IRSTI 27.39.19 DOI: https://doi.org/10.26577/JMMCS.2021.v111.i3.07

M. A. Sadybekov[1*], **B. O. Derbissaly**[2]
[1]Institute of Mathematics and Mathematical Modeling, Kazakhstan, Almaty
[2]Al-Farabi Kazakh National University, Kazakhstan, Almaty
*e-mail: sadybekov@math.kz

ON GREEN'S FUNCTION OF DARBOUX PROBLEM FOR HYPERBOLIC EQUATION

It is well known that the Darboux problem for the hyperbolic equation is correct, both in the sense of classical and generalized solutions. An integral form of the solution of the Darboux problem in a characteristic triangle for a general two-dimensional hyperbolic equation of the second order is represented in the article. It is shown that the solution to this problem can be written in terms of the Green function. It is also shown that the Riemann-Green function of the hyperbolic equation is not defined in the entire domain. To construct the Riemann-Green function of this equation, it is important to have the Riemann-Green function of this problem that was defined at all points of the domain. For that, the coefficients of the general hyperbolic equation have been continued odd. The definition of the Green function of the Darboux problem is given. To show that a Green function exists and is unique, we divide the domain into several subdomains. Its existence and uniqueness have been proven. An explicit form of the Green's function is presented. It is shown that the Green's function can be represented by the Riemann–Green function. There is given a method for constructing the Green function of such a problem. The main fundamental difference of this paper is that it is devoted to the study of Green's function for the hyperbolic problem. In contrast to the (well-developed) theory of Green's function for self-adjoint elliptic problems, this theory has not been developed.
Key words: Hyperbolic equation, initial-boundary value problem, Darboux problem, boundary condition, Green function, a characteristic triangle, Riemann–Green function.

М. А. Садыбеков[1*], Б. О. Дербісалы[2]
[1]Математика және математикалық моделдеу институты , Қазақстан, Алматы қ.
[2]Әл-Фараби атындағы Қазақ ұлттық университеті, Қазақстан, Алматы қ.
*e-mail: sadybekov@math.kz

ГИПЕРБОЛАЛЫҚ ТЕҢДЕУ ҮШІН ДАРБУ ЕСЕБІНІҢ ГРИН ФУНКЦИЯСЫ

Дарбудың гиперболалық теңдеуге арналған есебі классикалық және жалпыланған шешімдер тұрғысынан да дұрыс екендігі белгілі. Мақалада екінші ретті жалпы екі өлшемді гиперболалық теңдеу үшін сипаттамалық үшбұрыштағы Дарбу есебін шешудің интегралды түрі берілді. Бұл есептің шешімін Грин функциясын қолдана отырып жазуға болатындығы көрсетілді. Сондай-ақ, гиперболалық теңдеудің Риман-Грин функциясы бүкіл облыста анықталмағаны көрсетілді. Бұл теңдеудің Риман-Грин функциясын құру үшін осы есептің облыстың барлық нүктелерінде анықталған Риман-Грин функциясы болуы керек. Ол үшін жалпы гиперболалық теңдеудің коэффициенттері тақ түрде жалғастырылды. Дарбу есебі үшін Грин функциясының анықтамасы берілді. Грин функциясы бар және жалғыз екенін көрсету үшін облыс бірнеше кіші облыстарға бөлінді. Оның бар екендігі және жалғыздығы дәлелденді. Грин функциясының нақты түрі берілді. Грин функциясы Риман-Грин функциясымен берілуі мүмкін екендігі көрсетілді. Мұндай есептер үшін Грин функциясын құру әдісі берілді. Бұл жұмыстың басқа жұмыстардан негізгі түбегейлі айырмашылығы-бұл гиперболалық есеп үшін Грин функциясын зерттеуге арналған жұмыс. Бұл теорияның өз-3зіне түйіндес эллиптикалық есептер үшін Грин функциясы (жақсы дамыған) теориясынан айырмашылығы, әлі дұрыс зерттелмегенінде.

Түйін сөздер: Гиперболалық теңдеу, бастапқы-шекаралық есеп, Дарбу есебі, шекаралық шарт, Грин функциясы, характеристикалық үшбұрыш, Риман–Грин функциясы.

