IRSTI 41.03.02 DOI: https://doi.org/10.26577/JMMCS.2022.v113.i1.10

M.Zh. Minglibayev , **A.K. Kushekbay***
Al-Farabi Kazakh National University, Kazakhstan, Almaty
*e-mail: kkabylay@gmail.com

SECULAR PERTURBATIONS OF TRANSLATIONAL-ROTATIONAL MOTION IN THE NON-STATIONARY THREE-BODY PROBLEM

Modern observational data in astronomy show that real space systems are non-stationary, their masses, sizes, shapes and a few other physical characteristics change over time during evolution. In this connection, the creation of mathematical models of the motion of non-stationary celestial bodies becomes relevant. We consider a non-stationary three-body problem with axisymmetric dynamical structure, shape and variable compression. The Newtonian interaction force is characterized by an approximate expression of the force function accurate to the second harmonic. The masses of bodies change isotropically at different rates. The axes of inertia of the proper coordinate system of non-stationary axisymmetric three bodies coincide with the major axes of inertia of the bodies, and it is assumed that their relative orientations remain unchanged in the process of evolution. Differential equations of translation-rotational motion of three non-stationary axisymmetric bodies with variable masses and dimensions in the relative coordinate system, with the origin in the center of the more massive body, are presented. The analytical expression for the Newtonian force function of the interaction of three bodies with variable masses and dimensions is given. The canonical equations of translational-rotational motion of three bodies in Delaunay-Andoyer analogues are obtained. The equations of secular perturbations of translational-rotational motion of non-stationary axisymmetric three-bodies in the Delaunay-Andoyer analogues of osculating elements have been obtained. The new results obtained can be used to analyze the dynamic evolution of the translation-rotational motion of the three-body problem. The problem is investigated by methods of perturbation theory.
Key words: celestial mechanics, three-body problem, variable mass, secular perturbation, axisymmetric body, translational-rotational motion.

М.Дж. Минглибаев, А.К. Кушекбай*
Әл-Фараби атындағы Қазақ ұлттық университеті, Қазақстан, Алматы қ.
*e-mail: kkabylay@gmail.com

Бейстационар үш дене есебінің ілгерілемелі-айналмалы қозғалысының ғасырлық ұйытқулары

Астрономиядағы заманауи бақылау деректері нақты ғарыштық жүйелердің бейстационар екендігін, олардың массалары, өлшемдері, пішіні және басқа да бірқатар физикалық сипаттамалары эволюция барысында өзгеретіндігін көрсетеді. Осыған байланысты бейстационар аспан денелерінің қозғалысының математикалық модельдерін жасау актуалды мәселе болып табылады. Динамикалық құрылымы мен формасы өстік симметриялы және сығылуы ауыспалы бейстационар үш дене есебі қарастырылған. Ньютондық өзара әрекеттесу күші күш функциясының екінші гармоникаға дәл келетін жуық мәнімен өрнегімен сипатталады. Дене массалары әр түрлі жылдамдықта изотропты түрде өзгереді. Бейстационар өстік симметриялы үш дененің өзіндік координаттар жүйесінің өстері денелердің негізгі инерция өстерімен сәйкес келеді және эволюция барысында олардың салыстырмалы бағдары өзгеріссіз қалады деп есептеледі. Массалары мен өлшемдері айнымалы, бейстационар, өстік-симметриялы үш дененің ілгерілемелі-айналмалы қозғалысының дифференциалдық теңдеулері массасы үлкенірек дененің центрінен басталатын салыстырмалы координаталар жүйесінде келтірілген.

Массалары мен өлшемдері айнымалы үш дененің Ньютондық өзара әрекеттесуінің күштік функциялары үшін аналитикалық өрнек келтірілген. Үш дененің ілгерілемелі-айналмалы қозғалысының канондық теңдеулері Делоне-Андуайе оскуляциялаушы элементтерінің аналогтарында алынған. Бейстационар өстік-симметриялы үш дененің ілгерілемелі-айналмалы қозғалысының ғасырлық ұйытқу теңдеулері Делоне-Андуайе оскуляциялаушы элементтерінің аналогтарында алынған. Алынған жаңа нәтижелерді үш дене есебінің ілгерілемелі-айналмалы қозғалысының динамикалық эволюциясын талдауға пайдалануға болады. Мәселе ұйытқу теориясының әдістерімен зерттеледі.
Түйін сөздер: Аспан механикасы, үш дене есебі, айнымалы масса, ғасырлық ұйытқу, өстік симметриялық дене, ілгерілемелі-айналмалы қозғалыс.

