\partial r} = -\int_0^t \frac{r^\nu \tau^{1-\nu}(r-\tau)}{4a^4(t-\tau)^2} \exp\left[-\frac{r^2+\tau^2}{4a^2(t-\tau)}\right] I_\nu\left(\frac{r\tau}{2a^2(t-\tau)}\right) \varphi(\tau)\, d\tau +$$
$$+\int_0^t \frac{r^\nu \tau^{2-\nu}}{4a^4(t-\tau)^2} \exp\left[-\frac{r^2+\tau^2}{4a^2(t-\tau)}\right] I_{\nu-1,\nu}\left(\frac{r\tau}{2a^2(t-\tau)}\right) \varphi(\tau)\, d\tau + \frac{\partial \tilde{q}(r,t)}{\partial r},$$

where the notation $I_{\nu-1,\nu}(z) = I_{\nu-1}(z) - I_\nu(z)$ is used, and satisfying the boundary condition (6), we obtain a singular Volterra integral equation of the second kind with respect to the required density $\varphi(t)$:

$$\varphi(t) - \int_0^t \frac{t^\nu \tau^{2-\nu}}{2a^2(t-\tau)^2} \exp\left[-\frac{t\tau}{2a^2(t-\tau)}\right] I_{\nu-1,\nu}\left(\frac{t\tau}{2a^2(t-\tau)}\right) e^{-\frac{t-\tau}{4a^2}} \varphi(\tau)\, d\tau -$$
$$-\int_0^t \frac{3t^\nu \tau^{1-\nu}}{2a^2(t-\tau)} \exp\left[-\frac{t\tau}{2a^2(t-\tau)}\right] I_\nu\left(\frac{t\tau}{2a^2(t-\tau)}\right) e^{-\frac{t-\tau}{4a^2}} \varphi(\tau)\, d\tau = F(t), \tag{10}$$

where

$$F(t) = 2t^{2\nu-1}g(t).$$

We write equation (10) in the following form:

$$\varphi(t) - \int_0^t \left[\frac{\tau^{2-\nu}}{t^{2-\nu}} \cdot e^{-\frac{t-\tau}{4a^2}}\right] N_1(t,\tau)\,\varphi(\tau)\,d\tau - \int_0^t \left[\frac{\tau^{2-\nu}}{t^{2-\nu}} \cdot e^{-\frac{t-\tau}{4a^2}}\right] N_2(t,\tau)\,\varphi(\tau)\,d\tau = F(t), \tag{11}$$

where

$$N_1(t,\tau) = \frac{1}{2a^2}\frac{t^2}{(t-\tau)^2}\exp\left[-\frac{t\tau}{2a^2(t-\tau)}\right] I_{\nu-1,\nu}\left(\frac{t\tau}{2a^2(t-\tau)}\right), \tag{12}$$

$$N_2(t,\tau) = \frac{3}{2a^2}\frac{t^2}{\tau(t-\tau)}\exp\left[-\frac{t\tau}{2a^2(t-\tau)}\right] I_{\nu}\left(\frac{t\tau}{2a^2(t-\tau)}\right). \tag{13}$$

Remark 2 *[21, P. 215] Let the solution of the integral equation*

$$y(t) + \int_a^t K(t,\tau)\,y(\tau)\,d\tau = f(t)$$

have the form

$$y(t) = f(t) + \int_a^t R(t,\tau)\,f(\tau)\,d\tau.$$

Then the solution of the more complicated integral equation

$$y(t) + \int_a^t K(t,\tau)\,\frac{g(\tau)}{g(t)}y(\tau)\,d\tau = f(t)$$

has the form

$$y(t) = f(t) + \int_a^t R(t,\tau)\,\frac{g(\tau)}{g(t)}f(\tau)\,d\tau.$$

According to this remark, we will seek a solution of the following equation

$$\varphi(t) - \int_0^t N_1(t,\tau)\,\varphi(\tau)\,d\tau - \int_0^t N_2(t,\tau)\,\varphi(\tau)\,d\tau = F(t). \tag{14}$$

Note the following property of the kernel $N(t,\tau) = N_1(t,\tau) + N_2(t,\tau)$, from which it follows that the integral equation (14), and together with it equation (11) are singular and to them the method of successive approximations cannot be applied.