М. А. Садыбеков[1*], Б. О. Дербисалы[2]
[1]Институт математики и математического моделирования, Казахстан, г. Алматы
[2]Казахский Национальный университет им. аль-Фараби, Казахстан, г. Алматы
*e-mail: sadybekov@math.kz

О ФУНКЦИИ ГРИНА ЗАДАЧИ ДАРБУ ДЛЯ ГИПЕРБОЛИЧЕСКОГО УРАВНЕНИЯ

Хорошо известно, что задача Дарбу для гиперболического уравнения корректна как в смысле классических, так и обобщенных решений. В статье представлена интегральная форма решения задачи Дарбу в характеристическом треугольнике для общего двумерного гиперболического уравнения второго порядка. Показано, что решение этой задачи может быть записано с помощью функции Грина. Также показано, что функция Римана-Грина гиперболического уравнения не определена во всей области. Чтобы построить функцию Римана-Грина этого уравнения, важно иметь функцию Римана-Грина той задачи, которая была определена во всех точках области. Для этого было продолжено нечетно коэффициенты общего гиперболического уравнения. Дано определение функции Грина задачи Дарбу. Чтобы показать, что функция Грина существует и единственна, мы разделяем область на несколько подобластей. Его существование и единственность были доказаны. Представлена явная форма функции Грина. Показано, что функция Грина может быть представлена функцией Римана–Грина. Дан метод построения функции Грина для такой задачи. Основное принципиальное отличие этой работы состоит в том, что она посвящена изучению функции Грина для гиперболической задачи. В отличие от (хорошо развитой) теории функции Грина для самосопряженных эллиптических задач, эта теория не была разработана.

Ключевые слова: Гиперболическое уравнение, начально-краевая задача, задача Дарбу, граничное условие, функция Грина, характеристический треугольник, функция Римана–Грина.

1 Introduction

An explicit form of the Green's function in the sector for biharmonic and triharmonic equations is given in [1], [2]. The Green's function of the Neumann problem for the Poisson equation in the half-space R_n^+ is explicitly constructed in [3], and the Green's function for the Robin problem in the circle in [4], [5], [6]. We also note the articles [7], [8], which are devoted to the construction of the Green's function for the Dirichlet problem for the polyharmonic equation in the unit ball. In [9], [10] a representation of the Green's function for the classical external and internal Neumann problems for the Poisson equation in the unit ball is given.

2 Formulation of the problem

Let $\Omega = \{(\xi,\eta):\ 0 \le \xi \le 1,\ \xi \le \eta \le 1\}$. The following hyperbolic equation is considered in Ω:

$$\frac{\partial^2 u}{\partial\xi\partial\eta} + a(\xi,\eta)\frac{\partial u}{\partial\xi} + b(\xi,\eta)\frac{\partial u}{\partial\eta} + c(\xi,\eta)u = f(\xi,\eta), \quad (\xi,\eta)\in\Omega, \tag{1}$$

with the initial condition

$$u(\xi,\xi) = \tau_0(\xi),\ 0 \le \xi \le 1, \tag{2}$$

and the boundary condition

$$u(0,\xi_0) = \tau_1(\xi_0),\ 0 \le \xi_0 \le 1. \tag{3}$$

We will assume that $a, b, a_\xi, b_\eta, c, f \in C\left(\overline{\Omega}\right)$; $\tau_0, \tau_1 \in C^1([0,1])$ and

$$a(\xi,\xi) = b(\xi,\xi), \ 0 \le \xi \le 1. \tag{4}$$

If (4) is not fulfilled then we will rewrite the function $u(\xi,\eta)$ in the following form:

$$u(\xi,\eta) = U(\xi,\eta)\cdot\gamma(\eta), \ (\xi,\eta) \in \Omega. \tag{5}$$

Substituting (5) in (1) we get

$$\frac{\partial^2 U}{\partial\xi\partial\eta} + a_1(\xi,\eta)\frac{\partial U}{\partial\xi} + b_1(\xi,\eta)\frac{\partial U}{\partial\eta} + c_1(\xi,\eta)U = f_1(\xi,\eta), \quad (\xi,\eta) \in \Omega,$$

where

$$a_1 = \frac{\gamma'}{\gamma} + a, \ b_1 = b, \ c_1 = \frac{\gamma'}{\gamma} + c, \ f_1 = \frac{f}{\gamma}.$$

We choose $\gamma(\eta), \ \xi \le \eta \le 1$, in such a way that

$$a_1(\xi,\xi) = b_1(\xi,\xi), \ 0 \le \xi \le 1, \tag{6}$$

holds. From (6) we get equation

$$\frac{\gamma'(\xi)}{\gamma(\xi)} + a(\xi,\xi) = b(\xi,\xi), \ 0 \le \xi \le 1 \tag{7}$$

Solving (7) we get

$$\gamma(\xi) = \exp\left(-\int_0^\xi \big(a(s,s) - b(s,s)\big)ds\right).$$

Therefore, by condition (4), we can always get.

Also, we assume that

$$a_\xi(\xi,\xi) = b_\eta(\xi,\xi), \ 0 \le \xi \le 1. \tag{8}$$

3 Proof of correctness of problem (1)-(3)

For the sake of completeness, we present here a proof of the correctness of the considered problem (1)-(3).