М.Дж. Минглибаев, А.К. Кушекбай*
Казахский национальный университет имени аль-Фараби, Казахстан, г. Алматы
*e-mail: kkabylay@gmail.com

Вековые возмущения поступательно-вращательного движения в нестационарной задаче трех тел

Современные данные наблюдений в астрономии показывают, что реальные космические системы являются нестационарными, их массы, размеры, форма и ряд других физических характеристик изменяются с течением времени в процессе эволюции. В связи с этим, становится актуальным создание математических моделей движения нестационарных небесных тел. Рассматривается нестационарная задача трех тел, обладающих осесимметричным динамическим строением, формой и переменным сжатием. Ньютоновская сила взаимодействия характеризуется приближенным выражением силовой функции с точностью до второй гармоники. Массы тел изменяются изотропно в различных темпах. Оси инерции собственной системы координат нестационарных осесимметричных трех тел совпадают с главными осями инерции тел, и предполагается, что в ходе эволюции их относительная ориентация остаются неизменными. Приведены дифференциальные уравнения поступательно-вращательного движения трех нестационарных осесимметричных тел с переменными массами и размерами в относительной системе координат, с началом в центре более массивного тела. Приведены аналитическое выражение силовой функций ньютоновского взаимодействия трех тел с переменными массами и размерами. Получены канонические уравнения поступательно-вращательного движения трех тел в аналогах оскулирующих элементов Делоне-Андуайе. Получены уравнения вековых возмущений поступательно-вращательного движения нестационарных осесимметричных трех тел в аналогах оскулирующих элементов Делоне-Андуайе. Полученные новые результаты могут быть использованы для анализа динамической эволюции поступательно-вращательного движения задачи трех тел. Задача исследована методами теории возмущений.

Ключевые слова: Небесная механика, задача трех тел, переменная масса, вековое возмущение, осесимметричное тело, поступательно-вращательное движение.

1 Introduction

In classical celestial mechanics, real celestial bodies are considered as material points moving in absolutely empty space under the action of the forces of mutual attraction according to Newton's law of universal gravitation [1]. However, it is not always possible to be satisfied with this first approximation. In other cases, it is impossible to consider real celestial bodies as material points and we have to take into account the influence of their shape by considering them as rigid bodies.

But in reality celestial bodies are not material points (spherically symmetric bodies), but they are not, of course, absolutely rigid bodies either, but always possess a certain degree of plasticity or even are liquid (or gaseous, or dusty) formations [2]. Modern observational data in astronomy show that the real space systems are non-stationary, their masses, sizes, shapes and a number of other physical characteristics change over time during the evolution [5, 12]. In this connection, the creation of mathematical models of the motion of nonstationary celestial bodies becomes actual.

The goal of this work is to obtain differential equations of secular perturbations of the translational-rotational motion of nonstationary axisymmetric three-body dynamic structure, shape, and variable compression. The solution of this problem is associated with rather cumbersome symbolic calculations, which are best performed using computer algebra systems [9, 11].

2 Problem formulation and equations of motion in the relative coordinate system

Let us consider the motion of three non-stationary axisymmetric celestial bodies T_0, T_1, T_2 with variable masses, sizes and variable compression moving in a absolutely empty space under the action of mutual attraction forces according to Newton's law of universal gravitation.

Let the shapes of bodies T_0, T_1, T_2 are different, axisymmetric, and have their own equatorial symmetry plane. Let also assume that the compressions of the bodies with respect to the equatorial plane are variable. The initial locations of the main axes of inertia and the center of inertia in the body of axisymmetric bodies remain unchanged during evolution and are directed along the intersection of the three mutually perpendicular planes.

