1-бөлім	Раздел 1	Section 1
Математика	Математика	Mathematics

IRSTI 27.29.17 DOI: https://doi.org/10.26577/JMMCS.2022.v114.i2.01

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ON BOUNDED SOLUTIONS OF DIFFERENTIAL SYSTEMS

The question of the existence of bounded solutions on an infinite interval of a linear inhomogeneous system of ordinary differential equations in a finite-dimensional space is considered. The study of bounded solutions of systems of ordinary differential equations is one of the most important problems in the qualitative theory of differential equations. In the study of the asymptotic behavior of solutions to differential systems, the works of A. Poincaré and A.M. Lyapunov. Various conditions for the existence of bounded solutions of a linear system of ordinary differential equations have been obtained by many authors. Note the works of O. Perron, A. Walter, H. Shpet, D. Caligo, N.I. Gavrilova, M. Hukukara, M. Nagumo, M. Caratheodori, U. Barbouti, N.Ya. Lyashchenko, B.P. Demidovich, A. Wintner, R. Bellman, Yu.S. Bogdanov, Z. Vazhevsky, N. Levinson, M. Markus, L. Cesari and others. In this paper, we establish sufficient conditions for the boundedness of all solutions of a linear inhomogeneous system of differential equations on an infinite interval. A coefficient criterion for the boundedness of all solutions on an infinite interval of a linear inhomogeneous system of differential equations in a certain class of differential systems is given. Applied methods of differential equations and function theory. The results obtained are used in applications of differential equations and are of practical value.
Key words: solution, boundedness, system, linear, differential equation.

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Дифференциалдық жүйелердің шектеулі шешімдері туралы

Ақырлы өлшемді кеңістіктегі қарапайым дифференциалдық теңдеулердің сызықтық біртекті емес жүйесінің шексіз аралықта шектелген шешімдердің болуы туралы мәселе қарастырылады. Қарапайым дифференциалдық теңдеулер жүйесінің шектеулі шешімдерін зерттеу дифференциалдық теңдеулердің сапалық теориясының маңызды мәселелерінің бірі болып табылады. Дифференциалдық жүйелерге арналған шешімдердің асимптотикалық мінез-құлқын зерттеу барысында А. Пуанкаре мен А.М. Ляпуновтың жұмыстары негізін қалаушы болып табылады. Қарапайым дифференциалдық теңдеулердің сызықтық жүйесінің шектелген шешімдерінің болуының әр түрлі шарттары көптеген авторлармен алынған. О. Перрон, А. Вальтер, Х. Шпет, Д. Калиго, Н.И. Гаврилова, М. Хукукара, М. Нагумо, М. Каратеодори, У. Барбути, Н.Я. Лященко, Б.П. Демидович, А. Винтнер, Р. Беллман, Ю.С. Богданов, З. Важевский, Н. Левинсон, М. Маркус, Л. Сезари және т.б. Бұл жұмыста біз дифференциалдық теңдеулердің сызықтық біртекті емес жүйесінің барлық шешімдерінің шексіз аралықта шектелуіне жеткілікті шарттар орнатамыз. Дифференциалдық жүйелердің белгілі бір класындағы дифференциалдық теңдеулердің сызықтық біртекті емес жүйесінің шексіз аралықтағы барлық шешімдердің шектелуінің коэффициент критерийі келтірілген. Дифференциалдық теңдеулер мен функциялар теориясының әдістері қолданылған. Алынған нәтижелер дифференциалдық теңдеулерді қолдану кезінде қолданылады және практикалық маңызы бар.
Түйін сөздер: шешім, шектілік, жүйе, сызықтық, дифференциалдық теңдеу.

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Об ограниченных решениях дифференциальных систем

Рассматривается вопрос существования ограниченных решений на бесконечном интервале линейной неоднородной системы обыкновенных дифференциальных уравнений в конечномерном пространстве. Изучение ограниченных решений систем обыкновенных дифференциальных уравнений является одной из важнейших задач качественной теории дифференциальных уравнений. В исследовании асимптотического поведения решений дифференциальных систем основополагающими являются работы А. Пуанкаре и А.М. Ляпунова. Разнообразные условия существования ограниченных решений линейной системы обыкновенных дифференциальных уравнений получены многими авторами. Отметим работы О. Перрона, А. Вальтера, Х. Шпета, Д. Калиго, Н.И. Гаврилова, М. Хукукара, М. Нагумо, М. Каратеодори, У. Барбути, Н.Я. Лященко, Б.П. Демидовича, А. Винтнера, Р. Беллмана, Ю.С. Богданова, З. Важевского, Н. Левинсона, М. Маркуса, Л. Чезари и другие. В данной работе установлены достаточные условия ограниченности всех решений линейной неоднородной системы дифференциальных уравнений на бесконечном интервале. Приведен коэффициентный признак ограниченности всех решений на бесконечном интервале линейной неоднородной системы дифференциальных уравнений в определенном классе дифференциальных систем. Примененяюся методы дифференциальных уравнений и теории функций. Полученные результаты находят применения в приложениях дифференциальных уравнений и представляет собой, практическую ценность.

