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THE METHOD OF VARIATION OF ARBITRARY CONSTANTS IN THE CASE OF A SYSTEM OF LINEAR DIFFERENTIAL EQUATIONS OF DIFFERENT ORDERS

The paper considers structures consisting of rods connected in one node.Longitudinal and transverse vibrations of such a structure are described by systems of linear differential equations on star graphs.The noted system of equations consists of three linear differential equations of different orders.Two equations correspond to two transverse vibrations, and the third equation describes the longitudinal vibrations of the bar.Moreover, the system of three linear differential equations in the general case does not decompose. In this work, a fundamental system of solutions of a homogeneous system is constructed when the conjugation conditions are satisfied at the point of connection of the rods. Also, by the method of variation of arbitrary constants, a particular solution of an inhomogeneous system is constructed, which is subject to the conjugation conditions at the point of connection of the rods. In subsequent works, the authors intend to investigate the natural frequencies of longitudinal and transverse vibrations of a structure consisting of many rods.
Key words: boundary conditions, boundary value problems, canonical problems, star graph, fundamental system of solutions of a homogeneous system, boundary problem, private solution, public solution, wronskian determinant.

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Әр түрлі ретті сызықтық дифференциалдық теңдеулер жүйесі жағдайында еркін тұрақтыларды вариациялау әдісі

Бұл мақалада стерженнің бір түйінге қосылған конструкциясы көрсетілген. Жұлдызды графта бойлық және бүйірлік тербелістер сызықты дифференциалдық теңдеулер жүйесімен берілген. Жүйеде әр түрлі реттегі үш өлшемді сызықтық дифференциалдық теңдеу берілген. Жүйедегі бастапқы екеуі екі көлденең тербеліске сәйкес келеді, ал үшінші теңдеу жолақтың бойлық тербелістерін сипаттайды. Сонымен қатар, жалпы жағдайда үш сызықтық дифференциалдық теңдеулер жүйесі жіктелмейді. Бұл жұмыста стержендердің бір нүктеде байланысқанда біртекті жүйенің фундаменталдық шешімі құрылған. Сондай-ақ, еркін тұрақтыларды вариациялау әдісімен стерженьдердің қосылу нүктесіндегі конъюгация шарттарына бағынатын біртекті емес жүйенің белгілі бір шешімі құрылды. Кейінгі жұмыстарда авторлар көптеген стержендердің жиынтық құрылымның бойлық және көлденең тербелістердің меншікті жиіліктерін зерттеуге ниетті.
Түйін сөздер: шекаралық шарттар, шекаралық есептер, канондық мәселелер, жұлдызды граф, сызықты дифференциалдық теңдеудің іргелі шешімі, шекаралық есеп, нақты шешім, вронскиян анықтауышы.

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Метод вариации произвольных постоянных в случае системы линейных дифференциальных уравнений разных порядков

В работе рассмотрены конструкции, состоящие из стержней соединенные в одном узле. Продольные и поперечные колебания подобных конструкции описываются системами линейных дифференциальных уравнений на звездных графах. Отмеченная система уравнений состоит из трех линейных дифференциальных уравнений, имеющих разные порядки. Два уравнения соответствуют двум поперечным колебаниям, а третье уравнение описывает продольные колебания стержня. Причем система трех линейных дифференциальных уравнений в общем случае не распадается. В работе построена фундаментальная система решений однородной системы при выполнении условий сопряжений в точке соединения стержней. Также методом вариации произвольных постоянных построено частное решение неоднородной системы, которое подчинено условиям сопряжения в точке соединения стержней. В последующих работах авторы намерены исследовать собственные частоты продольных и поперечных колебаний конструкции, состоящих из множества стержней.

Ключевые слова: граничные условия, краевые задачи, канонические проблемы, звездный граф, фундаментальная система решений однородной системы, граничная задача, частное решение, определитель Вронскиана.

1 Introduction

Under natural simplifying assumptions, the equation of transverse vibrations of the rod [1] has the form

$$\rho(x)A(x)\frac{\partial^2 w(x,t)}{\partial t^2}+\frac{\partial^2}{\partial x^2}\left(E(x)f(x)\frac{\partial^2 w(x,t)}{\partial x^2}\right)=f(x,t),\quad -\frac{l}{2}<x<\frac{l}{2} \tag{1}$$

The parameters included in equation (1) have a physical meaning:
$\rho(x)$ – material density;
$A(x)$ – cross-sectional area;
$w(x,t)$ – transverse movement;
$E(x)$ – Young's modulus;
$I(x)$ – moment of inertial of the cross-sectional area about the neutral axis;
$f(x,t)$ – lateral load.