Let $m_i = m_i(t_0)\nu_i$ be the mass, $l_i = l_i(t_0)\chi_i$ be the characteristic linear dimension, and A_i, B_i, C_i be the second-order principal moments of inertia of the bodies T_i, t_0 be the initial time, ν_i, χ_i $(i = 0, 1, 2)$ be the dimensionless known time functions.

Let us make the following assumptions:

1. Bodies with variable masses $m_i = m_i(t)$ have equatorial symmetry planes and characteristic linear sizes $l_i = l_i(t)$. The second order moments of inertia of the considered bodies are variable

$$A_i = A_i(t), \quad B_i = B_i(t), \quad C_i = C_i(t). \tag{1}$$

2. Bodies are axisymmetric and remain axisymmetric with respect to their own equatorial symmetry planes during evolution

$$A_i(t) = B_i(t) \neq C_i(t) \tag{2}$$

3. The axes of inertia of the proper coordinate system $G_i\widetilde{\xi}_i\widetilde{\eta}_i\widetilde{\zeta}_i$ coincide with the main axes of inertia and this position in the process of evolution is saved.

4. Masses and characteristic sizes of bodies change at different specific rates

$$\frac{\dot{m}_0(t)}{m_0(t)} \neq \frac{\dot{m}_1(t)}{m_1(t)} \neq \frac{\dot{m}_2(t)}{m_2(t)}, \quad \frac{\dot{l}_0(t)}{l_0(t)} \neq \frac{\dot{l}_1(t)}{l_1(t)} \neq \frac{\dot{l}_2(t)}{l_2(t)}. \tag{3}$$

5. Let us assume that the masses of the bodies change isotropically and there are no reactive forces as well as additional rotational moments.

$$\vec{F}_{(reac)} = 0, \quad \vec{M}^{(add)} = 0 \tag{4}$$

6. In the expression for the force function, we restrict the approximation to the second harmonic inclusive.

$$U \approx U^{(0)} + U^{(2)} \tag{5}$$

If the above assumptions are satisfied, the translational motions of bodies T_1 and T_2 in the gravitational field of the "central" body T_0 in the relative coordinate system (Fig. 1) are described by the equations [1-3]:

$$\ddot{x}_i = \frac{1}{\mu_i(t)}\frac{\partial U_{i0}^{(0)}}{\partial x_i} + \frac{\partial V_i}{\partial x_i}\ddot{y}_i = \frac{1}{\mu_i(t)}\frac{\partial U_{i0}^{(0)}}{\partial y_i} + \frac{\partial V_i}{\partial y_i}\ddot{z}_i = \frac{1}{\mu_i(t)}\frac{\partial U_{i0}^{(0)}}{\partial z_i} + \frac{\partial V_i}{\partial z_i} \tag{6}$$

where x_i, y_i, z_i coordinates of the center of mass of the body T_1 and T_2 in the relative coordinate system G_0xyz with origin in the center of the body T_0, $\mu_i(t) = m_0 m_i/(m_i + m_i)$ – reduced masses, V_i – perturbing functions have the form[2]

$$V_i = \frac{1}{\mu_i}U_{i0}^{(2)} + \frac{1}{m_i}U_{ij} + \frac{1}{m_0}\left[x_i\frac{\partial U_{j0}}{\partial x_j} + y_i\frac{\partial U_{j0}}{\partial y_j} + z_i\frac{\partial U_{j0}}{\partial z_j}\right], \quad (i,j = 1,2), \quad (i \neq j) \tag{7}$$