Remark 3 *For any value* ν, $\quad 0<\nu<1$,

$$\lim_{t\to 0}\int\limits_0^t N_1(t,\tau)=1 \quad \textit{and} \quad \lim_{t\to 0}\int\limits_0^t N_2(t,\tau)=0,$$

moreover

$$\int\limits_0^t N_1(t,\tau)=1, \quad \int\limits_0^t N_2(t,\tau)=\frac{3}{2a^2}\cdot\frac{\Gamma(\nu)}{\Gamma(1+\nu)}\cdot t, \quad \forall t>0.$$

7 Characteristic integral equation

In order to find a solution of the integral equation (14), we first seek a solution of the following "truncated"integral equation

$$\varphi_1(t)-\int\limits_0^t N_1(t,\tau)\,\varphi_1(\tau)\,d\tau=\Phi(t), \tag{15}$$

which, by Remark 3, is characteristic for the equation (14).

Remark 4 *If a solution of equation* (15) *is found, then the solution of equation* (14) *will be obtained by the Carleman–Vekua regularization method.*

We change the variables $t=\frac{1}{y}$, $\tau=\frac{1}{x}$ and introduce new functions:

$$\varphi(t)=\varphi\left(\frac{1}{y}\right)=\psi(y), \quad \Phi(t)=\Phi\left(\frac{1}{y}\right)=p(y),$$

then equation (15) reduces to the following integral equation with a difference kernel with respect to the unknown function $\psi(y)$:

$$\psi(y)-\int\limits_y^\infty k(y-x)\,\psi(x)\,dx=p(y), \tag{16}$$

where

$$k(y-x)=\frac{1}{2a^2}\frac{1}{(x-y)^2}\exp\left[-\frac{1}{2a^2(x-y)}\right]I_{\nu-1,\nu}\left(\frac{1}{2a^2(x-y)}\right).$$

We introduce the corresponding one-sided functions for ψ and p by the formulas:

$$\theta_+(z)=\begin{cases}\theta(z), & z>0,\\ 0, & z\le 0;\end{cases} \qquad \theta_-(z)=\begin{cases}0, & z\ge 0,\\ -\theta(z), & z<0,\end{cases}$$

and for the function k according to the formulas:

$$k_+(z)=\begin{cases} k(-z), & z>0,\\ 0, & z\le 0;\end{cases} \qquad k_-(z)=\begin{cases} 0, & z\ge 0,\\ k(z), & z<0.\end{cases}$$

Then the solution of equation (16) can be written as follows:

$$\mathbf{k}\psi_+ \equiv (I-\mathbf{k}_-)\,\psi_+ = \psi_+(y) - \int_{-\infty}^{+\infty} k_-(y-x)\,\psi_+(x)\,dx = p_+(y)+\psi_-(y)\,. \tag{17}$$

Equation (17), defined on the entire real axis, for $y>0$ coincides with equation (16) and, as will be shown below, the solution of equation (17) does not depend on the method of completing the definition of the equation on the negative semiaxis, i.e. does not depend on function $\psi_-(y)$.