In what follows, the conditions $w(x,t)$ for fixing the ends of the bar are set to the equation with respect to the transverse displacement, and the initial shape of the axis of the bar is considered known.

At the same time, the differential equation of free longitudinal vibrations of the rod [2] has the form

$$\rho\frac{\partial^2 u}{\partial t^2}=E\frac{\partial^2 u}{\partial x^2} \tag{2}$$

In many engineering calculations, it is assumed that the motions are separated: lateral vibrations do not affect longitudinal vibrations and vice versa.However, such a separation of the rod movements is not always justified.In the work of S.A. Nazarov. [3] shows a system of three differential equations

$$\begin{cases} l_1(Y) = \dfrac{d^2}{dx^2}\left(\mu(x)a(x)\dfrac{d^2y_1(x)}{dx^2}\right) + \dfrac{d^2}{dx^2}\left(\mu(x)b(x)\dfrac{d^2y_2(x)}{dx^2}\right) - \dfrac{d^2}{dx^2}\left(\mu(x)d(x)\dfrac{dy_3(x)}{dx}\right) \\ l_2(Y) = \dfrac{d^2}{dx^2}\left(\mu(x)b(x)\dfrac{d^2y_1(x)}{dx^2}\right) + \dfrac{d^2}{dx^2}\left(\mu(x)c(x)\dfrac{d^2y_2(x)}{dx^2}\right) - \dfrac{d^2}{dx^2}\left(\mu(x)f(x)\dfrac{dy_3(x)}{dx}\right), \\ l_3(Y) = \dfrac{d}{dx}\left(\mu(x)d(x)\dfrac{d^2y_1(x)}{dx^2}\right) + \dfrac{d}{dx}\left(\mu(x)f(x)\dfrac{d^2y_2(x)}{dx^2}\right) - \dfrac{d}{dx}\left(\mu(x)\dfrac{dy_3(x)}{dx}\right) \end{cases} \tag{3}$$

which describes the joint transverse, longitudinal vibrations of the bar. Since the reduced system does not disintegrate, the longitudinal vibrations affect the lateral ones and vice versa.In this way, in the general case, system (3) does not always split into equations of the type (1) and (2).

System (3) of linear differential equations consists of three differential equations of different orders.In standard textbooks on the theory of differential equations [4], such systems have not been studied in detail. In particular, when the system consists of linear differential equations of different orders, the method of variations of arbitrary constants requires substantiation.In this paper, we have developed a method of variations of arbitrary constants for system (3) on star graphs.

2 Statement of the problem

Consider a star graph Γ (Figure 1)consisting of many vertices and many bows.We number the vertices of the graph with natural numbers starting from 0 up to $m+1$. The bows of the graph are numbered through $e_1, \ldots, e_{m+1}$, moreover e_j – bow, directed towards the vertex j. On every bow e_j the vector function.

$$Y_j(x_j) = \begin{bmatrix} y_{1j}(x_j) & y_{2j}(x_j) & y_{3j}(x_j) \end{bmatrix}^T, \quad x_j \in e_j,$$

Which satisfies the following system of linear differential equations

$$\begin{cases} \frac{d^2}{dx_j^2}\left(\mu_j(x_j)a_j(x_j)\frac{d^2y_{1j}(x_j)}{dx_j^2}\right) + \frac{d^2}{dx_j^2}\left(\mu_j(x_j)b_j(x_j)\frac{d^2y_{2j}(x_j)}{dx_j^2}\right) - \frac{d^2}{dx_j^2}\left(\mu_j(x_j)d_j(x_j)\frac{dy_{3j}(x_j)}{dx_j}\right) = g_{1j}(x_j), \\ \frac{d^2}{dx_j^2}\left(\mu_j(x_j)b_j(x_j)\frac{d^2y_{1j}(x_j)}{dx_j^2}\right) + \frac{d^2}{dx_j^2}\left(\mu_j(x_j)c_j(x_j)\frac{d^2y_{2j}(x_j)}{dx_j^2}\right) - \frac{d^2}{dx_j^2}\left(\mu_j(x_j)f_j(x_j)\frac{dy_{3j}(x_j)}{dx_j}\right) = g_{2j}(x_j), \\ \frac{d}{dx_j}\left(\mu_j(x_j)d_j(x_j)\frac{d^2y_{1j}(x_j)}{dx_j^2}\right) + \frac{d}{dx_j}\left(\mu_j(x_j)f_j(x_j)\frac{d^2y_{2j}(x_j)}{dx_j^2}\right) \frac{d}{dx_j}\left(\mu_j(x_j)\frac{dy_{3j}(x_j)}{dx_j}\right) = g_{3j}(x_j). \end{cases} \tag{4}$$