Ключевые слова: решение, ограниченность, система, линейная, дифференциальное уравнение.

1 Introduction

The question of the existence of bounded solutions of differential systems on an infinite interval is considered. The study of bounded solutions of systems of ordinary differential equations is one of the most important problems in the qualitative theory of differential equations. In the study of the asymptotic behavior of solutions of differential systems, the works of A. Poincaré [1] and A.M. Lyapunov [2]. Conditions for the existence of bounded solutions of a linear system of ordinary differential equations were obtained by the authors: Dunkel O., Hukuhara M., Nagumo M., Caccioppoli R., Caratheodory M., Dini U., Spath H., Weyl H., Wiman A., Ascoli G ., Gavrilov N.I., Gusarova R.S., Conti R., Barbuti U., Lyashchenko N.Ya., Demidovich B.P., Faedo S., Wilkins I.E, Ghizzetti A., Sobol I.M. , Haupt O., Boas M., Boas R.P., Wintner A., Bellman R., Bogdanov Yu.S., Butlewski Z., Bylov B.F., Wazewski T., Walter A., Caligo D., Kitamura T., Landau E., Levinson N., Marcus M., Perron O., Cesari L., Spath H., Shtokalo I.Z., Sobol I.M., et al. For detailed references, see the book by Cesari L. [3]. General information is available in the books: [4] V.V. Nemytsky and V.V. Stepanov, [5] Erugin N.P., [6] Sansone G., [7] Pliss V.A., [8] Bylov B.F., Vinograd R.E., Grobman D.M., Nemytskiy V.V., [9] Izobov N.A., [10] Coddington E.A. and Levinson N., [11] Demidovich B.P., [12] Lefschetz S., [13] Massera H.L., Scheffer H.H., [14] Bellman R., [15] Coppel W.A., [16] Daletskiy Yu.L., Kerin M.G. We also note the works: [17–20] Wintner A., [21] Yoshizawa T., [22] Bihari I., [23] Hartman Ph., [24] Hale J., Onuchic N.

In this paper, sufficient conditions are established boundedness of all solutions of a linear inhomogeneous system of differential equations on an infinite interval. A coefficient criterion for the boundedness on an infinite interval of all solutions of a linear inhomogeneous system of differential equations in a certain class of differential systems is given. Applied methods of differential equations and function theory. The results obtained are used in applications of differential equations and are of practical value.

2 Materials and research methods

A linear inhomogeneous system of differential equations is considered

$$\dot{x} = A(t)x + f(t) \tag{1}$$

where

$$t \in I \equiv (1, +\infty), \quad A(t) \in (I), \quad f(t) \in (I);$$

and the corresponding linear homogeneous system of differential equations

$$\dot{x} = A(t)x \tag{2}$$

Theorem 1 *If the conditions*

$$\|A(t)\| \le K\gamma t^{\gamma-1}, \quad 0 < \gamma < 1, \quad K > 0, \quad \|f(t)\| \le K\gamma t^{\gamma-1}, \quad t \in [t_0, +\infty); \quad t_0 \in I;$$

and the linear homogeneous system (2) *is generalized correct, has negative upper generalized Lyapunov exponents with respect to* t^γ, *then any solution to the linear inhomogeneous system of differential equations* (1) *on the interval* $[t_0, +\infty)$ *limited.*

Proof. From (1) multiplying scalarly by $x(t)$ get

$$(x', x) = (A(t)x, x) + (f(t), x). \tag{3}$$

From (3) get

$$|(x', x) \le |(A(t)x, x)| + |(f(t), x)|. \tag{4}$$

From (4) get

$$|(x', x)| \le \|A(t)\|\|x\|^2 + \|f(t)\|\|x\|. \tag{5}$$

From (5) get

$$-\|A(t)\|v^2 - \|f(t)\|v \le v'\|A(t)\|v^2 + \|f(t)\|v \tag{6}$$

where $v(t) = \|x(t)\|$. From (6) get

$$de^{-Kt^\gamma} \le \|x(t)\| \le De^{Kt^\gamma} \tag{7}$$

where $d > 0$, $D > 0$. From (7) we obtain that any nonzero solution to the linear inhomogeneous system (1) has a finite upper generalized Lyapunov exponent with respect to t^γ. In the linear homogeneous system (2) we take the largest upper generalized Lyapunov exponent $\lambda_1 < 0$. Let's take $\alpha \in (0, |\lambda_1|)$) and in the linear inhomogeneous system of differential equation (1) we perform the transformation

$$x = ye^{\alpha t^\gamma}, \quad x(t_0) = y(t_0). \tag{8}$$

Then from (1) we obtain

$$\dot{y} = [A(t) + \alpha\gamma t^{\gamma-1}E]y + e^{\alpha t^\gamma} f(t) \tag{9}$$

where the corresponding linear homogeneous system of differential equations

$$\dot{y} = [A(t) + \alpha\gamma t^{\gamma-1}E]y \tag{10}$$

is generalized correct and has negative upper generalized Lyapunov exponents.