Theorem 2 *[22] The Fourier integrals of the right and left one-sided functions are the boundary values of the function, which are analytic, respectively, in the upper and lower half-planes.*

Applying the Fourier transform to equation (17), we get

$$\Psi^+(s) - K^-(s)\cdot\Psi^+(s) = P^+(s)+\Psi^-(s)\,, \tag{18}$$

where the corresponding Fourier images are denoted in capital letters. Under the condition

$$A(s) = 1-K^-(s)\neq 0, \quad \forall s\in\mathbb{R}$$

from (18), we obtain the following Riemann boundary value problem

$$\Psi^+(s) = \frac{1}{A(s)}\cdot\Psi^-(s) + R^-(s)\cdot P^+(s) + P^+(s)\,, \tag{19}$$

where

$$R^-(s) = \frac{K^-(s)}{A(s)}.$$

We calculate $K^-(s)$:

$$K^-(s) = \int_{-\infty}^{+\infty} k_-(\eta)\cdot e^{is\eta}d\eta = \int_{-\infty}^{0} \frac{e^{-\frac{1}{2a^2(-\eta)}+is\eta}}{(-\eta)^2}\cdot I_{\nu-1,\nu}\left(\frac{1}{2a^2(-\eta)}\right)d\eta =$$

$$= \int_{0}^{\infty} \frac{e^{-\frac{1}{2a^2\eta}-is\eta}}{\eta^2}\cdot I_{\nu-1,\nu}\left(\frac{1}{2a^2\eta}\right)d\eta = 1-2\frac{\sqrt{is}}{a}I_\nu\left(\frac{\sqrt{is}}{a}\right)K_{\nu-1}\left(\frac{\sqrt{is}}{a}\right).$$

Hence,

$$A(s) = 2\frac{\sqrt{is}}{a} I_\nu \left(\frac{\sqrt{is}}{a} \right) K_{\nu-1} \left(\frac{\sqrt{is}}{a} \right).$$

The coefficient of the Riemann problem

$$\frac{1}{A(s)} = \prod_{-\infty}^{+\infty} \frac{1}{s - s_k} \cdot \frac{1}{\tilde{A}(s)}$$

has simple poles at the points $s_k = i(a\alpha_k)^2$, $k \in \mathbb{Z}$. The Riemann problem (19) has an index equal to the number of zeros of the function $A(s)$ in the lower half-plane, including the axis itself, i.e. equals 1. $\frac{C}{z}$ is the main part of the expansion of the function $[A(z)]^{-1} \cdot \Psi^-(z)$ in powers of z. Then

$$\chi(z) = \frac{\Psi^-(z)}{A(z)} - \frac{C}{z}$$

will be a function whose original is zero at $y > 0$.

Now equality (19) can be represented as

$$\Psi^+(s) = P^+(s) + R^-(s) \cdot P^+(s) + \frac{C}{s} + \chi(s), \quad \forall s \in \mathbb{R}. \tag{20}$$

Passing in relation (20) to the originals for $y > 0$, we obtain the general solution of the integral equation (16)

$$\psi(y) = p(y) + \int_y^\infty r_-(x-y)\, p(x)\, dx + C.$$

Note that $\psi_0(y) = C$, $C = \text{const}$ is a solution of the corresponding homogeneous equation

$$\psi(y) - \int_y^\infty k(y-x)\, \psi(x)\, dx = 0.$$

Here $r_-(\eta)$ is the restriction to the negative semiaxis of the original Fourier transform $R^-(s)$ and is determined according to the theory of residues. By closing the contour of integration over the semicircle in the upper half-plane and using the theory of residues and Jordan's lemma, we have

$$r_-(\eta) = \frac{a^2}{2\sqrt{\pi}} \cdot \frac{1}{\eta^{\frac{3}{2}}} \sum_{k \in \mathbb{Z} \setminus \{0\}} A_{\nu,k} \int_0^\infty \xi e^{-\frac{\xi^2}{4\eta}} \cdot e^{-i\alpha_k a^2 \xi} d\xi. \tag{21}$$

7.1 Estimation of the resolvent $r_-(\eta)$

Let us prove the following lemma.

