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ASYMPTOTIC BEHAVIOR OF THE SOLUTION OF THE INTEGRAL BOUNDARY VALUE PROBLEM FOR SINGULARLY PERTURBED INTEGRO-DIFFERENTIAL EQUATIONS

The work is devoted to clarifying asymptotic with respect to a small parameter behavior of the solution of the integral boundary value problem for singularly perturbed linear integro-differential equation. We study the boundary value problem for singularly perturbed integro-differential equations with the phenomena of the so-called boundary jumps, when the fast solution variable becomes unbounded at both boundaries. The exceptions of the qualitative influence of integral terms on the asymptotic behavior of the solutions for singularly perturbed integro-differential equations are shown. The presence of integral terms will significantly change the degenerate equation: the solution of the assumed singularly perturbed integro-differential equation does not tend to the solution of the usual degenerate equation, obtained from the supposed equation with the zero value of a small parameter and will tend to solve a specially modified degenerate integro-differential equation with an additional term called the jump of the integral term. Boundary and initial functions are defined; their existence and uniqueness are proved. On the basis of the constructed boundary and initial functions are obtained analytical formula and asymptotic estimates of the solution for the integral boundary value problem. It is established that the solution of the considered boundary value problem at the ends of a given segment has the phenomena of boundary jumps of the same orders. A modified degenerate boundary value problem is constructed, to the solution of which approaches the solution of assumed singularly perturbed integral boundary value problem. The value of the jump of integral terms is found. An example was made based on the initial results.

Key words: singular perturbation, small parameter, the initial jump, asymptotics.

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Сингулярлы ауытқыған интегралды-дифференциалдық теңдеуге арналған шекаралық есеп шешімінің асимптотикалық сипаты

Жұмыс сингулярлы ауытқыған интегро-дифференциалдық теңдеу үшін шекаралық есеп шешімінің кіші параметр бойынша асимптотикалық құбылысын анықтауға арналған. Біз сингулярлы ауытқыған интегро-дифференциалдық теңдеуді, кіші параметр нөлге ұмтылғанда шешім кесіндінің екі жақ шетінде шексіздікке ұмтылатын шекаралық секіріс құбылысымен қарастырамыз. Жұмыста интегралдық мүшенің сингулярлы ауытқыған интегро-дифференциалдық теңдеу үшін шекаралық есеп шешімінің асимптотикалық сипатына әсері көрсетілген. Интегралдық мүше ауытқымаған шекаралық есепті айтарлықтай өзгертеді. Кіші параметр нөлге ұмтылған кезде сингулярлы ауытқыған шекаралық есеп шешімі қарапайым ауытқымаған шекаралық есеп шешіміне ұмтылмайды. Сингулярлық ауытқыған шекаралық есеп шешімі интегралдық мүшенің секірісі бар өзгертілген ауытқы-маған есеп шешіміне ұмтылады.

Жұмыста интегралдық шекаралық есеп шешімінің аналитикалық формуласы мен асимпто-
тикалық бағалаулары алынды. Берілген кесіндінің екі жақ шетінде де қарастырылып оты-
рған шекаралық есеп шешімінің бастапқы секіріс құбылысының реті бірдей екендігі анықтал-
ды. Осы сингулярлы ауытқыған шекаралық есебіне сәйкес өзгертілген ауытқымаған шекара-
лық есеп құрылған. Берілген сингулярлы ауытқыған шекаралық есеп шешімінің өзгертілген
ауытқымаған шекаралық есебіне ұмтылатыны дәлелденген. Интегралдық мүшенің бастапқы
секірісінің шамасы анықталды. Мысал бастапқы нәтижелер негізінде жасалды.

Түйін сөздер: сингулярлық ауытқу, кіші параметр, бастапқы секіріс, асимптотика.

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**Асимптотическое поведение решения интегральной краевых задач для сингулярно
возмущенных интегро-дифференциальных уравнений**

Работа посвящена выяснению асимптотического по малому параметру поведения решения
интегральной краевой задачи для сингулярного возмущенного линейного интегро-
дифференциального уравнения. Мы изучаем краевую задачу для сингулярно возмущенных
интегро-дифференциальных уравнений с явлениями так называемых граничных скачков,
когда переменная быстрого решения становится неограниченной на обеих границах. Показа-
ны исключения качественного влияния интегральных членов на асимптотическое поведение
решений сингулярно возмущенных интегро-дифференциальных уравнений. Наличие инте-
гральных членов существенно изменит вырожденное уравнение. Решение предполагаемого
сингулярно возмущенного интегро-дифференциального уравнение не стремится к решению
обычного вырожденного уравнения, полученного из предполагаемого уравнения с нулевым
значением малого параметра, и будет стремиться специально модифицированное вырожден-
ное интегро-дифференциальное уравнение с дополнительным членом, называемым скачком
интегрального члена. В работе получены аналитическая формула и асимптотические оценки
решения интегральной краевой задачи. Установлено, что решение рассматриваемой краевой
задачи на концах заданного отрезка имеет явления граничных скачков тех же порядков.
Построена модифицированная вырожденная краевая задача, к решению которой стремится
решение исходной сингулярно возмущенной интегральной краевой задачи. Определена
величина начального скачка интегрального члена. Пример был сделан на основе начальных
результатов.

Ключевые слова: малый параметр, сингулярное возмущение, асимптотика, начальный ска-
чок.

1 Introduction

Singularly perturbed equations is called equations that contain a small parameter with higher
derivatives. Many applied problems in physics, mechanics, technology, etc. are modeled using
this type of equations in mathematics. In the following work of the authors L. Schlesinger
[1], G.D. Birkhoff [2], P. Noaillon [3], W. Wasow [4], A.H. Nayfeh [5], A. N. Tikhonov [6,7],
M.I. Vishik, L.A. Lusternik [8,9], N.N. Bogolyubov, U.A Mitropolsky [10], A.B. Vasilieva and
V.F. Butuzov [11,12], R.E. O'Malley [13,14], D.R. Smith [15], W. Eckhaus [16], K. W. Chang
and F. A. Howes [17], J. Kevorkian and J.D. Cole [18], Sanders and F. Verhulst [19], E.F.
Mischenko, N. Rozov [20], S.A. Lomov [21], M.I. Imanaliev [22], K.A.Kassymov [23-25] was
appeared and devoloped the theory of such type equations.

In the study of certain singularly perturbed problems, it can be found that the fast
variable of the solution near the boundary of the set takes on an infinitely large value, since

the small parameter tends to zero. The study of the Cauchy problem with an initial jump for a single nonlinear ODE was began by authors' work L.A. Lusternik and M.I. Vishik [26] and K.A.Kassymov [27]. The most common cases of the Cauchy problem with initial jump were studied by K.A. Kassymov. A characteristic exception of such problems is that the solution of this problem approaches to solution of degenerate equation with modified initial conditions when the small parameter goes to zero. In this instance, it is said that the phenomenon of the initial jump for the solution is valid.

In [28-31] was investigated boundary value problems (BVP) for singularly perturbed ODE and integro-differential equations with initial jumps.