Let us call a function from the class $u(\xi,\eta), \ u_{\xi\eta} \in C\left(\overline{\Omega}\right)$ a *regular solution* to the problem, converting equation (1), initial condition (2) and boundary condition (3) into an identity.

Theorem 1 *Let* $a, b, a_\xi, b_\eta, c, f \in C\left(\overline{\Omega}\right)$; $\tau_0, \tau_1 \in C^1([0,1])$. *Then problem* (1)-(3) *has a unique regular solution.*

3.1 Proof of existence of solution of problem (1)-(3)

Let

$$u(\xi,\eta)=\zeta(\xi,\eta)\cdot\omega(\xi,\eta). \tag{9}$$

Then (1) has the form

$$\begin{gathered}\frac{\partial^2\zeta}{\partial\xi\partial\eta}\cdot\omega+\frac{\partial^2\omega}{\partial\xi\partial\eta}\cdot\zeta+\left[\frac{\partial\zeta}{\partial\xi}+b\zeta\right]\cdot\frac{\partial\omega}{\partial\eta}+\left[\frac{\partial\zeta}{\partial\eta}+a\zeta\right]\cdot\frac{\partial\omega}{\partial\xi}+\\+\left[b\frac{\partial\zeta}{\partial\eta}+a\frac{\partial\zeta}{\partial\xi}+c\zeta\right]\cdot\omega=f.\end{gathered} \tag{10}$$

We choose $\zeta(\xi,\eta)$ in such a way that

$$\frac{\partial\zeta(\xi,\eta)}{\partial\eta}+a(\xi,\eta)\cdot\zeta(\xi,\eta)=0, \tag{11}$$

holds. From (11) we get

$$\zeta(\xi,\eta)=\exp\left(-\int_0^\xi a(\xi,s)ds\right). \tag{12}$$

Dividing equation (10) by ζ, we have the following problem

$$\frac{\partial^2\omega}{\partial\xi\partial\eta}+b_2(\xi,\eta)\cdot\frac{\partial\omega}{\partial\xi}+c_2(\xi,\eta)\cdot\omega=f_2,\ (\xi,\eta)\in\Omega, \tag{13}$$

$$\omega(\xi,\xi)=\tau_2(\xi),\ 0\le\xi\le 1, \tag{14}$$

$$\omega(0,\xi_0)=\tau_3(\xi_0),\ 0\le\xi_0\le 1, \tag{15}$$

where

$$b_2=\frac{1}{\zeta}\cdot\zeta_\xi+b,\ f_2(\xi,\eta)=\frac{f(\xi,\eta)}{\zeta(\xi,\eta)},$$

$$c_2=\frac{1}{\zeta}\left(\zeta_{\xi\eta}+a\zeta_\xi+b\zeta_\eta+c\right),\ \tau_2(\xi)=\frac{\tau_0(\xi)}{\zeta(\xi,\xi)},\ \tau_3(\xi_0)=\frac{\tau_1(\xi_0)}{\zeta(0,\xi_0)},$$

Let us introduce a new notation

$$\frac{\partial\omega}{\partial\eta}=\upsilon.$$

Then equation (13) is equivalent to the following system of equations:

$$\left\{\begin{array}{l}\dfrac{\partial\upsilon}{\partial\xi}=f_2(\xi,\eta)-b_2(\xi,\eta)\cdot\upsilon(\xi,\eta)-c_2(\xi,\eta)\cdot\omega(\xi,\eta),\\ \dfrac{\partial\omega}{\partial\eta}=\upsilon(\xi,\eta).\end{array}\right. \tag{16}$$

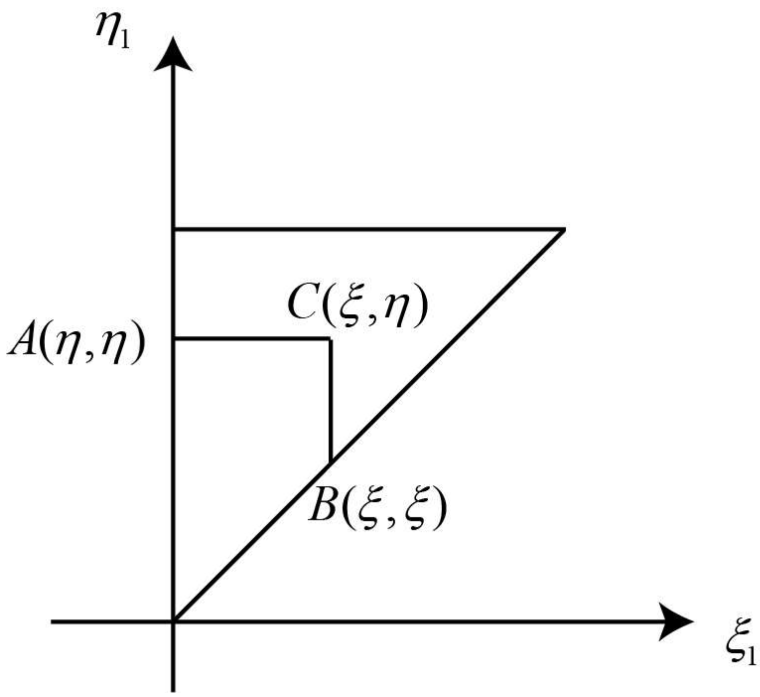

Figure 1: The domain Ω.