From the linear system of differential equations (9) we obtain

$$y(t) = Y(t, t_0)y(t_0) + \int\limits_{t_0}^{t} Y(t, s)e^{\alpha s^\gamma} f(s)ds \tag{11}$$

where $Y(t, t_0) = Y(t)Y^{-1}(t_0)$ – the Cauchy matrix of a linear homogeneous system of differential equations (10). By virtue of the generalized correctness of the linear system (10), for any $\varepsilon \in (0, |\alpha|)$ exists $D_\varepsilon(t_0) > 0$ and the inequality

$$\|Y(t, \tau)\| \leq D_\varepsilon(t_0)e^{\varepsilon\tau^\gamma} \tag{12}$$

at $t \geq \tau \geq t_0$. From (11), (12) get

$$\|y(t)\| \leq D_\varepsilon(t_0)\|y(t_0)\| + \int\limits_{t_0}^{t} D_\varepsilon(t_0)e^{\varepsilon s^\gamma} K\gamma s^{\gamma-1}ds. \tag{13}$$

From (8), (13) get

$$\|x(t)\| \leq e^{-\alpha t^\gamma} D_\varepsilon(t_0) \left(\|x(t_0)\| + K\frac{e^{\varepsilon t^\gamma}}{\varepsilon} \right). \tag{14}$$

From (14), using arbitrary smallness ε, directing $\varepsilon \to 0$ get

$$\|x(t)\| \leq e^{-\alpha t^\gamma} D_\varepsilon(t_0)(\|x(t_0)\| + Kt^\gamma) \tag{15}$$

at $t \geq t_0$.

It follows from (15) that any solution of the linear inhomogeneous system of differential equations (1), on the interval $[t_0, +\infty)$ limited. Theorem 1 is proved.

Consider a linear inhomogeneous system of differential equations

$$\frac{dy_i}{dt} = \sum_{k=1}^{n} p_{ik}(t)y_i + f_i(t), \quad i = \overline{1, n}; \tag{16}$$

where $p_{ik}(t)$, $f_i(t)$, $i = \overline{1, n}$; $k = \overline{1, n}$; continuous real functions on the interval $(1, +\infty)$, $t_0 \in (1, +\infty)$.

Theorem 2 *If for* $1 < t_0 \le t$ *conditions are met:*

$$1)\quad p_{i-1,i-1}(t) \ge p_{ii}(t) + \beta\gamma t^{\gamma-1},\quad i = \overline{2,n};\quad \beta > 0, \gamma > 0;$$

$$2) \lim t \to +\infty \frac{|p_{ik}(t)|}{\gamma t^{\gamma-1}} = 0,\quad i \not\equiv k,\quad i = \overline{1,n};\quad k = \overline{1,n};$$

$$3) \lim t \to +\infty \frac{1}{t^\gamma} \int\limits_{t_0}^{t} p_{ii}(s)ds = \beta_i < 0,\quad i = \overline{1,n};$$

$$4) |f_i(t)| \le K\gamma t^{\gamma-1},\quad i = \overline{1,n};\quad K > 0;$$

then any solution to the linear inhomogeneous system of differential equations (16) *on the interval* $[t_0, +\infty)$ *limited.*

Proof. The corresponding linear homogeneous system of differential equations

$$\frac{dy_i}{dt} = \sum_{k=1}^{n} p_i k(t) y_i,\quad i = \overline{1,n}; \tag{17}$$

under conditions: 1), 2) and 3) is generalized correct and has negative generalized upper Lyapunov exponents with respect to t^γ. The largest generalized upper Lyapunov exponent of the linear homogeneous system of differential equations (17) is $\beta_1 < 0$. Using condition 4) and similarly to Theorem 1, we obtain that any solution to the linear inhomogeneous system of differential equations (16) bounded. Theorem 2 is proved.

Let's look at an example. In system $x_1' = -\frac{1}{4\sqrt{t}} x_1 + \frac{1}{4\sqrt{t}}$, $x_2' = -\frac{1}{2\sqrt{t}} x_2 - \frac{1}{4\sqrt{t}}$; $1 < t_0 \le t$; where $\gamma = \frac{1}{2}$, $0 < \beta \le \frac{1}{2}$, $\beta_1 = -\frac{1}{2}$, $\beta_2 = -1$, $\frac{1}{2} \le K$, the conditions of Theorem 2 are satisfied; therefore, any solution to a linear inhomogeneous system of differential equations is bounded.

3 Result

In this work, sufficient conditions for the boundedness of solutions of a linear inhomogeneous system of differential equations are obtained.

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