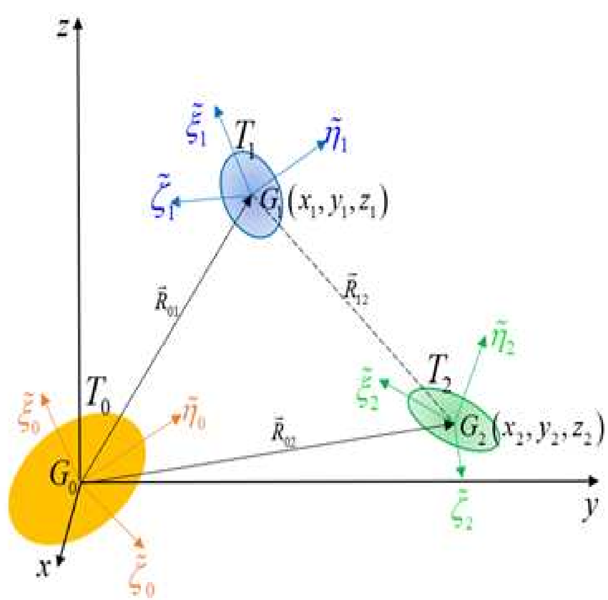

Figure 1: Bodies in a relative coordinate system G_0xyz.

$G_0\widetilde{\xi}_0\widetilde{\eta}_0\widetilde{\zeta}_0$, $G_1\widetilde{\xi}_1\widetilde{\eta}_1\widetilde{\zeta}_1$, $G_2\widetilde{\xi}_2\widetilde{\eta}_2\widetilde{\zeta}_2$ – own coordinate systems.

The expression of the Newtonian force function of the interaction of three non-stationary bodies has the form [1, 2]

$$U = \frac{1}{2}\sum_{i=0}^{2}\sum_{j=0,i\neq j}^{2} U_{ij} \tag{8}$$

U_{ij} – the force function of the mutual attraction of the two bodies T_i and T_j is

$$U_{ij} \approx U_{ij}^{(0)} + U_{ij}^{(2)} \tag{9}$$

$U_{ij}^{(0)}$ – the first term of the force function decomposition is

$$U_{ij}^{(0)} = f\frac{m_i m_j}{R_{ij}} \tag{10}$$

$U_{ij}^{(2)}$ – the second term of the force function decomposition is equal to

$$U_{ij}^{(2)} = fm_i\frac{2A_j + C_j - 3I_j^{(j,i)}}{2R_{ij}^3} + fm_j\frac{2A_i + C_i - 3I_i^{(i,j)}}{2R_{ij}^3} \tag{11}$$

where $R_{ij} = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2 + (z_j - z_i)^2}$ – the mutual distance between the centers of inertia of the considered bodies, f gravitational constant, $I_i^{(i,j)}$ and$I_j^{(j,i)}$ – moments of inertia of bodies T_i and T_j with relative to vector G_iG_j, connecting the centers of mass of two bodies, is defined by the expression

$$I_i^{(i,j)} = A_i(\alpha_{ij}^2 + \beta_{ij}^2) + C_i\gamma_{ij}^2 \quad I_j^{(i,j)} = A_j(\alpha_{ji}^2 + \beta_{ji}^2) + C_j\gamma_{ji}^2 \tag{12}$$

Where $\alpha_{ij}, \beta_{ij}, \gamma_{ij}$ – the directional cosines of the vector G_iG_j with the main central axes of inertia of the body T_i. The rotational motions of bodies T_0, T_1, T_2 around their own center of masses in Euler variables are described by the equations [1]

$$\begin{aligned}
&\frac{d}{dt}(A_jp_j) - (A_j - C_j)q_jr_j = \left[\frac{\partial U}{\partial\psi_j} - cos\theta_j\frac{\partial U}{\partial\varphi_j}\right]\frac{sin\varphi_j}{sin\theta_j} + cos\varphi_j\frac{\partial U}{\partial\theta_j},\\
&\frac{d}{dt}(A_jq_j) - (C_j - A_j)p_jr_j = \left[\frac{\partial U}{\partial\psi_j} - cos\theta_j\frac{\partial U}{\partial\varphi_j}\right]\frac{cos\varphi_j}{sin\theta_j} - sin\varphi_j\frac{\partial U}{\partial\theta_j},\\
&\frac{d}{dt}(C_jr_j) = \frac{\partial U}{\partial\varphi_j} \quad j = 0, 1, 2
\end{aligned} \tag{13}$$

where

$$p_j = \dot\psi_j\sin\theta_j\sin\varphi_j + \dot\theta_j\cos\varphi_j, \quad q_j = \dot\psi_j\sin\theta_j\cos\varphi_j - \dot\theta_j\sin\varphi_j, \quad r_j = \dot\psi_j\cos\theta_j + \varphi_j \tag{14}$$

p_j, q_j, r_j – the projections of the angular velocities of bodies T_j on the axes of their own coordinate systems $G_j\widetilde{\xi}_j\widetilde{\eta}_j\widetilde{\zeta}_j$, $\varphi_j, \psi_j, \theta_j$ – Euler angles [6].