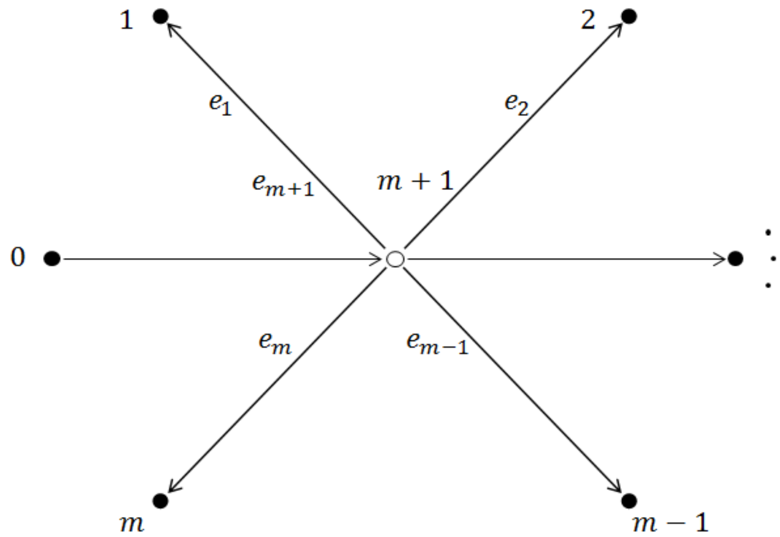

Figure 1: Star graph

In what follows, we assume that the coefficients $\mu_j(x_j), a_j(x_j), b_j(x_j), c_j(x_j), d_j(x_j), f_j(x_j)$ represent given real continuous functions. The indicated coefficients have a physical meaning. For example, $\mu_j(x_j)$ – Young's modulus, other coefficients $a_j(x_j), b_j(x_j), c_j(x_j), d_j(x_j), f_j(x_j)$ – static moments of the cross-sectional area of the bar j. The bow length e_j is assumed to be equal l_j for $j = 1, \ldots, m+1$. It is convenient to introduce the notation for $j = 1, \ldots, m+1$.

$$V_{1j}^{(1)}(Y_j;\xi) = \left[\frac{d}{dx_j}\left(\mu_j a_j \frac{d^2 y_{1j}}{dx_j^2}\right) + \frac{d}{dx_j}\left(\mu_j b_j \frac{d^2 y_{2j}}{dx_j^2}\right) - \frac{d}{dx_j}\left(\mu_j d_j \frac{dy_{3j}}{dx_j}\right)\right]\Bigg|_{x_j=\xi}$$

$$V_{2j}^{(1)}(Y_j;\xi) = \left[\mu_j a_j \frac{d^2 y_{1j}}{dx_j^2} + \mu_j b_j \frac{d^2 y_{2j}}{dx_j^2} - \mu_j d_j \frac{dy_{3j}}{dx_j}\right]\Bigg|_{x_j=\xi}$$

$$V_{1j}^{(2)}(Y_j;\xi) = \left[\frac{d}{dx_j}\left(\mu_j b_j \frac{d^2 y_{1j}}{dx_j^2}\right) + \frac{d}{dx_j}\left(\mu_j c_j \frac{d^2 y_{2j}}{dx_j^2}\right) - \frac{d}{dx_j}\left(\mu_j f_j \frac{dy_{3j}}{dx_j}\right)\right]\Bigg|_{x_j=\xi}$$

$$V_{2j}^{(2)}(Y_j;\xi) = \left[\mu_j b_j \frac{d^2 y_{1j}}{dx_j^2} + \mu_j c_j \frac{d^2 y_{2j}}{dx_j^2} - \mu_j f_j \frac{dy_{3j}}{dx_j}\right]\Bigg|_{x_j=\xi}$$