In [32, 33] was studied BVP for integro-differential equations third-order with a small parameter at two higher derivatives, when hold the so-called phenomena of boundary jumps, i.e. when some derivatives of the solution for sufficiently small values of the parameter become infinitely large at both ends of the interval. But as this takes place at the ends of the considered interval of solving these problems, there were jumps of different orders. Singularly perturbed differential equations with a piecewise constant argument considered in [34].

In the present work, we investigate integral BVP for singularly perturbed linear integro-differential equations third-order, solution that at the ends of a given segment has jumps of the same order. The main goal of this paper is to establish the asymptotic behavior of the solution on a small parameter and the construction of the modified degenerate problem. A similar work but without integral conditions considered in [35].

2 Statement of the problem and auxiliary materials

Consider the following singularly perturbed linear integro-differential equation

$$L_\mu y \equiv \mu^2 y''' + \mu A_0(t)y'' + A_1(t)y' + A_2(t)y = F(t) + \int_0^1 \sum_{i=0}^2 H_i(t,x)y^{(i)}(x,\mu)dx \qquad (1)$$

with integral boundary conditions

$$h_1 y \equiv y(0,\mu) = \alpha, \quad h_2 y \equiv y'(0,\mu) = \beta, \quad h_3 y \equiv y'(1,\mu) - \int_0^1 \sum_{i=0}^2 a_i(x)y^{(i)}(x,\mu)dx = \gamma, \quad (2)$$

where $\mu > 0$ is a small parameter, α, β, γ are known constants independent of μ.

Assume that following conditions hold:

C1) Functions $A_i(t), i = \overline{0,2}$, $F(t)$, $a_i(t), i = \overline{0,2}$ are defined on the interval $0 \leq t \leq 1$, $H_0(t,x), H_1(t,x), H_2(t,x)$ are defined in the domain $D = \{0 \leq t \leq 1, 0 \leq x \leq 1\}$ and sufficiently smooth.

C2) The roots $\kappa_i(t)$, $i = 1,2$ of "additional characteristic equation" $\kappa^2 + A_0(t)\kappa + A_1(t) = 0$ satisfy the following inequalities $\kappa_1(t) \leqslant -\gamma_1 < 0$, $\kappa_2(t) \geqslant \gamma_2 > 0$.

C3) $a_2(1) \neq 1$.

We consider homogeneous singularly perturbed equation associated with (1):

$$L_\varepsilon y \equiv \mu^2 y''' + \mu A_0(t)y'' + A_1(t)y' + A_2(t)y = 0. \qquad (3)$$

The fundamental set of solutions of equation (3) takes the form [35]

$$
y_1^{(q)}(t,\mu) = \frac{1}{\mu^q} e^{\frac{1}{\mu} \int\limits_0^t \kappa_1(x)dx} \cdot \left(\kappa_1^q(t)y_{10}(t) + O(\mu)\right), \qquad q = \overline{0,2},
$$

$$
y_2^{(q)}(t,\mu) = \frac{1}{\mu^q} e^{-\frac{1}{\mu} \int\limits_t^1 \kappa_2(x)dx} \cdot \left(\kappa_2^q(t)y_{20}(t) + O(\mu)\right), \qquad q = \overline{0,2}, \tag{4}
$$

$$
y_3^{(q)}(t,\mu) = y_{30}^{(q)}(t) + O(\mu), \qquad q = \overline{0,2},
$$

where $y_{30}(t) = e^{-\int\limits_0^t \frac{A_2(x)}{A_1(x)}dx}$, functions $y_{i0}(t), i = 1, 2$ are solutions of the problem:

$$
p_i(t) \cdot y_{i0}'(t) + q_i(t) \cdot y_{i0}(t) = 0, \quad y_{i0}(0) = 1, \quad i = 1, 2,
$$

where

$$
p_i(t) = (A_0(t) + 2\kappa_i(t))\kappa_i(t), \quad q_i(t) = 3\kappa_i(t)\kappa_i'(t) + A_0(t)\kappa_i'(t) + A_2(t).
$$

Wronskian, composed of the fundamental system of solutions of equation (3) has the form:

$$
W(t,\mu) = \frac{1}{\mu^3} e^{\frac{1}{\mu} \int\limits_0^t \kappa_1(x)dx - \frac{1}{\mu} \int\limits_t^1 \kappa_2(x)dx} (y_{10}(t)y_{20}(t)y_{30}(t)\kappa_1(t)\kappa_2(t) \cdot
$$

$$
\cdot (\kappa_2(t) - \kappa_1(t)) + O(\mu)). \tag{5}
$$

We introduce the functions [35]

$$
K_0(t,s,\mu) = \frac{P_0(t,s,\mu)}{W(s,\mu)}; \quad K_1(t,s,\mu) = \frac{P_1(t,s,\mu)}{W(s,\mu)}, \tag{6}
$$

where $P_0(t,s,\mu)$, $P_1(t,s,\mu)$ are the third order determinant derived from the Wronskian $W(s,\mu)$ by replacing the third row with $y_1(t,\mu), 0, y_3(t,\mu)$ and $0, y_2(t,\mu), 0$ respectively. Sum of $K_0(t,s,\mu)$ and $K_1(t,s,\mu)$ are the Cauchy function. Therefore, these functions have the following properties:

1. With respect to the variable t satisfy equation (3), i.e.

$$
L_\mu K_0(t,s,\mu) = 0, \quad L_\mu K_1(t,s,\mu) = 0, \quad t \in [0,1], \quad t \neq s.
$$

2. When $t = s$ satisfy the conditions:

$$
K_0(s,s,\mu) + K_1(s,s,\mu) = 0, \quad K_0'(s,s,\mu) + K_1'(s,s,\mu) = 0, \quad K_0''(s,s,\mu) + K_1''(s,s,\mu) = 1.
$$

For the functions $K_0(t,s,\mu)$, $K_1(t,s,\mu)$ in view (4), (5), (6) the following asymptotic representation hold as $\mu \to 0$ [35]:

$$
K_0^{(q)}(t,s,\mu) = \mu^2 \left(\frac{y_{30}^{(q)}(t)}{y_{30}(s)\kappa_1(s)\kappa_2(s)} - \frac{\kappa_1^q(t)y_{10}(t)}{\mu^q y_{10}(s)\kappa_1(s)(\kappa_2(s) - \kappa_1(s))} e^{\frac{1}{\mu} \int\limits_s^t \kappa_1(x)dx} + \right.
$$

$$
\left. + O(\mu)\right), \quad t \geq s, \quad q = \overline{0,2}, \tag{7}
$$

$$K_1^{(q)}(t,s,\mu) = \mu^2 \left(\frac{\kappa_2^q(t)y_{20}(t)}{\mu^q y_{20}(s)\kappa_2(s)\,(\kappa_2(s) - \kappa_1(s))} e^{-\frac{1}{\mu}\int\limits_t^s \kappa_2(x)dx} + O(\mu) \right), t \le s, q = \overline{0,2}.$$