In the domain Ω, take an arbitrary point $C(\xi,\eta)$ and draw the characteristics through it to the boundary of the domain Ω (see Figure (1)). Integrating the first equation of system (16) by AC, the second by BC and using conditions (14), (15), we get

$$\begin{cases} \upsilon(\xi,\eta) = \tau_3'(\eta) + \int\limits_0^{\xi} \Big[f_2(\xi_1,\eta) - b_2(\xi_1,\eta)\upsilon(\xi_1,\eta) - c_2(\xi_1,\eta)\omega(\xi_1,\eta) \Big] d\xi_1, \\ \omega(\xi,\eta) = \tau_2(\xi) + \int\limits_{\xi}^{\eta} \upsilon(\xi,\eta_1)d\eta_1, (\xi,\eta)\in\Omega, \end{cases} \tag{17}$$

Substituting the second equation of the system (15) into the first equation we get

$$\upsilon(\xi,\eta)=\tau_3'(\eta)+\int\limits_0^\xi f_2(\xi_1,\eta)d\xi_1-\int\limits_0^\xi b_2(\xi_1,\eta)\upsilon(\xi_1,\eta)d\xi_1-\int\limits_0^\xi c_2(\xi_1,\eta)\tau_2(\xi_1)d\xi_1-$$
$$-\int\limits_0^\xi d\xi_1\int\limits_{\xi_1}^\eta c_2(\xi_1,\eta)\upsilon(\xi_1,\eta_1)d\eta_1$$

It is easy to show that, if $\upsilon(\xi,\eta)$ is solution to system (17), then $\omega(\xi,\eta)$ is a solution to problem (13)-(15). Therefore, system (17) is equivalent to problem (13)-(15).

We will search the solution to system (17) using the method of successive approximations. We choose an initial approximation

$$\upsilon_0(\xi,\eta)=0.$$

We construct the following approximation using the formula

$$\upsilon_n(\xi,\eta)=\tau_3'(\eta)+\int\limits_0^\xi f_2(\xi_1,\eta)d\xi_1-\int\limits_0^\xi b_2(\xi_1,\eta)\upsilon_{n-1}(\xi_1,\eta)d\xi_1-\int\limits_0^\xi c_2(\xi_1,\eta)\tau_2(\xi_1)d\xi_1-$$
$$-\int\limits_0^\xi d\xi_1\int\limits_{\xi_1}^\eta c_2(\xi_1,\eta)\upsilon_{n-1}(\xi_1,\eta_1)d\eta_1 \tag{18}$$

Show that difference $|\upsilon_n-\upsilon_{n-1}|$ satisfy the inequality

$$|\upsilon_n-\upsilon_{n-1}|\le K^{n-1}\cdot A\cdot\frac{\xi^{n-1}}{(n-1)!}, \tag{19}$$

where

$$K=\max_{\overline{\Omega}}\,[|b_2|+|c_2|],$$
$$A\le\|f_2\|_{C(\overline{\Omega})}+\|\tau_3\|_{C'([0,1])}+\|\tau_2\|_{C'([0,1])}.$$

We prove the validity of inequality (19) by mathematical induction. For $n=1$, as is easy to see from (18), the estimate of (19) is correct.

We show that these inequality will remain valid when n is replaced by $n+1$. From equality (18), according to the classical method we have

$$|\upsilon_{n+1}-\upsilon_n|=\left|\int\limits_0^\xi b_2(\xi_1,\eta)\big(\upsilon_n-\upsilon_{n-1}\big)(\xi_1,\eta)d\xi_1\right|+\left|\int\limits_0^\xi d\xi_1\int\limits_{\xi_1}^\eta c_2(\xi_1,\eta)\big(\upsilon_n-\upsilon_{n-1}\big)(\xi_1,\eta_1)d\eta_1\right|\le$$
$$\le K^{n-1}\cdot A\cdot\int_0^\xi|b_2|\frac{\xi_1^{n-1}}{(n-1)!}d\xi_1+K^{n-1}\cdot A\cdot\int_0^\xi|c_2|\frac{\xi_1^{n-1}}{(n-1)!}d\xi_1\le\frac{K^n}{n!}\cdot A\cdot\xi^n,$$