At the inner top $(m+1)$ add the following conjugation conditions to system (4):

$$\begin{cases} y_{11}(0) = y_{12}(0) = \ldots = y_{1m}(0) = y_{1,m+1}(l_{m+1}), \\ y_{21}(0) = y_{22}(0) = \ldots = y_{2m}(0) = y_{2,m+1}(l_{m+1}), \\ y_{31}(0) = y_{32}(0) = \ldots = y_{3m}(0) = y_{3,m+1}(l_{m+1}), \\ \frac{d}{dx_1} y_{11}(0) = \frac{d}{dx_2} y_{12}(0) = \ldots = \frac{d}{dx_m} y_{1,m}(0) = \frac{d}{dx_{m+1}} y_{1,m+1}(l_{m+1}), \\ \frac{d}{dx_1} y_{21}(0) = \frac{d}{dx_2} y_{22}(0) = \ldots = \frac{d}{dx_m} y_{2,m}(0) = \frac{d}{dx_{m+1}} y_{2,m+1}(l_{m+1}), \\ \frac{d}{dx_{m+1}} y_{3,m+1}(l_{m+1}) = \frac{d}{dx_1} y_{3,1}(0) + \ldots + \frac{d}{dx_m} y_{3m}(0) \\ V^{(1)}_{1,m+1}(Y_{m+1}; l_{m+1}) = V^{(1)}_{1,1}(Y_1; 0) + \ldots + V^{(1)}_{1,m}(Y_m; 0), \\ V^{(1)}_{2,m+1}(Y_{m+1}; l_{m+1}) = V^{(1)}_{2,1}(Y_1; 0) + \ldots + V^{(1)}_{2,m}(Y_m; 0), \\ V^{(2)}_{1,m+1}(Y_{m+1}; l_{m+1}) = V^{(2)}_{1,1}(Y_1; 0) + \ldots + V^{(2)}_{1,m}(Y_m; 0), \\ V^{(2)}_{2,m+1}(Y_{m+1}; l_{m+1}) = V^{(2)}_{1,1}(Y_1; 0) + \ldots + V^{(2)}_{1,m}(Y_m; 0). \end{cases} \tag{5}$$

The total number of conjugation conditions of the form (5) is equal to $5m + 5$. For the correct statement of the problem, to system (4) with conjugation conditions (5), we must add one more $5m + 5$ boundary conditions at the boundary vertices $0, 1, \ldots, m$. The boundary conditions at the boundary vertices correspond to the conditions for fixing the rods at their ends.

Usually, when solving boundary value problems for systems of linear differential equations, you need to know:

1. a fundamental system of solutions to a homogeneous system;
2. a particular solution to an inhomogeneous system.

The further goal of the article is to first prove the existence of a fundamental system of solutions to the homogeneous system (4) for $g_{1j} = g_{2j} = g_{3j} = 0$, $j = 1, \ldots, m + 1$. After that, using the constructed fundamental system of solutions, by the method of variations of arbitrary constants, construct a particular solution of systems (4) and (5).

3 Construction of a fundamental system of solutions

Let $j-$ a fixed number from the set $\{1, 2, \ldots, m + 1\}$. In the system (4) we put $g_{1j} = g_{2j} = g_{3j} = 0$. Then, with respect to the homogeneous system (4), the following statement is true: for arbitrary constants $\theta_1, \theta_2, \ldots, \theta_{10}$ solution $Y_j(x_j) = \left[\begin{array}{ccc} y_{1j}(x_j) & y_{2j}(x_j) & y_{3j}(x_j) \end{array}\right]^T$, $x_j \in e_j$, of the system of three homogeneous equations (4) with the Cauchy conditions at the point $x_j = \xi_j$.