Let functions $\Phi_i(t,\mu), i = 1,2,3$ are solutions for the following problem:

$$L_\mu \Phi_i(t,\mu) = 0, \quad h_k \Phi_i(t,\mu) = \delta_{ki}, \qquad i,k = 1,2,3,$$

where δ_{ki} is Kronecker symbol. Functions $\Phi_i(t,\mu),\ i = 1,2,3$ are called boundary functions. The boundary functions are determined by the formula

$$\Phi_i(t,\mu) = \frac{J_i(t,\mu)}{J(\mu)}, \tag{8}$$

where $J(\mu)$ is the determinant consisting of the fundamental solution system of equation (3):

$$J(\mu) = \begin{vmatrix} h_1 y_1(t,\mu) & h_1 y_2(t,\mu) & h_1 y_3(t,\mu) \\ h_2 y_1(t,\mu) & h_2 y_2(t,\mu) & h_2 y_3(t,\mu) \\ h_3 y_1(t,\mu) & h_3 y_2(t,\mu) & h_3 y_3(t,\mu) \end{vmatrix}, \tag{9}$$

$J_i(t,\mu)$ is the determinant derived from $J(\mu)$ by replacing the i-th row by the fundamental set of solutions $y_1(t,\mu), y_2(t,\mu), y_3(t,\mu)$ of the equation (3). From (9), by virtue of (2), (3), we obtain for $J(\mu)$ the asymptotic representation:

$$J(\mu) = \frac{1}{\mu^2}\left(\kappa_2(1)y_{20}(1)\kappa_1(0)(1 - a_2(1)) + O(\mu)\right) \ne 0. \tag{10}$$

Then from (8) in view (4), (10) we get asymptotic representation as $\mu \to 0$ for boundary functions $\Phi_i(t,\mu),\ i = 1,2,3$:

$$\Phi_1^{(q)}(t,\mu) = y_{30}^{(q)}(t) - \frac{y_{30}'(0)\kappa_1^q(t)y_{10}(t)}{\mu^{q-1}\kappa_1(0)} e^{\frac{1}{\mu}\int\limits_0^t \kappa_1(x)dx} -$$

$$-\frac{\kappa_2^q(t)y_{20}(t)\,(h_3 y_{30}(t) - a_2(0)y_{30}'(0))}{\mu^{q-1}\kappa_2(1)y_{20}(1)(1 - a_2(1))} e^{-\frac{1}{\mu}\int\limits_t^1 \kappa_2(x)dx} +$$

$$+O\left(\mu + \frac{1}{\mu^{q-2}}e^{-\frac{t}{\mu}\gamma_1} + \frac{1}{\mu^{q-2}}e^{-\frac{1-t}{\mu}\gamma_2}\right),\ q = \overline{0,2}, \tag{11}$$

$$\Phi_2^{(q)}(t,\mu) = -\mu\frac{y_{30}^{(q)}(t)}{\kappa_1(0)} + \frac{\kappa_1^q(t)y_{10}(t)}{\mu^{q-1}\kappa_1(0)} e^{\frac{1}{\mu}\int\limits_0^t \kappa_1(x)dx} - \frac{\kappa_2^q(t)y_{20}(t)a_2(0)}{\mu^{q-1}\kappa_2(1)y_2(1)(1 - a_2(1))} \cdot$$

$$\cdot e^{-\frac{1}{\mu}\int\limits_t^1 \kappa_2(x)dx} + O\left(\mu^2 + \frac{1}{\mu^{q-2}}e^{-\frac{t}{\mu}\gamma_1} + \frac{1}{\mu^{q-2}}e^{-\frac{1-t}{\mu}\gamma_2}\right),\ q = \overline{0,2},$$

$$\Phi_3^{(q)}(t,\mu) = \frac{\kappa_2^q(t)y_{20}(t)}{\mu^{q-1}\kappa_2(1)y_{20}(1)(1 - a_2(1))} e^{-\frac{1}{\mu}\int\limits_t^1 \kappa_2(x)dx} + O\left(\frac{1}{\mu^{q-2}}e^{-\frac{1-t}{\mu}\gamma_2}\right).$$

3 Main results

We are looking for a solution of the boundary value problem (1),(2) in the form:

$$y(t, \mu) = C_1 \Phi_1(t, \mu) + C_2 \Phi_2(t, \mu) + C_3 \Phi_3(t, \mu) +$$

$$+ \frac{1}{\mu^2} \int\limits_0^t K_0(t, s, \mu) v(s, \mu) ds + \frac{1}{\mu^2} \int\limits_1^t K_1(t, s, \mu) v(s, \mu) ds, \tag{12}$$

where $\Phi_i(t, \mu), i = 1, 2, 3$ are boundary functions and expressed by the formula (8), $K_0(t, s, \mu), K_1(t, s, \mu)$ are auxiliary functions, determined by the formula (6) and having an asymptotic representation (7), $C_i, i = 1, 2, 3$ are unknown constants, $v(t, \mu)$ is an unknown function.

Replacing (12) into equation (1) we get that $v(t, \mu)$ satisfies the following Fredholm integral equation of the second kind:

$$v(t, \mu) = p(t, \mu) + \int\limits_0^1 H(t, s, \mu) v(s, \mu) ds, \tag{13}$$

where

$$p(t, \mu) = F(t) + C_1 \int\limits_0^1 \sum_{i=0}^2 H_i(t, x) \Phi_1^{(i)}(x, \mu) dx + C_2 \int\limits_0^1 \sum_{i=0}^2 H_i(t, x) \Phi_2^{(i)}(x, \mu) dx +$$

$$+ C_3 \int\limits_0^1 \sum_{i=0}^2 H_i(t, x) \Phi_3^{(i)}(x, \mu) dx, \tag{14}$$

$$H(t, s, \mu) = \frac{1}{\mu^2} \int\limits_s^1 \sum_{i=0}^2 H_i(t, x) K_0^{(i)}(x, s, \mu) dx - \frac{1}{\mu^2} \int\limits_0^s \sum_{i=0}^2 H_i(t, x) K_1^{(i)}(x, s, \mu) dx.$$

C4) 1 is not an eigenvalue of the kernel $H(t, s, \mu)$.