The estimate (19) show absolute and uniform convergence over $\overline{\Omega}$ of the following series

$$\upsilon_0 + \sum_{n=1}^{\infty} (\upsilon_n - \upsilon_{n-1})$$

members of which are less than the absolute value of the members of the uniformly converging series

$$A + A \cdot \sum_{n=1}^{\infty} K^{n-1} \frac{\xi^{n-1}}{(n-1)!} = A\big(1 + \exp(K\xi)\big).$$

Consequently, successive approximation of υ_n on $\overline{\Omega}$ uniformly tend, respectively, to certain limit of υ, which is continuous on $\overline{\Omega}$. Passing to the limit in equality (18), we obtain that the limit function of $\upsilon(\xi, \eta)$ satisfy system (17). In this case, the function υ is continuous on $\overline{\Omega}$. Since we have proved the existence of the solution in $\overline{\Omega}$. The solution to problem (1)-(3) is found by substituting ω, ζ for (9).

3.2 Proof of uniqueness of the solution to problem (1)-(3)

Assume that system (17) has different solutions $\upsilon_1,\ \upsilon_2$. Denote $V = \upsilon_1 - \upsilon_2$. Then V satisfy the following equation

$$V(\xi, \eta) = -\int_0^{\xi} b_2(\xi_1, \eta) V(\xi_1, \eta) d\xi_1 - \int_0^{\xi} d\xi_1 \int_{\xi_1}^{\eta} c_2(\xi_1, \eta) V(\xi_1, \eta_1) d\eta_1$$

Let us prove that $V \equiv 0$. Function V is continuous and bounded as the difference of continuous functions in a closed domain $\overline{\Omega}$. Therefore, there is a positive constant B such that

$$|V(\xi, \eta)| \leq B.$$

Then we have

$$V \leq \int_0^{\xi} |b_2(\xi_1, \eta)| B d\xi_1 + \int_0^{\xi} |c_2(\xi_1, \eta)|(\eta - \xi_1) B d\xi_1 \leq$$
$$\leq \int_0^{\xi} |b_2(\xi_1, \eta)| B d\xi_1 + \int_0^{\xi} |c_2(\xi_1, \eta)| B d\xi_1 \leq \frac{K \cdot B \cdot \xi}{1!}.$$

By mathematical induction for any n, we obtain the following estimate

$$|V| \leq B K^n \frac{\xi^n}{n!}.$$

Since these inequality is met for any n, then it follows that $V \equiv 0$, i.e. $\upsilon_1 = \upsilon_2$.

3.3 Proof of stability of the solution to problem (1)-(3)

In order to prove the stability of the solution to problem (1)-(3) we need a stability estimate for υ. Since

$$\upsilon(\xi,\eta)=\lim_{N\to\infty}\upsilon_N=\lim_{N\to\infty}\left[\upsilon_0+\sum_{n=1}^{N}(\upsilon_n-\upsilon_{n-1})\right]=\upsilon_0+\sum_{n=1}^{\infty}(\upsilon_n-\upsilon_{n-1}), \tag{20}$$

then using the estimate of (19) from (20), we get

$$|\upsilon(\xi,\eta)|\le A\cdot\sum_{n=1}^{\infty}\frac{K^{n-1}}{(n-1)!}\xi^{n-1}=A\cdot\exp(K). \tag{21}$$

Hence, using (19) from (21) we obtain the estimate of the stability of the solution to problem (1)-(3):

$$||\upsilon||_{C(\Omega)}\le\exp(K)\left(\|f_2\|_{C(\overline{\Omega})}+\|\tau_3\|_{C'([0,1])}+\|c_1\|_{C([0,1])}\cdot\|\tau_2\|_{C'([0,1])}\right)\le$$
$$\le\exp(2K)\left(\|f_2\|_{C(\overline{\Omega})}+\|\tau_3\|_{C'([0,1])}+\|\tau_2\|_{C'([0,1])}\right).$$

4 Green's function of the problem (1)-(3)

Definition 1 *Green's function of the problem* (1)-(3) *let us call the function* $G(\xi,\eta;\xi_1,\eta_1)$, *which for every fixed* $(\xi_1,\eta_1)\in\Omega$, *satisfies the homogeneous equation*

$$L_{(\xi,\eta)}G(\xi,\eta;\xi_1,\eta_1)=0,\ (\xi,\eta)\in\Omega,\ at\,\xi\ne\xi_1,\ \ \eta\ne\eta_1,\ \ \eta\ne\xi_1,\ \ \xi\ne\eta_1; \tag{22}$$

and the next boundary conditions

$$G(\xi,\xi;\xi_1,\eta_1)=0,\ 0\le\xi\le1,\ (\xi_1,\eta_1)\in\Omega; \tag{23}$$

$$G(0,\xi_0;\xi_1,\eta_1)=0,\ 0\le\xi_0\le1,\ (\xi_1,\eta_1)\in\Omega; \tag{24}$$