$$\begin{gathered} y_{kj}(\xi_j) = \theta_k,\ \frac{dy_{kj}(\xi_j)}{dx_j} = \theta_{k+3},\ k = 1, 2, 3, \\ \frac{d^2 y_{ij}(\xi_j)}{dx_j^2} = \theta_{i+6}, \quad \frac{d^3 y_{ij}(\xi_j)}{dx_j^3} = \theta_{i+8}, \quad i = 1, 2 \end{gathered} \tag{6}$$

exists and is unique. This statement follows from the general theorem on the existence of a solution to the Cauchy problem for systems of linear differential equations [4]. We choose the point ξ_j as follows: $\xi_j = 0$ for $j = 1, \ldots, m$ and $j = m + 1$ at point $\xi_j = l_j$. Let for $i \in \{1, 2, \ldots, 10\}$ a fixed value $\theta_i = 1$, and the rest $\theta_k = 0$, $\forall k \neq i$. Let us denote the solution to problem (4), (6) by $Y_j^{(i)}(x_j)$. The general solution of the homogeneous system (4) has the form

$$Y_j(x_j) = \sum_{i=1}^{10} \theta_{ij} Y_j^{(i)}(x_j), \tag{7}$$

where are θ_{ij} – arbitrary constants.

Let us choose the numbers $\{\theta_{i\,j}\}$ so that the right-hand side of relation (7) satisfies the conjugation conditions (5). We substitute relation (7) into conjugation conditions (5). As a result, we get equalities:

$$\theta_{i\,1} = \theta_{i\,2} = \ldots = \theta_{i\,m} = \theta_{i\,m+1}, \qquad i = 1, 2, 3, 4, 5$$

So, $\theta_{i\,j}$ it is convenient to denote by θ_i at $i = 1, 2, 3, 4, 5$. Similarly, we obtain

$$A\tau = h, \tag{8}$$

where

$$\alpha_{9j}^{(1)} = \left[\mu_j \left(a_j - d_j^2\right)\right]_{x_j = \xi_j}, \qquad \alpha_{9j}^{(2)} = \left[\mu_j \left(b_j - f_j^2\right)\right]_{x_j = \xi_j},$$

$$\alpha_{7j}^{(1)} = \frac{d}{dx_j}\left[\mu_j \left(a_j - d_j^2\right)\right]_{x_j = \xi_j}, \qquad \alpha_{7j}^{(2)} = \frac{d}{dx_j}\left[\mu_j \left(b_j - f_j^2\right)\right]_{x_j = \xi_j},$$

$$\alpha_{10j}^{(1)} = \left[\mu_j \left(b_j - d_j f_j\right)\right]_{x_j = \xi_j}, \qquad \alpha_{10j}^{(2)} = \left[\mu_j \left(c_j - f_j^2\right)\right]_{x_j = \xi_j},$$

$$\alpha_{8j}^{(1)} = \frac{d}{dx_j}\left[\mu_j \left(b_j - d_j f_j\right)\right]_{x_j = \xi_j}, \qquad \alpha_{8\,j}^{(2)} = \frac{d}{dx_j}\left[\mu_j \left(c_j - f_j^2\right)\right]_{x_j = \xi_j},$$

$$\alpha_{6j}^{(1)} = \frac{d}{dx_j}\left[\mu_j\left(d_j+1\right)\right]_{x_j=\xi_j}, \qquad \alpha_{6j}^{(2)} = \frac{d}{dx_j}\left[\mu_j\left(f_j+1\right)\right]_{x_j=\xi_j},$$

$$\alpha_{7j}^{(3)} = \mu_j a_j\Big|_{x_j=\xi_j}, \qquad \alpha_{7j}^{(4)} = \mu_j b_j\Big|_{x_j=\xi_j}, \qquad \alpha_{9j}^{(3)} = \alpha_{10j}^{(3)} = 0,$$

$$\alpha_{8j}^{(3)} = \mu_j b_j\Big|_{x_j=\xi_j}, \qquad \alpha_{8j}^{(4)} = \mu_j c_j\Big|_{x_j=\xi_j}, \qquad \alpha_{9j}^{(4)} = \alpha_{10j}^{(4)} = 0,$$

$$\alpha_{6j}^{(3)} = \mu_j d_j\Big|_{x_j=\xi_j}, \qquad \alpha_{6j}^{(4)} = \mu_j f_j\Big|_{x_j=\xi_j}, \qquad h_1 = \sum_{j=1}^{m}\theta_{6j},$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \alpha_{6m+1}^{(1)} & \alpha_{7m+1}^{(1)} & \alpha_{8m+1}^{(1)} & \alpha_{9m+1}^{(1)} & \alpha_{10m+1}^{(1)} \\ \alpha_{6m+1}^{(2)} & \alpha_{7m+1}^{(2)} & \alpha_{8m+1}^{(2)} & \alpha_{9m+1}^{(2)} & \alpha_{10m+1}^{(2)} \\ \alpha_{6m+1}^{(3)} & \alpha_{7m+1}^{(3)} & \alpha_{8m+1}^{(3)} & \alpha_{9m+1}^{(3)} & \alpha_{10m+1}^{(3)} \\ \alpha_{6m+1}^{(4)} & \alpha_{7m+1}^{(4)} & \alpha_{8m+1}^{(4)} & \alpha_{9m+1}^{(4)} & \alpha_{10m+1}^{(4)} \end{bmatrix}$$