Taking into account the condition (C4) integral equation (13) has an unique solution, that can be represented in the form

$$v(t, \mu) = p(t, \mu) + \int\limits_0^1 R_\mu(t, s, 1) p(s, \mu) ds, \tag{15}$$

where $R_\mu(t, s, 1)$ is a resolvent of the kernel $H(t, s, \mu)$, representable by (14) as an asymptotic formula $R_\mu(t, s, 1) = R_0(t, s, 1) + O(\mu)$. Substituting (15) taking account of (14) into (12), we obtain solution of the boundary value problem (1),(2) in the form:

$$y(t, \mu) = \sum_{i=1}^3 C_i T_i(t, \mu) + Q(t, \mu), \tag{16}$$

where

$$T_i(t,\mu) = \Phi_i(t,\mu) + \frac{1}{\mu^2}\int\limits_0^t K_0(t,s,\mu)\overline{\phi}_i(s,\mu)ds + \frac{1}{\mu^2}\int\limits_1^t K_1(t,s,\mu)\overline{\phi}_i(s,\mu)ds,$$

$$Q(t,\mu) = \frac{1}{\mu^2}\int\limits_0^t K_0(t,s,\mu)\overline{f}(s,\mu)ds + \frac{1}{\mu^2}\int\limits_1^t K_1(t,s,\mu)\overline{f}(s,\mu)ds, \tag{17}$$

$$\overline{\phi}_i(s,\mu) = \int\limits_0^1 \sum_{j=0}^2 \overline{H}_j(s,x,\mu)\Phi_i^{(j)}(x,\mu)dx, \quad \overline{f}(s,\mu) = F(s) + \int\limits_0^1 R_\mu(s,p,1)F(p)dp,$$

$$\overline{H}_j(s,x,\mu) = H_j(s,x) + \int\limits_0^1 R_\mu(s,p,1)H_j(p,x)dp.$$

For the functions $\overline{H}_j(s,x,\mu)$, $\overline{f}(s,\mu)$ are valid asymptotic representations:

$$\overline{H}j(s,x,\mu) = Hj(s,x) + \int\limits_0^1 R_0(s,p,1)H_j(p,x)dp + O(\mu) \equiv \overline{H}_j(s,x) + O(\mu),$$

$$\overline{f}(s,\mu) = F(s) + \int\limits_0^1 R_0(s,p,1)F(p)dp + O(\mu) \equiv \overline{f}(s) + O(\mu). \tag{18}$$

For the functions $\overline{\phi}_i(s,\mu)$ applying (11), (18), we can obtain the asymptotic formula:

$$\overline{\phi}_i(s,\mu) = \overline{\phi}_i(s) + O(\mu), \quad i = \overline{1,3}, \tag{19}$$

where

$$\overline{\phi}_1(s) = \int\limits_0^1 \sum_{j=0}^2 \overline{H}_j(s,x)y_{30}^{(j)}(x)dx + \overline{H}_2(s,0)y_{30}'(0) - \frac{(h_3y_{30}(t) - a_2(0)y_{30}'(0))}{1 - a_2(1)}\overline{H}_2(s,1),$$

$$\overline{\phi}_2(s) = -\overline{H}_2(s,0) - \overline{H}_2(s,1)\frac{a_2(0)}{1-a_2(1)}, \quad \overline{\phi}_3(s) = \overline{H}_2(s,1)\frac{1}{1-a_2(1)}. \tag{20}$$

From (17) applying (7),(11), (18)-(20), for the functions $T_i(t,\mu)$, $i = 1,2,3$, $Q(t,\mu)$, we get asymptotic representations:

$$T_1^{(q)}(t,\mu) = y_{30}^{(q)}(t) + \int\limits_0^t \frac{y_{30}^{(q)}(t)\overline{\phi}_1(s)}{y_{30}(s)A_1(s)}ds - \frac{y_{10}(t)\kappa_1^q(t)}{\mu^{q-1}\kappa_1(0)}\left(y_{30}'(0) + \frac{\overline{\phi}_1(0)}{\kappa_1(0)(\kappa_2(0) - \kappa_1(0))}\right)\cdot$$

$$\cdot e^{\frac{1}{\mu}\int\limits_0^t \kappa_1(x)dx} - \frac{y_{20}(t)\kappa_2^q(t)}{\mu^{q-1}y_{20}(1)\kappa_2(1)}\left(\frac{h_3y_{30}(t) - a_2(0)y_{30}'(0)}{1 - a_2(1)} - \frac{\overline{\phi}_1(1)}{\kappa_2(1)(\kappa_2(1) - \kappa_1(1))}\right).$$

$$\cdot e^{\frac{1}{\mu}\int_1^t \kappa_2(x)dx} + \frac{\kappa_1^{q-2}(t) - \kappa_2^{q-2}(t)}{\mu^{q-1}(\kappa_2(t) - \kappa_1(t))} \cdot \overline{\phi}_1(t) +$$

$$+ O\left(\mu + \frac{1}{\mu^{q-2}} e^{\frac{1}{\mu}\int_0^t \kappa_1(x)dx} + \frac{1}{\mu^{q-2}} e^{\frac{1}{\mu}\int_1^t \kappa_2(x)dx} \right);$$

$$T_2^{(q)}(t,\mu) = \int_0^t \frac{y_{30}^{(q)}(t)\overline{\phi}_2(s)}{y_{30}(s)A_1(s)} ds + \frac{\kappa_1^{q-2}(t) - \kappa_2^{q-2}(t)}{\mu^{q-1}(\kappa_2(t) - \kappa_1(t))} \cdot \overline{\phi}_2(t) -$$

$$- \frac{y_{10}(t)\kappa_1^q(t)}{\mu^{q-1}\kappa_1(0)} \left(\frac{\overline{\phi}_2(0)}{\kappa_1(0)(\kappa_2(0) - \kappa_1(0))} - 1 \right) e^{\frac{1}{\mu}\int_0^t \kappa_1(x)dx} +$$

$$+ \frac{y_{20}(t)\kappa_2^q(t)}{\mu^{q-1}y_{20}(1)\kappa_2(1)} \left(\frac{\overline{\phi}_2(1)}{\kappa_2(1)(\kappa_2(1) - \kappa_1(1))} - \frac{a_2(0)}{1 - a_2(1)} \right) e^{\frac{1}{\mu}\int_1^t \kappa_2(x)dx} +$$

$$+ O\left(\mu + \frac{1}{\mu^{q-2}} e^{\frac{1}{\mu}\int_0^t \kappa_1(x)dx} + \frac{1}{\mu^{q-2}} e^{\frac{1}{\mu}\int_1^t \kappa_2(x)dx} \right);$$

$$T_3^{(q)}(t,\mu) = \int_0^t \frac{y_{30}^{(q)}(t)\overline{\phi}_3(s)}{y_{30}(s)A_1(s)} ds - \frac{\kappa_1^q(t)y_{10}(t)\overline{\phi}_3(0)}{\mu^{q-1}\kappa_1^2(0)(\kappa_2(0) - \kappa_1(0))} e^{\frac{1}{\mu}\int_0^t \kappa_1(x)dx} + \quad (21)$$

$$+ \frac{y_{20}(t)\kappa_2^q(t)}{\mu^{q-1}y_{20}(1)\kappa_2(1)} \left(\frac{\overline{\phi}_3(1)}{\kappa_2(1)(\kappa_2(1) - \kappa_1(1))} + \frac{1}{1 - a_2(1)} \right) e^{\frac{1}{\mu}\int_1^t \kappa_2(x)dx} +$$

$$+ \frac{\kappa_1^{q-2}(t) - \kappa_2^{q-2}(t)}{\mu^{q-1}(\kappa_2(t) - \kappa_1(t))} \cdot \overline{\phi}_3(t) + O\left(\mu + \frac{1}{\mu^{q-2}} e^{\frac{1}{\mu}\int_0^t \kappa_1(x)dx} + \frac{1}{\mu^{q-2}} e^{\frac{1}{\mu}\int_1^t \kappa_2(x)dx} \right);$$