Lemma 1 *The resolvent* $r_-(\eta)$ (21) *satisfies the estimate*

$$r_-(\eta) \leqslant C \cdot \frac{1}{\sqrt{\eta}}, \quad \eta > 0.$$

Proof.

$$\begin{aligned}
&r_-(\eta) \leqslant \left| \frac{a^2}{2\sqrt{\pi}} \cdot \frac{1}{\eta^{\frac{3}{2}}} \sum_{k\in\mathbb{Z}\setminus\{0\}} A_{\nu-1,k} \int_0^\infty \xi e^{-\frac{\xi^2}{4\eta} - i\alpha_k a^2 \xi} d\xi \right| = \\
&= \left| \frac{a^2}{2\sqrt{\pi}\eta^{\frac{3}{2}}} \left\{ \sum_{k=-\infty}^{-1} A_{\nu-1,k} \int_0^\infty \xi e^{-\frac{\xi^2}{4\eta} - i\alpha_k a^2 \xi} d\xi + \sum_{k=1}^{\infty} A_{\nu-1,k} \int_0^\infty \xi e^{-\frac{\xi^2}{4\eta} - i\alpha_k a^2 \xi} d\xi \right\} \right| = \\
&= \left| \frac{a^2}{2\sqrt{\pi}\eta^{\frac{3}{2}}} \left\{ \sum_{n=1}^{\infty} A_{\nu-1,-n} \int_0^\infty \xi e^{-\frac{\xi^2}{4\eta} - i\alpha_{-n} a^2 \xi} d\xi + \sum_{n=1}^{\infty} A_{\nu-1,n} \int_0^\infty \xi e^{-\frac{\xi^2}{4\eta} - i\alpha_n a^2 \xi} d\xi \right\} \right| = \\
&= \| z_n = -i\alpha_n, \quad z_{-n} = i\alpha_n \| \leq \\
&\leq \frac{a^2}{2\sqrt{\pi}\eta^{\frac{3}{2}}} \sum_{n=1}^{\infty} \left\{ \left| A_{\nu-1,-n} \int_0^\infty \xi e^{-\frac{\xi^2}{4\eta} + i\alpha_n a^2 \xi} d\xi \right| + \left| A_{\nu-1,n} \int_0^\infty \xi e^{-\frac{\xi^2}{4\eta} - i\alpha_n a^2 \xi} d\xi \right| \right\} \leq \\
&\leq \frac{a^2}{2\sqrt{\pi}\eta^{\frac{3}{2}}} \int_0^\infty \xi e^{-\frac{\xi^2}{4\eta}} d\xi \sum_{n=1}^{\infty} \left\{ |A_{\nu-1,-n}| + |A_{\nu-1,n}| \right\} = \frac{a^2}{2\sqrt{\eta}} \sum_{n=1}^{\infty} \left\{ |A_{\nu-1,-n}| + |A_{\nu-1,n}| \right\}.
\end{aligned}$$

Let's find the sum $\sum\limits_{n=1}^{\infty} \left\{ |A_{\nu-1,-n}| + |A_{\nu-1,n}| \right\}$:

$$\begin{aligned}
&S = \sum_{n=1}^{\infty} \left\{ \left| \frac{1}{2z_{-n} I_{\nu-1}(z_{-n}) K_{\nu-1}(z_{-n})} \right| + \left| \frac{1}{2z_n I_{\nu-1}(z_n) K_{\nu-1}(z_n)} \right| \right\} = \\
&= \| z_n = -i\alpha_n, \quad z_{-n} = i\alpha_n \| = \\
&= \sum_{n=1}^{\infty} \left\{ \left| \frac{1}{2i\alpha_n I_{\nu-1}(i\alpha_n) K_{\nu-1}(i\alpha_n)} \right| + \left| \frac{1}{-2i\alpha_n I_{\nu-1}(-i\alpha_n) K_{\nu-1}(-i\alpha_n)} \right| \right\} = \\
&= \left\| \begin{array}{rl} K_{\nu-1}(z) = \dfrac{\pi i}{2} e^{\frac{\pi}{2}(\nu-1)i} H^{(1)}_{\nu-1}(iz), & I_{\nu-1}(z) = e^{-\frac{\pi}{2}(\nu-1)i} J_{\nu-1}(iz) \\ J_{\nu-1}(-z) = e^{(\nu-1)\pi i} J_{\nu-1}(z), & H^{(1)}_{\nu-1}(-z) = -e^{-(\nu-1)\pi i} H^{(2)}_{\nu-1}(z) \\ H^{(1)}_{\nu-1}(z) = J_{\nu-1}(z) + iN_{\nu-1}(z), & H^{(2)}_{\nu-1}(z) = J_{\nu-1}(z) - iN_{\nu-1}(z) \end{array} \right\| =
\end{aligned}$$