and on the above characteristic lines, the following conditions must be met: the values of the derivatives of the Green function in directions parallel to these characteristics must coincide in adjacent regions; i.e.,

$$\frac{\partial G(\xi_1+0,\eta;\xi_1,\eta_1)}{\partial\eta}+a(\xi_1,\eta)G(\xi_1+0,\eta;\xi_1,\eta_1)=$$
$$=\frac{\partial G(\xi_1-0,\eta;\xi_1,\eta_1)}{\partial\eta}+a(\xi_1,\eta)G(\xi_1-0,\eta;\xi_1,\eta_1),\ at\ \eta\ne\eta_1,\ \ \eta\ne\xi_1; \tag{25}$$

$$\frac{\partial G(\eta_1+0,\eta;\xi_1,\eta_1)}{\partial\eta}+a(\eta_1,\eta)G(\eta_1+0,\eta;\xi_1,\eta_1)=$$
$$=\frac{\partial G(\eta_1-0,\eta;\xi_1,\eta_1)}{\partial\eta}+a(\eta_1,\eta)G(\xi_1-0,\eta;\xi_1,\eta_1),\ at\ \eta\ne\eta_1,\ \ \eta\ne\xi_1; \tag{26}$$

$$\frac{\partial G(\xi,\eta_1+0;\xi_1,\eta_1)}{\partial\xi}+b(\xi,\eta_1)G(\xi,\eta_1+0;\xi_1,\eta_1)=$$

$$=\frac{\partial G(\xi,\eta_1-0;\xi_1,\eta_1)}{\partial\xi}+b(\xi,\eta_1)G(\xi,\eta_1-0;\xi_1,\eta_1),\ at\ \xi\neq\xi_1\ \xi\neq\eta_1; \tag{27}$$

$$\frac{\partial G(\xi,\xi_1+0;\xi_1,\eta_1)}{\partial\xi}+b(\xi,\xi_1)G(\xi,\xi_1+0;\xi_1,\eta_1)=$$

$$=\frac{\partial G(\xi,\xi_1-0;\xi_1,\eta_1)}{\partial\xi}+b(\xi,\xi_1)G(\xi,\xi_1-0;\xi_1,\eta_1)\ at\ \xi\neq\xi_1\ \xi\neq\eta_1; \tag{28}$$

and the "corner condition"

$$G(\xi_1-0,\eta_1-0;\xi_1,\eta_1)-G(\xi_1+0,\eta_1-0;\xi_1,\eta_1)+$$

$$+G(\xi_1+0,\eta_1+0;\xi_1,\eta_1)-G(\xi_1-0,\eta_1+0;\xi_1,\eta_1)=1. \tag{29}$$

must be satisfied as the regions meet at $(\xi,\eta)=(\xi_1,\eta_1)$.

5 Existence and uniqueness of the Green's function of the problem (4)-(6)

Theorem 2 *The function* $G(\xi,\eta;\xi_1,\eta_1)$ *that satisfies the conditions* (22)-(29) *exists and is unique.*

Proof. To show that a function $G(\xi,\eta;\xi_1,\eta_1)$ which satisfies the conditions (22)-(29) exists and unique, we divide the domain Ω into several subdomains (see Figure (2)) and consider the following problems sequentially. Let (ξ_1,η_1) be an arbitrary point of the domain Ω.

In the domain $\Omega_1=\{(\xi,\eta):0<\xi<\xi_1,\xi<\eta<\xi_1\}$ we consider the problem

$$L_{(\xi,\eta)}G=0,\ (\xi,\eta)\in\Omega_1; \tag{30}$$

$$G(\xi,\xi;\xi_1,\eta_1)=0,\ 0\leq\xi\xi_1; \tag{31}$$

$$G(0,\xi_0;\xi_1,\eta_1)=0,\ 0\leq\xi_0\leq\xi_1,\ (\xi_1,\eta_1)\in\Omega_2. \tag{32}$$

The problem (30)-(32) is a Darboux problem and has a unique solution

$$G(\xi,\eta;\xi_1,\eta_1)\equiv 0,\ (\xi,\eta)\in\Omega_1. \tag{33}$$

In the domain $\Omega_2=\{(\xi,\eta):0\leq\xi\leq\xi_1,\xi_1\leq\eta\leq\eta\}$ let us consider the problem

$$L_{(\xi,\eta)}G=0,\ (\xi,\eta)\in\Omega_2; \tag{34}$$

$$G(0,\xi_0;\xi_1,\eta_1)=0,\ \xi\leq\xi_0\leq\eta_1,\ (\xi_1,\eta_1)\in\Omega_2. \tag{35}$$

From (33) we have the next equality

$$\frac{\partial G(\xi,\xi_1+0;\xi_1,\eta_1)}{\partial\xi}+b(\xi,\xi_1)G(\xi,\xi_1+0;\xi_1,\eta_1)=0,\ 0\leq\xi\leq\xi_1. \tag{36}$$