$$Q^{(q)}(t,\mu) = \int_0^t \frac{y_{30}^{(q)}(t)\overline{f}(s)}{y_{30}(s)A_1(s)} ds - \frac{\kappa_1^q(t)y_{10}(t)\overline{f}(0)}{\mu^{q-1}\kappa_1^2(0)(\kappa_2(0) - \kappa_1(0))} e^{\frac{1}{\mu}\int_0^t \kappa_1(x)dx} +$$

$$+ \frac{\kappa_2^q(t)y_{20}(t)\overline{f}(1)}{\mu^{q-1}\kappa_2^2(1)y_{20}(1)(\kappa_2(1) - \kappa_1(1))} e^{\frac{1}{\mu}\int_1^t \kappa_2(x)dx} + \frac{\kappa_1^{q-2}(t) - \kappa_2^{q-2}(t)}{\mu^{q-1}(\kappa_2(t) - \kappa_1(t))} \cdot \overline{f}(t) +$$

$$+ O\left(\mu + \frac{1}{\mu^{q-2}} e^{\frac{1}{\mu}\int_0^t \kappa_1(x)dx} + \frac{1}{\mu^{q-2}} e^{\frac{1}{\mu}\int_1^t \kappa_2(x)dx} \right).$$

To find the unknown constants C_i, $i = 1, 2, 3$ from (16), taking account of the boundary conditions (2), we get a system of algebraic equations:

$$\begin{cases} C_1 h_1 T_1(t,\mu) + C_2 h_1 T_2(t,\mu) + C_3 h_1 T_3(t,\mu) = \alpha - h_1 Q(t,\mu), \\ C_1 h_2 T_1(t,\mu) + C_2 h_2 T_2(t,\mu) + C_3 h_2 T_3(t,\mu) = \beta - h_2 Q(t,\mu). \\ C_1 h_3 T_1(t,\mu) + C_2 h_3 T_2(t,\mu) + C_3 h_3 T_3(t,\mu) = \gamma - h_3 Q(t,\mu) \end{cases} \quad (22)$$

For the elements of this system are valid asymptotic representations:

$$h_1 T_1(t, \mu) = 1 + O(\mu), \ h_1 T_i(t, \mu) = O(\mu), \ i = 2, 3, \ h_1 Q(t, \mu) = O(\mu),$$

$$h_2 T_i(t, \mu) = -\frac{\overline{\phi}_i(0)}{\kappa_2(0)(\kappa_2(0) - \kappa_1(0))} + O(\mu), \ \ i = 1, 3,$$

$$h_2 T_2(t, \mu) = 1 - \frac{\overline{\phi}_2(0)}{\kappa_2(0)(\kappa_2(0) - \kappa_1(0))} + O(\mu),$$

$$h_2 Q(t, \mu) = -\frac{\overline{f}(0)}{\kappa_2(0)(\kappa_2(0) - \kappa_1(0))} + O(\mu),$$

$$h_3 T_i(t, \mu) = [\overline{\phi}_i] + O(\mu), \ i = 1, 2, \ h_3 T_3(t, \mu) = 1 + [\overline{\phi}_3] + O(\mu), \ h_3 Q(t, \mu) = [\overline{f}] + O(\mu),$$

where

$$[\overline{\phi}_i] = \int_0^1 \frac{\overline{\phi}_i(s)}{y_{30}(s) A_1(s)} \left(y'_{30}(1) - \int_s^1 \sum_{j=0}^2 a_j(x) y_{30}^{(j)}(x) - a_1(s) y_{30}(s) \right) ds -$$

$$-\frac{a_2(0)\overline{\phi}_i(0)}{\kappa_1(0)(\kappa_2(0) - \kappa_1(0))} + \frac{\overline{\phi}_i(1)}{\kappa_1(1)(\kappa_2(1) - \kappa_1(1))} - \frac{a_2(1)\overline{\phi}_i(1)}{\kappa_2(1)(\kappa_2(1) - \kappa_1(1))}, \ \ i = \overline{1, 3},$$

$$[\overline{f}] = \int_0^1 \frac{\overline{f}(s)}{y_{30}(s) A_1(s)} \left(y'_{30}(1) - \int_s^1 \sum_{j=0}^2 a_j(x) y_{30}^{(j)}(x) - a_1(s) y_{30}(s) \right) ds - \quad (23)$$

$$-\frac{a_2(0)\overline{f}(0)}{\kappa_1(0)(\kappa_2(0) - \kappa_1(0))} + \frac{\overline{f}(1)}{\kappa_1(1)(\kappa_2(1) - \kappa_1(1))} - \frac{a_2(1)\overline{f}(1)}{\kappa_2(1)(\kappa_2(1) - \kappa_1(1))}.$$

Then for the main determinant of the system (22):

$$I(\mu) = \begin{vmatrix} h_1 T_1(t, \mu) & h_1 T_2(t, \mu) & h_1 T_3(t, \mu) \\ h_2 T_1(t, \mu) & h_2 T_2(t, \mu) & h_2 T_3(t, \mu) \\ h_3 T_1(t, \mu) & h_3 T_2(t, \mu) & h_3 T_3(t, \mu) \end{vmatrix},$$

we get the following asymptotic representation:

$$I(\mu) = (1 + [\overline{\phi}_3]) \left(1 - \frac{\overline{\phi}_2(0)}{\kappa_2(0)(\kappa_2(0) - \kappa_1(0))} \right) + \frac{\overline{\phi}_3(0)[\overline{\phi}_2]}{\kappa_2(0)(\kappa_2(0) - \kappa_1(0))} \equiv \overline{I} + O(\mu).$$

C5) $\overline{I} \neq 0$.

Then from the system (22), we determine the unknown constants C_i, $i = 1, 2, 3$. Thus, the following theorem will be true.

Theorem 1 *If conditions (C1)-(C5) are satisfied, then the boundary value problem (1),(2) on the interval $[0, 1]$ has a unique solution, expressed by formula (16), where $T_i(t, \mu)$, $i = 1, 2, 3$, $Q(t, \mu)$ are expressed by formula (17) and the constants C_i, $i = 1, 2, 3$ are determined from system (22).*

For C_i, $i = 1, 2, 3$ are valid asymptotic representations:

$$C_1 = \alpha + O(\mu); \quad C_2 = \frac{1}{\overline{\overline{I}}} \left[\alpha \cdot \frac{\overline{\phi}_1(0)(1 + [\overline{\phi}_3]) - \overline{\phi}_3(0)[\overline{\phi}_1]}{\kappa_2(0)(\kappa_2(0) - \kappa_1(0))} + \right.$$

$$\left. + \left(\beta + \frac{\overline{f}(0)}{\kappa_2(0)(\kappa_2(0) - \kappa_1(0))} \right) (1 + [\overline{\phi}_3]) + (\gamma - [\overline{f}]) \frac{\overline{\phi}_3(0)}{\kappa_2(0)(\kappa_2(0) - \kappa_1(0))} \right] + O(\mu); \quad (24)$$

$$C_3 = \frac{1}{\overline{\overline{I}}} \left[\alpha \left(\frac{\overline{\phi}_2(0)[\overline{\phi}_1] - \overline{\phi}_1(0)[\overline{\phi}_2]}{\kappa_2(0)(\kappa_2(0) - \kappa_1(0))} - [\overline{\phi}_1] \right) - \left(\beta + \frac{\overline{f}(0)}{\kappa_2(0)(\kappa_2(0) - \kappa_1(0))} \right) [\overline{\phi}_2] + \right.$$

$$\left. + (\gamma - [\overline{f}]) \left(1 - \frac{\overline{\phi}_2(0)}{\kappa_2(0)(\kappa_2(0) - \kappa_1(0))} \right) \right] + O(\mu).$$