The resulting equations (6) and (13) fully characterize the translational-rotational motion of bodies T_1 and T_2 and the rotational motion of body T_0 in the relative coordinate system G_0xyz in the considered statement.

The equations of perturbed motion in the form of Newton's equations, although they are the most general in the case when the perturbing forces admit a force function, are inconvenient. In this case the equations of perturbed motion in the form of the canonical Hamilton equations are preferable, which have, as in the classical problems, a number of advantages and elegance [5].

In the considered formulation of the problem is very complicated, so we will use the methods of perturbation theory for its investigation [1].

3 Equations of motion in osculating Delaunay-Andoyer elements

For our purposes, the canonical equations of perturbed motion in osculating analogues of Delaunay-Andoyer elements are preferable [1].

Let us consider the analogues of Delaunay-Andoyer elements.

$$L, G, H, l, g, h \quad \text{– Delaunay elements} \tag{15}$$

$$L', G', H', l', g', h' \quad \text{– Andoyerelements (see Fig. 2)} \tag{16}$$

Equations of translational motion of bodies T_1 and T_2 in osculating Delaunay elements have the form [3].

$$\dot{L}_i = \frac{\partial F_i}{\partial l_i}, \quad \dot{G}_i = \frac{\partial F_i}{\partial g_i}, \quad \dot{H}_i = \frac{\partial F_i}{\partial h_i}, \quad \dot{l}_i = -\frac{\partial F_i}{\partial L_i}, \quad \dot{g}_i = -\frac{\partial F_i}{\partial G_i}, \quad \dot{h}_i = -\frac{\partial F_i}{\partial H_i} \tag{17}$$

where

$$F_i = \frac{1}{\sigma_i^2}\frac{\mu_{0i}^2}{2L_i^2} + F_{ipert} \tag{18}$$

$$F_{ipert} = V_i - \frac{1}{2}b_i R_{i0}^2 \tag{19}$$

$$b_i = b_i(t_0) = \frac{\ddot{\sigma}_i}{\sigma_i} = (m_0 + m_i)\frac{d^2}{dt^2}\left(\frac{1}{m_0 + m_i}\right), \quad \sigma_i = \frac{m_0(t_0) + m_i(t_0)}{m_0(t) + m_i(t)}, \quad i = 1, 2 \tag{20}$$

Given the relation $\alpha_{ij}^2+\beta_{ij}^2+\gamma_{ij}^2=1$ and formulas (9) – (12), expression (19) has the form

$$F_{ipert}=\frac{1}{\mu_i}\left(fm_0(C_i-A_i)\left[\frac{1-3\gamma_{i0}^2}{2R_{i0}^3}\right]+fm_i(C_0-A_0)\left[\frac{1-3\gamma_{0i}^2}{R_{0i}^3}\right]\right)+\frac{1}{m_i}\times$$

$$\times\left(fm_im_j\left[\frac{1}{R_{ij}}\right]+fm_j(C_i-A_i)\left[\frac{1-3\gamma_{ij}^2}{2R_{ij}^3}\right]+fm_i(C_j-A_j)\left[\frac{1-3\gamma_{ji}^2}{2R_{ji}^3}\right]\right)+\frac{1}{m_0}\times \quad (21)$$

$$\times\left(x_i\frac{\partial}{\partial x_j}+y_i\frac{\partial}{\partial y_j}+z_i\frac{\partial}{\partial z_j}\right)\begin{pmatrix} fm_0m_j\left[\frac{1}{R_{0j}}\right]+fm_j(C_i-A_i)\left[\frac{1-3\gamma_{ij}^2}{2R_{ij}^3}\right]+ \\ +fm_i(C_j-A_j)\left[\frac{1-3\gamma_{ji}^2}{R_{ji}^3}\right] \end{pmatrix}-\frac{1}{2}b_i[R_{i0}^2]$$