$$= \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{\alpha_n\left|J_{\nu-1}\left(\alpha_n\right)\right|}\left\{\frac{1}{\left|J_{\nu-1}\left(\alpha_n\right)-iN_{\nu-1}\left(\alpha_n\right)\right|}+\frac{1}{\left|J_{\nu-1}\left(\alpha_n\right)+iN_{\nu-1}\left(\alpha_n\right)\right|}\right\} =$$

$$= \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{\alpha_n\left|J_{\nu-1}\left(\alpha_n\right)\right|}\cdot\frac{2}{\sqrt{J_{\nu-1}^2\left(\alpha_n\right)+N_{\nu-1}^2\left(\alpha_n\right)}} = \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{\alpha_n\left|J_{\nu-1}\left(\alpha_n\right)\right|}\cdot\frac{2}{\sqrt{J_{\nu-1}^2\left(\alpha_n\right)}} =$$

$$= \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{\alpha_n J_{\nu-1}^2\left(\alpha_n\right)} \le \frac{2}{\pi}\int\limits_{\alpha_1}^{\infty}\frac{d\left(\alpha_n\right)}{\alpha_n J_{\nu-1}^2\left(\alpha_n\right)} = \left.\frac{N_{\nu-1}\left(\alpha_n\right)}{J_{\nu-1}\left(\alpha_n\right)}\right|_1^{\infty} \le C\left(\alpha_1\right),$$

where we used the formula (1.8.4.1) [23, P.39] and $N_{\nu-1}(z)$ is a cylinder function of the second kind (the Neumann function). Then we get

$$r_-(\eta) \leqslant \frac{a^2}{2\sqrt{\eta}}\sum_{n=1}^{\infty}\left\{\left|A_{\nu-1,-n}\right|+\left|A_{\nu-1,n}\right|\right\} \le \frac{C\left(\alpha_1\right)a^2}{2}\cdot\frac{1}{\sqrt{\eta}}.$$

Lemma is proved.

7.2 Solution of the “characteristic” equation

We found a solution of equation (16), which has the form

$$\psi(y) = p(y) + \int\limits_y^{\infty} r_-(x-y)\,p(x)\,dx + C.$$

We make the reverse replacements

$$t = \frac{1}{y}, \quad \tau = \frac{1}{x}$$

and write the solution of the characteristic equation (15) as follows

$$\varphi(t) = \Phi(t) + \int\limits_0^{t}\tilde{R}(t,\tau)\,\Phi(\tau)\,d\tau + C,$$

where

$$\tilde{R}(t,\tau) \le C\frac{\sqrt{t}}{\tau^{\frac{3}{2}}\sqrt{t-\tau}}. \tag{22}$$

The last inequality follows from the Lemma 1.

Remark 5 *Since $\varphi_0(t) = C,\ C = const$ is a solution of the homogeneous equation*

$$\varphi(t) - \int\limits_0^{t} N_1(t,\tau)\,\varphi(\tau)\,d\tau = 0,$$

then

$$\varphi_0(t) = C \cdot \frac{1}{t^{2-\nu}} \cdot e^{-\frac{t}{4a^2}}, \quad C = const \tag{23}$$

is a solution of the homogeneous equation

$$\varphi(t) - \int\limits_0^t \frac{t^{\nu}\tau^{2-\nu}}{2a^2(t-\tau)^2} \exp\left[-\frac{t\tau}{2a^2(t-\tau)}\right] I_{\nu-1,\nu}\left(\frac{t\tau}{2a^2(t-\tau)}\right) e^{-\frac{t-\tau}{4a^2}} \varphi(\tau)\, d\tau = 0.$$