$$h_{s+1} = \sum_{j=1}^{m}\sum_{k=6}^{10}\alpha_{k\,j}^{(s)}\theta_{kj}, \quad s = 1,2,3,4,$$

$$\tau = \left[\theta_{6m+1}\ \theta_{7m+1}\ \theta_{8m+1}\ \ \theta_{9m+1}\ \ \theta_{10m+1}\right]^T, \qquad h = \left[h_1\ h_2\ h_3\ h_4\ h_5\right]^T$$

It follows from the system of linear algebraic equations (8) that

$$\tau = A^{-1}h, \tag{9}$$

if $\det A \neq 0$. Let us introduce the notation

$$\widehat{Y}_{m+1}(x_{m+1}) = \left[\ Y_{m+1}^{(6)}(x_{m+1})\ \ Y_{m+1}^{(7)}(x_{m+1})\ \ Y_{m+1}^{(8)}(x_{m+1})\ \ \ Y_{m+1}^{(9)}(x_{m+1})\ \ Y_{m+1}^{(10)}(x_{m+1})\ \right],$$

$$\beta_{kj} = \left[\delta_{6k}\ \alpha_{kj}^{(1)}\ \alpha_{kj}^{(2)}\ \ \alpha_{kj}^{(3)}\ \ \alpha_{kj}^{(4)}\right]^T,$$

where $\delta_{6k} = 1$, if $k = 6$; $\delta_{6k} = 0$ if $k > 6$. Then it follows from equality (9) that

$$\tau = \sum_{j=1}^{m} \sum_{k=6}^{10} A^{-1} \beta_{kj} \theta_{kj}. \tag{10}$$

In this way, the general solution of the homogeneous system (4) with conjugation conditions (5), taking into account relations (7), (10), takes the form

$$\begin{cases} Y_j(x_j) = \sum_{i=1}^{5} \theta_i Y_j^{(i)}(x_j) + \sum_{i=6}^{10} \theta_{ij} Y_j^{(i)}(x_j), \ j = 1, \dots, m, \\ Y_{m+1}(x_{m+1}) = \sum_{i=1}^{5} \theta_i Y_{m+1}^{(i)}(x_{m+1}) + \sum_{k=6}^{10} \sum_{j=1}^{m} \theta_{kj} \widehat{Y}_{m+1}(x_{m+1}) A^{-1} \beta_{kj} \end{cases} \tag{11}$$

where θ_{kj}, $k = 6, 7, 8, 9, 10$, $j = 1, \dots, m$, θ_1, θ_2, θ_3, θ_4, θ_5 – arbitrary constants.

Theorem 1 *Let be $g_{1j} = g_{2j} = g_{3j} = 0$ at $j = 1, \dots, m+1$. Let's pretend that* $\det A \neq 0$. *Then the general solution of homogeneous system* (4) *with conjugation conditions* (5) *has the form* (11) *for arbitrary constants θ_{kj}, $k = 6, 7, 8, 9, 10$, $j = 1, \dots, m$, θ_1, θ_2, θ_3, θ_4, θ_5. The number of arbitrary constants is* $5m + 5$.

In this way, theorem 1 implies that the homogeneous differential system of solutions to system (4) with conjugation conditions (5) has the form

$$\left\{ \begin{array}{l} Y_j^{(1)}(x_j),\, Y_j^{(2)}(x_j),\, Y_j^{(3)}(x_j),\, Y_j^{(4)}(x_j),\, Y_j^{(5)}(x_j);\, Y_j^{(k)}(x_j), \\ \qquad\qquad\qquad\qquad k = 6, 7, 8, 9, 10;\ j = 1, \dots, m; \\ Y_{m+1}^{(1)}(x_{m+1}),\, Y_{m+1}^{(2)}(x_{m+1}), Y_{m+1}^{(3)}(x_{m+1}), Y_{m+1}^{(4)}(x_{m+1}), Y_{m+1}^{(5)}(x_{m+1}); \\ \qquad\qquad\qquad\qquad \widehat{Y}_{m+1}(x_{m+1})\, A^{-1} \beta_{kj}, k = 6, 7, 8, 9, 10;\ j = 1, \dots, m. \end{array} \right\}$$

4 Method of variation of arbitrary constants

In standard textbooks on the theory of differential equations [4], the method of variation of arbitrary constants is adapted for systems of linear differential equations of the first order. In this subsection, this method is modified for the case of systems containing equations of different orders. Second, the application of this method to systems of differential equations on star graphs is shown.