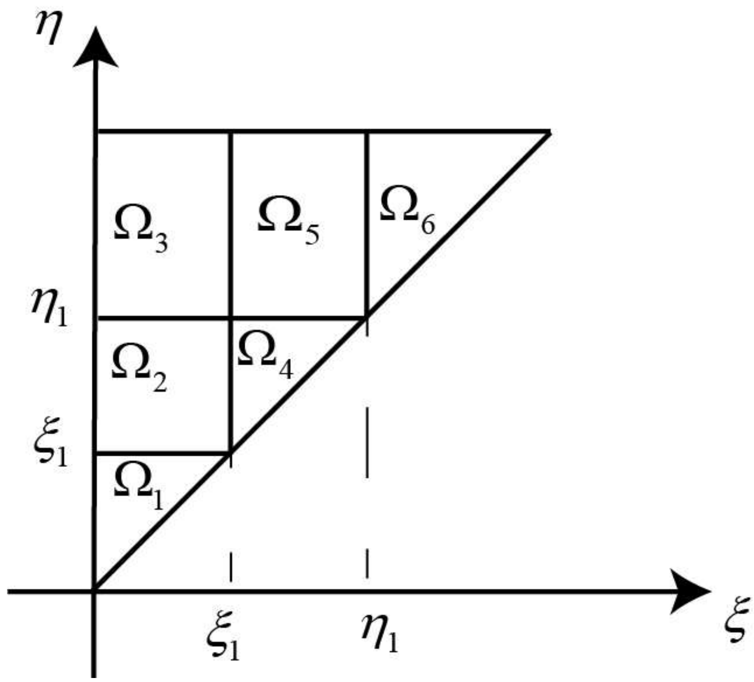

Figure 2: Splitting the domain Ω.

Integrating (36) by ξ we have

$$G(\xi,\xi_1+0;\xi_1,\eta_1)=\exp\left(-\int_0^\xi B(t,\xi_1)dt\right)C_1(\xi_1,\eta_1),\ 0\le\xi\le\xi_1. \tag{37}$$

Substituting $\xi=0$ in (37), using condition (23) we have that $C_1(\xi_1,\eta_1)\equiv 0$ and

$$G(\xi,\xi_1+0;\xi_1,\eta_1)=0,\ 0\le\xi\le\xi_1. \tag{38}$$

The problem (34),(35),(38) is a Goursat problem and has a unique solution

$$G(\xi,\eta;\xi_1,\eta_1)\equiv 0,\ (\xi,\eta)\in\Omega_2. \tag{39}$$

Therefore from (39) in the domain $\Omega_3=\{(\xi,\eta):0\le\xi\le\xi_1,\eta_1\le\eta\le 1\}$, we get the problem

$$L_{(\xi,\eta)}G=0,\ (\xi,\eta)\in\Omega_3; \tag{40}$$

$$G(0,\xi_0;\xi_1,\eta_1)=0,\ \eta_1\le\xi_0\le 1,\ (\xi_1,\eta_1)\in\Omega_3; \tag{41}$$

$$\frac{\partial G(\xi,\eta_1+0;\xi_1,\eta_1)}{\partial\xi}+b(\xi,\eta_1)\cdot G(\xi,\eta_1+0;\xi_1,\eta_1)=0,\ 0\le\xi\le\xi_1. \tag{42}$$

Integrating (42) by ξ we have

$$G(\xi,\eta_1+0;\xi_1,\eta_1)=\exp\left(-\int_0^{\xi} b(t,\eta_1)dt\right)C_2(\xi_1,\eta_1),\ 0\le\xi\le\xi_1. \tag{43}$$

Substituting $\xi=0$ in (43), using condition (23) we have that $C_2(\xi_1,\eta_1)\equiv 0$ and

$$G(\xi,\eta_1+0;\xi_1,\eta_1)=0,\ 0\le\xi\le\xi_1. \tag{44}$$

Therefore, the problem (40),(41),(44) is a Goursat problem and has a unique solution

$$G(\xi,\eta;\xi_1,\eta_1)\equiv 0,\ (\xi,\eta)\in\Omega_3. \tag{45}$$

In the domain $\Omega_4=\{(\xi,\eta):0\le\xi\le\xi_1,\xi\le\eta\le\eta_1\}$ we get the problem

$$L_{(\xi,\eta)}G=0,\ (\xi,\eta)\in\Omega_4; \tag{46}$$

$$G(\xi,\xi;\xi_1,\eta_1)=0,\ \xi_1\le\xi\le\eta_1. \tag{47}$$

From (35) we have

$$\frac{\partial G(\xi_1+0,\eta;\xi_1,\eta_1)}{\partial\eta}+a(\xi_1,\eta)G(\xi_1+0,\eta;\xi_1,\eta_1)=0,\ \xi_1\le\eta\le\eta_1. \tag{48}$$