Solution (23) *does not belong to class* (9)*.*

8 Solution of the "complete" integral equation. Carleman–Vekua regularization method

Theorem 3 *The original integral equation* (14) *for any function* $t^{-\frac{1}{2}-\nu} e^{\frac{t}{4a^2}} \cdot F(t) \in L_\infty(0,\infty)$ *has a unique solution in the class of functions*

$$t^{\frac{1}{2}-\nu} e^{\frac{t}{4a^2}} \cdot \varphi(t) \in L_\infty(0,\infty), \tag{24}$$

which can be found by the method of successive approximations.

Proof. To solve the original "complete"integral equation (14), we represent it as

$$\varphi(t) - \int\limits_0^t N_1(t,\tau)\varphi(\tau)\, d\tau = F(t) + \int\limits_0^t N_2(t,\tau)\varphi(\tau)\, d\tau$$

and apply the Carleman–Vekua regularization method. Assuming the right-hand side of equation (14) to be temporarily known, we write its solution

$$\varphi(t) = F(t) + \int\limits_0^t N_2(t,\tau)\varphi(\tau)\, d\tau + \int\limits_0^t \tilde{R}(t,\tau)\left\{F(\tau) + \int\limits_0^\tau N_2(\tau,\xi)\varphi(\xi)\, d\xi\right\} d\tau. \tag{25}$$

We change the order of integration in the iterated integral and, then, change the roles of the variables τ and ξ, hence equation (25) takes the form

$$\varphi(t) - \int\limits_0^t M(t,\tau)\varphi(\tau)\, d\tau = \tilde{F}(t), \tag{26}$$

where

$$M(t,\tau) = N_2(t,\tau) + \int\limits_\tau^t \tilde{R}(t,\tau) N_2(\xi,\tau)\, d\xi,$$

$$\tilde{F}(t) = F(t) + \int\limits_0^t \tilde{R}(t,\tau) F(\tau)\, d\tau.$$

By using (22), we obtain that the kernel $M(t,\tau)$ of the integral equation (14) has a weak singularity, since it satisfies the estimate

$$M(t,\tau) \le D_1 \cdot \frac{\sqrt{t}}{\sqrt{\tau}\sqrt{t-\tau}} + D_2.$$

This means that the solution of the integral equation (14) can be found by the method of successive approximations. The theorem is proved.

9 Solution of the boundary value problem (1)–(3). Proof of the Theorem 1

From the integral representation for the solution (8) of the boundary value problem (5)-(7), we get

$$\omega(r,t) = r^{2\nu-1}\frac{\partial u}{\partial r} = \int\limits_0^t \frac{r^{\nu}\cdot\tau^{1-\nu}}{2a^2(t-\tau)} \exp\left[-\frac{r^2+\tau^2}{4a^2(t-\tau)}\right] I_\nu\left(\frac{r\tau}{2a^2(t-\tau)}\right)\varphi(\tau)\,d\tau.$$

We estimate $\omega(r,t)$, taking into account that $t^{\frac{1}{2}-\nu}e^{\frac{t}{4a^2}}\varphi(t) \in L_\infty(0,\infty)$:

$$\omega(r,t) = \sqrt{t}\exp\left[-\frac{t}{4a^2}\right]\int\limits_0^t \frac{1}{2a^2}\cdot\frac{r^\nu\sqrt{\tau}}{(t-\tau)\sqrt{t}}\exp\left[-\frac{(r-\tau)(r+t-2\tau)}{4a^2(t-\tau)}\right]\times$$
$$\times\left\{\exp\left[-\frac{r\tau}{2a^2(t-\tau)}\right]I_\nu\left(\frac{r\tau}{2a^2(t-\tau)}\right)\right\}\cdot\left\{\tau^{\frac{1}{2}-\nu}\exp\left[\frac{\tau}{4a^2}\right]\varphi(\tau)\right\}d\tau.$$