The rotational motion of an axisymmetric body ($A=B$) around its center of inertia is described in the analogues of the Andoyerosculating elements. As noted above, the axes of the own coordinate system coincide with the main central axes of inertia of the body.
In the Euler variables, the kinetic energy of rotational motions of non-stationary axisymmetric bodies has the form

$$K_j^{rot}=\frac{1}{2}(A_j(p_j^2+q_j^2)+C_jr_j^2) \quad (22)$$

On the other hand, in the Andoyervariables we get [1]:

$$A_jp_j=\sqrt{G_j'^2-L_j'^2}\sin l_j'B_jq_j=\sqrt{G_j'^2-L_j'^2}\cos l_j'C_jr_j=L_j' \quad (23)$$

Therefore, the expression for kinetic energy (22) in the Andoyervariables can generally be written as

$$K_j^{rot}=\frac{1}{2}(G_j'^2-L_j'^2)\left[\frac{\sin^2 l_j'}{A_j}+\frac{\cos^2 l_j'}{B_j}\right]+\frac{L_j'^2}{2C_j} \quad (24)$$

In the case of an axisymmetric body, expression (24) is greatly simplified

$$K_j^{rot}=\frac{1}{2A_j}(G_j'^2-L_j'^2)+\frac{L_j'^2}{2C_j} \quad (25)$$

Hence, the Hamiltonian of rotational motions of axisymmetric bodies can be written in the form

$$F_j'=\frac{1}{2}(G_j'^2-L_j'^2)\frac{1}{A_j}+\frac{L_j'^2}{2C_j}+F_{jpert}' \quad (26)$$

where

$$F_{jpert}'=U^{(2)}, \quad j=0,1,2. \quad (27)$$

$$U^{(2)}=\frac{1}{2}\sum_{i=0,\ i\neq j}^{2}U_{ij}^{(2)} \quad (28)$$

Accordingly, the rotational motions of axisymmetric bodies T_0, T_1, T_2 around their own center of inertia are defined by the equations of perturbed motion in the Andoyerosculating elements of the form [3].

$$\dot{L}'_j = \frac{\partial F'_j}{\partial l'_j}, \quad \dot{G}'_j = \frac{\partial F'_j}{\partial g'_j}, \quad \dot{H}'_j = \frac{\partial F'_j}{\partial h'_j}, \quad \dot{l}'_j = -\frac{\partial F'_j}{\partial L'_j}, \quad \dot{g}'_j = -\frac{\partial F'_j}{\partial G'_j}, \quad \dot{h}'_j = -\frac{\partial F'_j}{\partial H'_j} \tag{29}$$

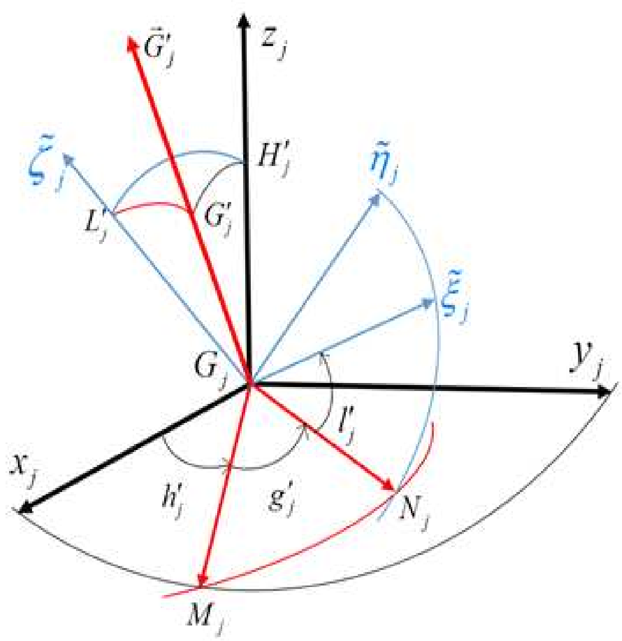

Figure 2: Andoyer variables

The geometric meaning of the analogues of the Andoyer variables are given in [10].