Theorem 2 *If conditions (C1)-(C5) are satisfied, then for solutions of problem (1),(2) are valid the following asymptotic representations:*

$$y^{(q)}(t, \mu) = \alpha \left[y_{30}^{(q)}(t) + \int_0^t \frac{y_{30}^{(q)}(t)\overline{\phi}_1(s)}{y_{30}(s)A_1(s)} ds - \frac{\kappa_2^{q-2}(t) - \kappa_1^{q-2}(t)}{\mu^{q-1}(\kappa_2(t) - \kappa_1(t))} \cdot \overline{\phi}_1(t) - \right.$$

$$- \frac{y_{10}(t)\kappa_1^q(t)}{\mu^{q-1}\kappa_1(0)} \left(y'_{30}(0) + \frac{\overline{\phi}_1(0)}{\kappa_1(0)(\kappa_2(0) - \kappa_1(0))} \right) e^{\frac{1}{\mu} \int_0^t \kappa_1(x)dx} - \frac{y_{20}(t)\kappa_2^q(t)}{\mu^{q-1}y_{20}(1)\kappa_2(1)} \cdot$$

$$\left. \cdot \left(\frac{h_3 y_{30}(t) - a_2(0)y'_{30}(0)}{1 - a_2(1)} - \frac{\overline{\phi}_1(1)}{\kappa_2(1)(\kappa_2(1) - \kappa_1(1))} \right) e^{\frac{1}{\mu} \int_1^t \kappa_2(x)dx} \right] +$$

$$+ \frac{1}{\overline{\overline{I}}} \left[\alpha \frac{\overline{\phi}_1(0)(1 + [\overline{\phi}_3]) - [\overline{\phi}_1]\overline{\phi}_3(0)}{\kappa_2(0)(\kappa_2(0) - \kappa_1(0))} + \right.$$

$$\left. + \left(\beta + \frac{\overline{f}(0)}{\kappa_2(0)(\kappa_2(0) - \kappa_1(0))} \right) (1 + [\overline{\phi}_3]) + \frac{(\gamma - [\overline{f}])\overline{\phi}_3(0)}{\kappa_2(0)(\kappa_2(0) - \kappa_1(0))} \right] \cdot$$

$$\cdot \left[\int_0^t \frac{y_{30}^{(q)}(t)\overline{\phi}_2(s)}{y_{30}(s)A_1(s)} ds - \frac{\kappa_2^{q-2}(t) - \kappa_1^{q-2}(t)}{\varepsilon^{q-1}(\kappa_2(t) - \kappa_1(t))} \cdot \overline{\phi}_2(t) - \right.$$

$$- \frac{y_{10}(t)\kappa_1^q(t)}{\mu^{q-1}\kappa_1(0)} \left(\frac{\overline{\phi}_2(0)}{\kappa_1(0)(\kappa_2(0) - \kappa_1(0))} - 1 \right) e^{\frac{1}{\mu} \int_0^t \kappa_1(x)dx} +$$

$$\left. + \frac{y_{20}(t)\kappa_2^q(t)}{\mu^{q-1}y_{20}(1)\kappa_2(1)} \left(\frac{\overline{\phi}_2(1)}{\kappa_2(1)(\kappa_2(1) - \kappa_1(1))} - \frac{a_2(0)}{1 - a_2(1)} \right) e^{\frac{1}{\mu} \int_1^t \kappa_2(x)dx} \right] +$$

$$+ \frac{1}{\overline{\overline{I}}} \left[\alpha \left(\frac{\overline{\phi}_2(0)[\overline{\phi}_1] - [\overline{\phi}_2]\overline{\phi}_1(0)}{\kappa_2(0)(\kappa_2(0) - \kappa_1(0))} - [\overline{\phi}_1] \right) - \left(\beta + \frac{\overline{f}(0)}{\kappa_2(0)(\kappa_2(0) - \kappa_1(0))} \right) [\overline{\phi}_2] + \right.$$

$$+ (\gamma - [\overline{f}]) \left(1 - \frac{\overline{\phi}_2(0)}{\kappa_2(0)(\kappa_2(0) - \kappa_1(0))} \right) \right] \cdot \left[\int\limits_0^t \frac{y_{30}^{(q)}(t)\overline{\phi}_3(s)}{y_{30}(s)A_1(s)} ds - \tag{25}$$

$$- \frac{\kappa_2^{q-2}(t) - \kappa_1^{q-2}(t)}{\mu^{q-1}(\kappa_2(t) - \kappa_1(t))} \overline{\phi}_3(t) - \frac{\kappa_1^q(t)y_{10}(t)\overline{\phi}_3(0)}{\mu^{q-1}\kappa_1^2(0)(\kappa_2(0) - \kappa_1(0))} e^{\frac{1}{\mu}\int\limits_0^t \kappa_1(x)dx} +$$

$$+ \frac{y_{20}(t)\kappa_2^q(t)}{\mu^{q-1}y_{20}(1)\kappa_2(1)} \left(\frac{\overline{\phi}_3(1)}{\kappa_2(1)(\kappa_2(1) - \kappa_1(1))} + \frac{1}{1 - a_2(1)} \right) e^{\frac{1}{\mu}\int\limits_1^t \kappa_2(x)dx} \right] +$$

$$+ \int\limits_0^t \frac{y_{30}^{(q)}(t)\overline{f}(s)}{y_{30}(s)A_1(s)} ds - \frac{\kappa_1^q(t)y_{10}(t)\overline{f}(0)}{\mu^{q-1}\kappa_1^2(0)(\kappa_2(0) - \kappa_1(0))} e^{\frac{1}{\mu}\int\limits_0^t \kappa_1(x)dx} +$$

$$+ \frac{\kappa_2^q(t)y_{20}(t)\overline{f}(1)}{\mu^{q-1}\kappa_2^2(1)y_{20}(1)(\kappa_2(1) - \kappa_1(1))} e^{\frac{1}{\mu}\int\limits_1^t \kappa_2(x)dx} + \frac{\kappa_1^{q-2}(t) - \kappa_2^{q-2}(t)}{\mu^{q-1}(\kappa_2(t) - \kappa_1(t))} \cdot \overline{f}(t)+$$

$$+ O \left(\mu + \frac{1}{\mu^{q-2}} e^{\frac{1}{\mu}\int\limits_0^t \kappa_1(x)dx} + \frac{1}{\mu^{q-2}} e^{\frac{1}{\mu}\int\limits_1^t \kappa_2(x)dx} \right), \quad q = \overline{0,2}.$$

The proof of the theorem follows from (16) taking account of (21), (24).