Then we have

$$|\omega(r,t)| \leqslant$$
$$\leqslant C_1\sqrt{t}\exp\left[-\frac{t}{4a^2}\right]\int\limits_0^t \frac{1}{2a^2}\cdot\frac{r^\nu}{t-\tau}\exp\left[-\frac{r\tau}{2a^2(t-\tau)}\right]I_\nu\left(\frac{r\tau}{2a^2(t-\tau)}\right)d\tau =$$
$$= \left\|\frac{r\tau}{2a^2(t-\tau)} = z\right\| = \frac{C_1\sqrt{t}\exp\left[-\frac{t}{4a^2}\right]r^\nu}{2a^2}\int\limits_0^\infty \frac{1}{\frac{r}{2a^2}+z}e^{-z}I_\nu(z)\,dz =$$
$$= \frac{C_1\sqrt{t}\exp\left[-\frac{t}{4a^2}\right]r^\nu}{2a^2}\cdot\frac{1}{\sqrt{\pi}}\cdot\frac{\Gamma(\nu)\Gamma\left(\frac{1}{2}\right)}{\Gamma(1+\nu)} = \frac{C_1\sqrt{t}\exp\left[-\frac{t}{4a^2}\right]r^\nu}{2a^2\nu},$$

where we used the formula (2.15.3.3) [23, P. 272]. Therefore,

$$|\omega(r,t)| \leqslant \frac{C_1\sqrt{t}\exp\left[-\frac{t}{4a^2}\right]r^\nu}{2a^2\nu}$$

or, taking into consideration (4):

$$\frac{\partial u(r,t)}{\partial r} \le \frac{C_1\sqrt{t}r^{1-\nu}}{2a^2\nu}\exp\left[-\frac{t}{4a^2}\right],$$

$$u(r,t) \le C_1\int\limits_0^r \frac{\sqrt{t}r^{1-\nu}}{2a^2\nu}\exp\left[-\frac{t}{4a^2}\right]dr = \frac{C_1\sqrt{t}}{2a^2\nu}\cdot\frac{r^{2-\nu}}{2-\nu}\exp\left[-\frac{t}{4a^2}\right].$$

This implies the validity of the main result – the Theorem 1.

10 Conclusion

The boundary value problem of heat conduction outside the cone is reduced by the method of heat potentials to the singular Volterra type integral equation of the second kind. We constructed for it a characteristic integral equation and found its explicit solution. Using the estimate for resolvent of the characteristic equation, we found a solution of the original integral equation by the Carleman-Vekua regularization method.

11 Acknowledgements

This work supported by the grants project AP09259780, 2021-2023 of the Ministry of Education and Science of the Republic of Kazakhstan.

References

[1] Cheblakova E.A. "Modeling convection in areas with free borders", *Computational Technologies,* 5:6 (2000): 87-98.

[2] Kheloufi A., Sadallah B.-K. "On the regularity of the heat equation solution in non-cylindrical domains: Two approaches", *Applied mathematics and computation,* 218 (2011): 1623-1633.

[3] Kheloufi A. "Existence and uniqueness results for parabolic equations with Robin type boundary conditions in a non-regular domain of $\mathbb{R}^3$", *Applied mathematics and computation,* 220 (2013): 756-769.

[4] Cherfaoui S., Kessab A., Kheloufi A. "Well-posedness and regularity results for a 2m-th order parabolic equation in symmetric conical domains of $\mathbb{R}^{N+1}$", *Mathematical methods in the applied sciences,* 40:5 (2017): 6035-6047.

[5] Kheloufi A. "On a Fourth Order Parabolic Equation in a Nonregular Domain of $\mathbb{R}^3$", *Mediterranean journal of mathematics,* 12 (2014): 803-820.