Let be j – a fixed number from the set $\{1, \dots, m+1\}$. In accordance with representation (11) of the general solution of the homogeneous system (4), we seek a particular solution in the form

$$Y_j(x_j) = \sum_{i=1}^{10} \theta_{ij}(x_j) Y_j^{(i)}(x_j), \tag{12}$$

Here is $\{\theta_{ij}(x_j)\}$ – an unknown set of functions to be defined. Let's pretend that

$$\begin{cases} \sum_{i=1}^{10} \frac{d\theta_{ij}(x_j)}{dx_j} Y_j^{(i)}(x_j) = 0, \\ \sum_{i=1}^{10} \frac{d\theta_{ij}(x_j)}{dx_j} \frac{dy_{1j}^{(i)}(x_j)}{dx_j} = 0, \\ \sum_{i=1}^{10} \frac{d\theta_{ij}(x_j)}{dx_j} \frac{dy_{2j}^{(i)}(x_j)}{dx_j} = 0, \\ \sum_{i=1}^{10} \frac{d\theta_{ij}(x_j)}{dx_j} \frac{d^2 y_{1j}^{(i)}(x_j)}{dx_j^2} = 0, \\ \sum_{i=1}^{10} \frac{d\theta_{ij}(x_j)}{dx_j} \frac{d^2 y_{2j}^{(i)}(x_j)}{dx_j^2} = 0, \\ \sum_{i=1}^{10} C_{ij} \frac{d\theta_{ij}(x_j)}{dx_j} = g_{1j}(x_j), \\ \sum_{i=1}^{10} D_{ij} \frac{d\theta_{ij}(x_j)}{dx_j} = g_{2j}(x_j), \\ \sum_{i=1}^{10} E_{ij} \frac{d\theta_{ij}(x_j)}{dx_j} = g_{3j}(x_j). \end{cases} \tag{13}$$

We introduce the Wronskian $W_j(x)$ by the formula

$$W_j(x) = \begin{vmatrix} y_{1j}^{(1)}(x_j) & y_{1j}^{(2)}(x_j) & \dots & y_{1j}^{(10)}(x_j) \\ y_{2j}^{(1)}(x_j) & y_{2j}^{(2)}(x_j) & \dots & y_{2j}^{(10)}(x_j) \\ y_{3j}^{(1)}(x_j) & y_{3j}^{(2)}(x_j) & \dots & y_{3j}^{(10)}(x_j) \\ \frac{dy_{1j}^{(1)}(x_j)}{dx_j} & \frac{dy_{1j}^{(2)}(x_j)}{dx_j} & \dots & \frac{dy_{1j}^{(10)}(x_j)}{dx_j} \\ \frac{dy_{2j}^{(1)}(x_j)}{dx_j} & \frac{dy_{2j}^{(2)}(x_j)}{dx_j} & \dots & \frac{dy_{2j}^{(10)}(x_j)}{dx_j} \\ \frac{d^2y_{1j}^{(1)}(x_j)}{dx_j^2} & \frac{d^2y_{1j}^{(2)}(x_j)}{dx_j^2} & \dots & \frac{d^2y_{1j}^{(10)}(x_j)}{dx_j^2} \\ \frac{d^2y_{2j}^{(1)}(x_j)}{dx_j^2} & \frac{d^2y_{2j}^{(2)}(x_j)}{dx_j^2} & \dots & \frac{d^2y_{2j}^{(10)}(x_j)}{dx_j^2} \\ C_{1j} & C_{2j} & \dots & C_{10j} \\ D_{1j} & D_{2j} & \dots & D_{10j} \\ E_{1j} & E_{2j} & \dots & E_{10j} \end{vmatrix}$$