From formula (25) for solutions of problem (1), (2), we obtain the following order of growth:

$$y''(0, \mu) = O\left(\frac{1}{\mu}\right), \quad y''(1, \mu) = O\left(\frac{1}{\mu}\right), \quad \mu \to 0.$$

It implies that at points $t = 0$ and $t = 1$ the solution of the considered problem (1), (2) have the first order boundary jumps.

4 Modified degenerate problem

Singularly perturbed boundary value problem (1), (2), we put in accordance with the following modified degenerate problem:

$$L_0\overline{y} \equiv A_1(t)\overline{y}'(t) + A_2(t)\overline{y}(t) = F(t) + \int\limits_0^1 \sum\limits_{i=0}^2 H_i(t,x)\overline{y}^{(i)}(x)dx + \Theta(t), \tag{26}$$

$$h_1\overline{y} \equiv \overline{y}(0) = \alpha, \quad h_2\overline{y} \equiv \overline{y}'(0) = \beta + \Theta_0,$$

$$h_3\overline{y} \equiv \overline{y}'(1) - \int\limits_0^1 \sum\limits_{i=0}^2 a_i(x)\overline{y}^{(i)}(x)dx = \gamma + a_2(0)\Theta_0 + (1 - a_2(1))\Theta_1, \tag{27}$$

where $\Theta(t)$ and Θ_0, Θ_1 are called jumps of the integral term and solution respectively.

For the difference $u(t, \mu)$ between the solution $y(t, \mu)$ of the singularly perturbed problem (1), (2) and the solution $\overline{y}(t)$ of the degenerate problem, we get the problem:

$$L_\mu u \equiv \mu^2 u''' + \mu A_0(t) u'' + A_1(t) u' + A_2(t) u = \int_0^1 \sum_{i=0}^2 H_i(t, x) u^{(i)}(x, \mu) dx - \Theta(t) -$$

$$-\mu^2 y''' - \mu A_0(t) y'', \tag{28}$$

$$h_1 u \equiv u(0, \mu) = 0, \quad h_2 u \equiv u'(0, \mu) = -\Theta_0,$$

$$h_3 u \equiv u'(1, \mu) - \int_0^1 \sum_{i=0}^2 a_i(x) u^{(i)}(x, \mu) dx = -a_2(0)\Theta_0 - (1 - a_2(1))\Theta_1. \tag{29}$$

The problem (28),(29) is of the same type as the problem (1),(2), by applying asymptotic formula (25). As a result, we obtain asymptotic representation for the solution of the problem (28), (29):

$$u^{(q)}(t, \mu) = \frac{1}{\overline{I}} \left[-\left(\Theta_0 + \frac{\overline{\Theta}(0)}{\kappa_2(0)(\kappa_2(0) - \kappa_1(0))} \right) (1 + [\overline{\phi}_3]) + \right.$$

$$- \frac{\left(a_2(0)\Theta_0 + (1 - a_2(1))\Theta_1 - [\overline{\Theta}] \right) \overline{\phi}_3(0)}{\kappa_2(0)(\kappa_2(0) - \kappa_1(0))} \right] \cdot \left[\int_0^t \frac{y_{30}^{(q)}(t)\overline{\phi}_2(s)}{y_{30}(s)A_1(s)} ds - \frac{\kappa_2^{q-2}(t) - \kappa_1^{q-2}(t)}{\mu^{q-1}(\kappa_2(t) - \kappa_1(t))} \overline{\phi}_2(t) - \right.$$

$$- \frac{\kappa_1^q(t) y_{10}(t)}{\mu^{q-1} \kappa_1(0)} \left(\frac{\overline{\phi}_2(0)}{\kappa_1(0)(\kappa_2(0) - \kappa_1(0))} - 1 \right) e^{\frac{1}{\mu} \int_0^t \kappa_1(x) dx} +$$

$$+ \frac{y_{20}(t) \kappa_2^q(t)}{\mu^{q-1} y_{20}(1) \kappa_2(1)} \left(\frac{\overline{\phi}_2(1)}{\kappa_2(1)(\kappa_2(1) - \kappa_1(1))} - \frac{a_2(0)}{1 - a_2(1)} \right) e^{\frac{1}{\mu} \int_1^t \kappa_2(x) dx} \right] +$$

$$+ \frac{1}{\overline{I}} \left[\left(\Theta_0 + \frac{\overline{\Theta}(0)}{\kappa_2(0)(\kappa_2(0) - \kappa_1(0))} \right) [\overline{\phi}_2] + \tag{30} \right.$$

$$- \left(a_2(0)\Theta_0 + (1 - a_2(1))\Theta_1 - [\overline{\Theta}] \right) \left(1 - \frac{\overline{\phi}_2(0)}{\kappa_2(0)(\kappa_2(0) - \kappa_1(0))} \right) \right] \cdot \left[\int_0^t \frac{y_{30}^{(q)}(t)\overline{\phi}_3(s)}{y_{30}(s)A_1(s)} ds - \right.$$

$$- \frac{\kappa_2^{q-2}(t) - \kappa_1^{q-2}(t)}{\mu^{q-1}(\kappa_2(t) - \kappa_1(t))} \overline{\phi}_3(t) - \frac{\kappa_1^q(t) y_{10}(t) \overline{\phi}_3(0)}{\mu^{q-1} \kappa_1^2(0)(\kappa_2(0) - \kappa_1(0))} e^{\frac{1}{\mu} \int_0^t \kappa_1(x) dx} +$$

$$+ \frac{y_{20}(t) \kappa_2^q(t)}{\mu^{q-1} y_{20}(1) \kappa_2(1)} \left(\frac{\overline{\phi}_3(1)}{\kappa_2(1)(\kappa_2(1) - \kappa_1(1))} + \frac{1}{1 - a_2(1)} \right) e^{\frac{1}{\mu} \int_1^t \kappa_2(x) dx} \right] -$$

$$-\int_0^t \frac{y_{30}^{(q)}(t)\overline{\Theta}(s)}{y_{30}(s)A_1(s)}ds + \frac{\kappa_1^q(t)y_{10}(t)\overline{\Theta}(0)}{\mu^{q-1}\kappa_1^2(0)(\kappa_2(0)-\kappa_1(0))}e^{\frac{1}{\mu}\int_0^t \kappa_1(x)dx} +$$

$$-\frac{\kappa_2^q(t)y_{20}(t)\overline{\Theta}(1)}{\mu^{q-1}\kappa_2^2(1)y_{20}(1)(\kappa_2(1)-\kappa_1(1))}e^{\frac{1}{\mu}\int_1^t \kappa_2(x)dx} + \frac{\kappa_2^{q-2}(t)-\kappa_1^{q-2}(t)}{\mu^{q-1}(\kappa_2(t)-\kappa_1(t))}\overline{\Theta}(t)+$$

$$+O\left(\mu + \frac{1}{\mu^{q-2}}e^{-\gamma_1\frac{t}{\mu}} + \frac{1}{\mu^{q-2}}e^{-\gamma_2\frac{1-t}{\mu}}\right), \quad q = \overline{0,2},$$

where

$$\overline{\Theta}(t) \equiv \Theta(t) + \int_0^1 R_0(t,s,1)\Theta(s)ds. \tag{31}$$