[6] Kheloufi A., Sadallah B.-K. "Study of the heat equation in a symmetric conical type domain of $\mathbb{R}^{N+1}$", *Mathematical methods in the applied sciences,* 37 (2014): 1807-1818.

[7] Chapko R., Johansson B.T., Vavrychuk V. "Numerical solution of parabolic Cauchy problems in planar corner domains", *Mathematics and computers in simulation,* 101 (2014): 1-12.

[8] Wang Y., Huang J., Wen X. "Two-dimensional Euler polynomials solutions of two-dimensional Volterra integral equations of fractional order", *Applied numerical mathematics,* 163 (2021): 77-95.

[9] Dehbozorgi R., Nedaiasl K. "Numerical solution of nonlinear weakly singular Volterra integral equations of the first kind: An hp-version collocation approach", *Applied numerical mathematics,* 161 (2021): 111-135.

[10] Wisniewski G., Habrych M., Miedzinski B. "Approach to prediction the transition of a small power low voltage switching arc into glowing", *20th International Symposium on Electrical Apparatus and Technologies (SIELA),* 3-6 June 2018, Bourgas, Bulgaria, Danvers, MA: IEEE.

[11] Miedzinski B., Wisniewski G., Kharin S.N., Nouri H., Grechanyuk M. "Arc-to-glow transition approachforpracticalusein dc low-power, low-voltageelectricgrids", *IEEE Transactions on Components Packagingand Manufacturing Technology,* 8:6 (2018): 932-938.

[12] Kharin S.N., Nouri H., Miedzinski B., Wisniewski G. "Transient phenomena of arc to glow discharge transformation at contact opening", *Proc. of 21st Int. Conf. on Electric contacts, Zurich, Switzerland,* (2002): 425-431.

[13] Kharin S. "Mathematical model of electrical contact bouncing", *AIP Conference Proceedings,* (2015): 1676.

[14] Sarsengeldin M.M., Kharin S.N., Rayev Z., Khairullin Y. "Mathematical model of heat transfer in opening electrical contacts with tunnel effect", *Filomat,* 32 (2018): 1003-1008.

[15] Kavokin A.A., Kulakhmetova A.T., Shpadi Y.R. "Application of Thermal Potentials to the Solution of the Problem of Heat Conduction in a Region Degenerates at the Initial Moment" , *Filomat,* 132 (2018): 825-836.

[16] Amangalieva M.M., Akhmanova D.M., Dzhenaliev M.T., Ramazanov M.I. "Boundary value problems for a spectrally loaded heat operator with load line approaching the time axis at zero or infinity" , *Differential Equations,* 47:2 (2011): 231-243.

[17] Jenaliyev M.T., Amangaliyeva M.M., Kosmakova M.T., Ramazanov M.I. "On a Volterra equation of the second kind with incompressible kernel" , *Advances in Difference Equations,* 2015 (2015): article number 71.

[18] Amangalieva M.M., Dzhenaliev M.T., Kosmakova M.T., Ramazanov M.I. "On one homogeneous problem for the heat equation in an infinite angular domain" , *Siberian Mathematical Journal,* 56:6 (2015): 982-995.

[19] Jenaliyev M.T., Ramazanov M.I. "On a Homogeneous Parabolic Problem in an Infinite Corner Domain" , *Filomat,* 32:3 (2018): 965-974.

[20] Baderko E.A. "Parabolic problems and boundary integral equations" , *Mathematical methods in the applied sciences,* 20 (1997): 449-459.

[21] Polianin A.D., Manzhirov A.V. *Handbook of integral equations* (Boca Raton, CRC Press, 2008).

[22] Gakhov F.D., Chersky Yu.I. *Convolution type equations* [Uravneniya tipa svyortki] (Moscow: Nauka, 1978).

[23] Prudnikov A.P., Brychkov Yu.A., Marichev O.I. *Integrals and Series. Volume 2. Special Functions* (Moscow: Fizmatlit, 2003).