Given the relation $\alpha_{ij}^2 + \beta_{ij}^2 + \gamma_{ij}^2 = 1$ and formulas (9) – (12), expression (27) has the form

$$F'_{jpert} = \frac{1}{2} \sum_{i=0,\ i \neq j}^{2} \left(f m_j (C_i - A_i) \left[\frac{1 - 3\gamma_{ij}^2}{2R_{ij}^3} \right] + f m_i (C_j - A_j) \left[\frac{1 - 3\gamma_{ji}^2}{R_{ji}^3} \right] \right) \tag{30}$$

The values in square brackets in the right-hand side of equation (21) and (30) must be expressed in terms of the Delaunay-Andoyerosculating elements.

4 Differential equations of secular perturbations

Let us consider the nonresonant case. By averaging the right-hand side of equation (17) and (29) over the variables g', l, we obtain the equations for secular perturbations of the translational-rotational motion of bodies T_1 and T_2 and the rotational motion of body T_0 in the problem under consideration [10, 11]. If we denote the secular parts of the perturbing functions F_i, F'_j as $F_{i\,\text{sec}}$, $F'_{j\,\text{sec}}$, then according to the Gaussian scheme we have

$$F_{isec} = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} F_i dl_i dg'_i, \quad F'_{jsec} = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} F'_j dl_j dg'_j, \quad i = 1,2, \quad j = 0,1,2. \tag{31}$$

Accordingly, it is possible to write

$$F'_{jsec} = \frac{1}{2A_j}(G'^2_j)_{sec} + \frac{1}{2}\left(\frac{1}{C_j} - \frac{1}{A_j}\right)(L'^2_j)_{sec} + \frac{1}{2}\sum_{i=0, i\neq j}^{2}\left(\begin{array}{c} fm_j(C_i - A_i)\left[\frac{1-3\gamma_{ij}^2}{2R_{ij}^3}\right]_{sec} + \\ + fm_i(C_j - A_j)\left[\frac{1-3\gamma_{ji}^2}{R_{ji}^3}\right]_{sec} \end{array}\right) \tag{32}$$

$$\begin{gathered} F_{isec} = \frac{\mu_{0i}^2}{2\sigma_i^2}\left(\frac{1}{L_i^2}\right)_{sec} + \frac{(m_0+m_i)}{m_0 m_i}\left(fm_0(C_i - A_i)\left[\frac{1-3\gamma_{i0}^2}{2R_{i0}^3}\right]_{sec} + fm_i(C_0 - A_0)\times\right. \\ \left.\times\left[\frac{1-3\gamma_{0i}^2}{R_{0i}^3}\right]_{sec}\right) + \frac{1}{m_i}\left(fm_i m_j\left[\frac{1}{R_{ij}}\right]_{sec} + fm_j(C_i - A_i)\left[\frac{1-3\gamma_{ij}^2}{2R_{ij}^3}\right]_{sec} + \right. \\ \left. + fm_i(C_j - A_j)\left[\frac{1-3\gamma_{ji}^2}{2R_{ji}^3}\right]_{sec}\right) + \frac{1}{m_0}\left(x_i\frac{\partial}{\partial x_j} + y_i\frac{\partial}{\partial y_j} + z_i\frac{\partial}{\partial z_j}\right)\times \\ \times\left(\begin{array}{l} fm_0 m_j\left[\frac{1}{R_{0j}}\right]_{sec} + fm_j(C_i - A_i)\left[\frac{1-3\gamma_{ij}^2}{2R_{ij}^3}\right]_{sec} + \\ + fm_i(C_j - A_j)\left[\frac{1-3\gamma_{ji}^2}{R_{ji}^3}\right]_{sec} \end{array}\right) - \frac{1}{2}b_i[R_{i0}^2]_{sec} \end{gathered} \tag{33}$$

5 Conclusion

The translational-rotational motion of three non-stationary axisymmetric mutually gravitating bodies according to Newton's law is studied by perturbation theory methods. Equations for secular perturbations are obtained. Further it is planned to express the values in square brackets in the right part of equations (32) and (33) through the Delaunaye-Andoyer osculating elements.

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