If the function $\overline{\Theta}(t)$ is selected in the form

$$\overline{\Theta}(t) = -\Theta_0\left(\overline{\phi}_2(t) + a_2(0)\overline{\phi}_3(t)\right) - \Theta_1(1-a_2(1))\overline{\phi}_3(t), \tag{32}$$

it can be presented that $u(t,\mu) \to 0$, $\mu \to 0$. From (32) in view of (18),(20), (31) to determine the jump of integral terms $\Theta(t)$, we obtain the following formula:

$$\Theta(t) = \Theta_0 H_2(t,0) - \Theta_1 H_2(t,1). \tag{33}$$

Theorem 3 *If conditions (C1)-(C5) are satisfied, then for the solution $y(t,\mu)$ of singularly perturbed integral boundary value problem (1),(2) are valid the limiting equalities:*

$$\lim_{\mu\to 0} y(t,\mu) = \overline{y}(t), \quad 0 \le t \le 1,$$

$$\lim_{\mu\to 0} y^{(i)}(t,\mu) = \overline{y}^{(i)}(t), \quad i = 1,2, \ 0 < t < 1,$$

where $\overline{y}(t)$ is a solution of modified degenerate boundary value problem (26),(27), the jump of the integral terms $\Theta(t)$ is determined by formula (33).

Example. To illustrate the results, we give an example. For simplicity, we consider the integral boundary value problem for a singularly perturbed integro-differential equation with constant coefficients:

$$L_\mu y \equiv \mu^2 y''' - \mu y'' - 2y' = 1 + \int_0^1 \delta y''(x,\mu)dx, \ \delta \ne 0,$$

$$h_1 y \equiv y(0,\mu) = \alpha, \ h_2 y \equiv y'(0,\mu) = \beta, \ h_3 y \equiv y'(1,\mu) - \int_0^1 \nu y''(x,\mu)dx = \gamma, \ \nu \ne 0, \ \nu \ne 1,$$

$$\tag{34}$$

where

$$A_0(t) = -1, \ A_1(t) = -2, \ A_2(t) = 0, \ F(t) = 1, \ H_i(t,x) = 0, \ i = 0,1,$$

$$H_2(t,x) = \delta, \ a_i(x) = 0, \ i = 0,1, \ a_2(x) = \nu.$$

The exact solution to this problem is:

$$y(t,\mu) = \mu \cdot \frac{\delta(\gamma - \beta) + (2\beta + 1)(1 - \nu) + [\beta(\delta + 2\nu) - \gamma(\delta + 2) + \nu - 1]e^{-\frac{2}{\mu}}}{2(1 - \nu)\left(e^{-\frac{3}{\mu}} - 1\right)} \cdot e^{-\frac{t}{\mu}} +$$

$$+\mu \cdot \frac{\beta(\delta + 2\nu) - \gamma(\delta + 2) + \nu - 1 + [\delta(\gamma - \beta) + (2\beta + 1)(1 - \nu)]e^{-\frac{1}{\mu}}}{4(1 - \nu)\left(e^{-\frac{3}{\mu}} - 1\right)} \cdot e^{-\frac{2(t-1)}{\mu}} + \alpha -$$

$$-\mu \cdot \frac{2\left[\delta(\gamma - \beta) + (2\beta + 1)(1 - \nu)\right] + 3\left[\beta(\delta + 2\nu) - \gamma(\delta + 2) + \nu - 1\right]e^{-\frac{2}{\mu}}}{4(1 - \nu)\left(e^{-\frac{3}{\mu}} - 1\right)} -$$

$$-\mu \cdot \frac{[\delta(\gamma - \beta) + (2\beta + 1)(1 - \nu)]e^{-\frac{3}{\mu}}}{4(1 - \nu)\left(e^{-\frac{3}{\mu}} - 1\right)} + \frac{\delta(\beta - \gamma)}{2(1 - \nu)} \cdot t - \frac{t}{2}.$$

Hence, passing to the limit at $\mu \to 0$, we get

$$\lim_{\mu \to 0} y(t,\mu) = \alpha + \frac{\delta(\beta - \gamma) + \nu - 1}{2(1 - \nu)}t \equiv \overline{y}(t), \ \ 0 \le t \le 1, \tag{35}$$

where $\overline{y}(t)$ is not a solution to the usual degenerate equation, obtained from (1) at $\mu = 0$, and is a solution of modified degenerate boundary value problem:

$$-2\overline{y}' = 1 + \int_0^1 \delta\overline{y}''(x)dx + \Theta(t), \tag{36}$$

$$h_1\overline{y} \equiv \overline{y}(0) = \alpha, \ \ h_2\overline{y} \equiv \overline{y}'(0) = \beta + \Theta_0,$$

$$h_3\overline{y} \equiv \overline{y}'(1) - \int_0^1 \nu\overline{y}''(x)dx = \gamma + \nu\Theta_0 + (1 - \nu)\Theta_1, \tag{37}$$

where Θ_0, Θ_1 are jumps of the solution at points $t = 0$ and $t = 1$ respectively, determined from problem (4.11), (4.12) and the value of the jump of the integral terms $\Delta(t)$ in accordance with formula (4.8) should be determined in the form:

$$\Theta(t) = \delta(\Theta_0 - \Theta_1). \tag{38}$$

The exact solution of the problem (36), (37) according to the condition (38) has the form

$$\overline{y}(t) = \alpha + \frac{\delta(\beta - \gamma) + \nu - 1}{2(1 - \nu)}t.$$

Therefore, the passage to the limit (35) will be valid. The limit transitions are similarly valid.

$$\lim_{\mu \to 0} y^{(i)}(t,\mu) = \overline{y}^{(i)}(t), \ \ i = 1,2, \ \ 0 < t < 1.$$

5 Conclusion

In this paper, we study the boundary value problem for singularly perturbed integro-differential equations with the phenomena of the so-called boundary jumps, when the fast solution variable becomes unbounded at both boundaries. The exceptions of the qualitative influence of integral terms on the asymptotic behavior of the solutions for singularly perturbed integro-differential equations are shown. The presence of integral terms will significantly change the degenerate equation: the solution of the assumed singularly perturbed integro-differential equation does not tend to the solution of the usual degenerate equation, obtained from the supposed equation with the zero value of a small parameter and will tend to solve a specially modified degenerate integro-differential equation with an additional term called the jump of the integral term. Moreover, the value of the jump of the integral term is determined differently than in [33] because of occurence of jumps in solutions of the same order. In addition, in the boundary conditions of the modified degenerate problem, there is also a change. The so-called jumps of the first derivative of the solution appear $\Theta_0 = \overline{y}'(0) - y'(0, \mu) \neq 0$ and $\Theta_1 = \overline{y}'(1) - y'(1, \mu) \neq 0$ at the ends of considered interval.

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