C_{ij}, D_{ij}, E_{ij} – here are some given constants. For example,

$$C_{ij} = \frac{d}{dx_j}\left(\mu_j a_j \frac{d^2y_{1j}^{(i)}(x_j)}{dx_j^2}\right) + \frac{d}{dx_j}\left(\mu_j b_j \frac{d^2y_{2j}^{(i)}(x_j)}{dx_j^2}\right) - \frac{d}{dx_j}\left(\mu_j d_j \frac{d^2y_{3j}^{(i)}(x_j)}{dx_j}\right)\Bigg|_{x_j=\xi_j},$$

$$D_{ij} = \frac{d}{dx_j}\left(\mu_j b_j \frac{d^2y_{1j}^{(i)}(x_j)}{dx_j^2}\right) + \frac{d}{dx_j}\left(\mu_j c_j \frac{d^2y_{2j}^{(i)}(x_j)}{dx_j^2}\right) - \frac{d}{dx_j}\left(\mu_j f_j \frac{dy_{3j}^{(i)}(x_j)}{dx_j}\right)\Bigg|_{x_j=\xi_j}, \quad (14)$$

$$E_{ij} = \mu_j d_j^2 \frac{d^2y_{1j}^{(i)}(x_j)}{dx_j^2} + \mu_j f_j \frac{d^2y_{2j}^{(i)}(x_j)}{dx_j^2} - \mu_j \frac{dy_{3j}^{(i)}(x_j)}{dx_j}\Bigg|_{x_j=\xi_j}.$$

Along with the Wronskian $W_j(x)$, we introduce for $s = 1,2,3$ the following determinants:$W_{kj}^{(s)}(x_j,t_j)$ – the determinant, which is obtained from the determinant by replacing that $k-$ string with a string $[y_{sj}^{(1)}(x_j)\ y_{sj}^{(2)}(x_j)\ \dots\ y_{sj}^{(10)}(x_j)]$. From relations (12) and (13) $s = 1, 2, 3$ we have in the representation

$$y_{sj}(x_j) = \int_0^{x_j} \frac{W_{8j}^{(s)}(x_j t_j)}{W_j(t_j)} g_{1j}(t_j) dt_j + \int_0^{x_j} \frac{W_{9j}^{(s)}(x_j t_j)}{W_j(t_j)} g_{2j}(t_j) dt_j +$$
$$+ \int_0^{x_j} \frac{W_{10j}^{(s)}(x_j t_j)}{W_j(t_j)} g_{3j}(t_j) dt_j, \ j = 1, \ldots, m. \tag{15}$$

For $j = m + 1$ and $s = 1, 2, 3$, we have a different representation of the solution

$$y_{sm+1}(x_{m+1}) = \int_{x_{m+1}}^{l_{m+1}} \frac{W_{8j}^{(s)}(x_j t_j)}{W_j(t_j)} g_{1j}(t_j) dt_j + \int_{x_{m+1}}^{l_{m+1}} \frac{W_{9j}^{(s)}(x_j t_j)}{W_j(t_j)} g_{2j}(t_j) dt_j +$$
$$+ \int_{x_{m+1}}^{l_{m+1}} \frac{W_{10j}^{(s)}(x_j t_j)}{W_j(t_j)} g_{3j}(t_j) dt_j, \ j = 1, \ldots, m. \tag{16}$$

Theorem 2 *Let be $j = 1, \ldots, m+1$. For $g_{1j}(x_j)$, $g_{2j}(x_j)$, $g_{3j}(x_j)$ any of the class $L_2(e_j)$, the inhomogeneous system of linear differential equations* (4) *on a star graph Γ has a particular solution given by formulas* (15) *and* (16)*.Moreover, for the indicated particular solution at the inner vertex $m + 1$ of the graph, relations* (5) *are valid.*

5 Conclusion

In engineering calculations, it is usually assumed that the longitudinal vibrations of the rods do not affect their lateral vibrations.However, this assumption is not always fulfilled. Therefore, in recent years, some authors have proposed models in which transverse and longitudinal vibrations interact with each other.It turns out that such models are described by systems of linear differential equations of different orders.Standard textbooks on the theory of differential equations study systems of differential equations consisting of first-order equations.Therefore, it seems relevant to adapt the standard methods of the theory of differential equations to systems consisting of equations of different orders. In this paper, the method of variation of arbitrary constants is modified for the indicated systems.

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