Integrating (48) by η we get

$$G(\xi_1+0,\eta;\xi_1,\eta_1)=\exp\left(-\int_{\xi_1}^{\eta} a(\xi_1,t)dt\right)C_3(\xi_1,\eta_1),\ \xi_1\le\eta\le\eta_1; \tag{49}$$

Substituting $\eta=\xi_1$ in (49), using condition (23) we have that $C_3(\xi_1,\eta_1)\equiv 0$ and

$$G(\xi_1+0,\eta;\xi_1,\eta_1)=0,\ \xi_1\le\eta\le\eta_1. \tag{50}$$

This problem (46),(47),(50) is a Darboux problem and has a unique solution

$$G(\xi,\eta;\xi_1,\eta_1)\equiv 0,\ (\xi,\eta)\in\Omega_4. \tag{51}$$

Therefore, from (45), (51) in the domain $\Omega_5=\{(\xi,\eta):\xi_1\le\xi\le\eta_1,\eta_1\le\eta\le 1\}$ our problem is the Cauchy problem

$$L_{(\xi,\eta)}G=0,\ (\xi,\eta)\in\Omega_5; \tag{52}$$

$$\frac{\partial G(\xi_1+0,\eta;\xi_1,\eta_1)}{\partial\eta}+a(\xi_1,\eta)G(\xi_1+0,\eta;\xi_1,\eta_1)=0,\ \eta_1\le\eta\le 1; \tag{53}$$

$$\frac{\partial G(\xi,\eta_1+0;\xi_1,\eta_1)}{\partial\xi}+b(\xi,\eta_1)G(\xi,\eta_1+0;\xi_1,\eta_1)=0,\ \xi_1\le\xi\le\eta_1; \tag{54}$$

$$G(\xi_1+0,\eta_1+0;\xi_1,\eta_1)=1. \tag{55}$$

The problem (51)-(55) is a Goursat problem and it has a unique solution, and it is easy to see that its solution coincides with the Riemann-Green function, that is,

$$G(\xi,\eta;\xi_1,\eta_1)=R(\xi,\eta;\xi_1,\eta_1),\ (\xi,\eta)\in\Omega_5. \tag{56}$$

Therefore from (56) in the domain $\Omega_6=\{(\xi,\eta):\eta_1\le\xi\le 1,\xi\le\eta\le 1\}$ we get the problem

$$L_{(\xi,\eta)}G=0,\ (\xi,\eta)\in\Omega_6; \tag{57}$$

$$G(\xi,\xi;\xi_1,\eta_1)=0,\ \eta_1\le\xi\le 1; \tag{58}$$

$$\frac{\partial G(\eta_1+0,\eta;\xi_1,\eta_1)}{\partial\eta}+b(\eta_1,\eta)G(\eta_1+0,\eta;\xi_1,\eta_1)=$$
$$=\frac{\partial R(\eta_1,\eta;\xi_1,\eta_1)}{\partial\eta}+b(\eta_1,\eta)R(\eta_1,\eta;\xi_1,\eta_1). \tag{59}$$

Let us rewrite condition (59) in the following form

$$\left[\frac{\partial}{\partial\eta}\left(G(\eta_1+0,\eta;\xi_1,\eta_1)\exp\left(\int_\xi^\eta a(\eta_1,t)dt\right)\right)\right]\exp\left(-\int_\xi^\eta a(\eta_1,t)dt\right)=$$
$$=\left[\frac{\partial}{\partial\eta}\left(R(\eta_1,\eta;\xi_1,\eta_1)\exp\left(\int_\xi^\eta a(\eta_1,t)dt\right)\right)\right]\exp\left(-\int_\xi^\eta a(\eta_1,t)dt\right). \tag{60}$$

Integrating (60) by η we get

$$G(\eta_1+0,\eta;\xi_1,\eta_1)=R(\eta_1,\eta;\xi_1,\eta_1)+C_4(\xi_1,\eta_1)\exp\left(-\int_\xi^\eta a(\eta_1,t)dt\right). \tag{61}$$

Using condition (23), from (61) we have that

$$G(\eta_1+0,\eta;\xi_1,\eta_1)=R(\eta_1,\eta;\xi_1,\eta_1)-R(\eta_1,\eta_1;\xi_1,\eta_1)\exp\left(-\int_{\eta_1}^\eta a(\eta_1,t)dt\right),\ \xi\le\eta\le 1. \tag{62}$$

The problem (57), (58), (62) is a Darboux problem and has a unique solution.

Thus, we have shown that for any $(\xi_1,\eta_1)\in\Omega$ and $(\xi,\eta)\in\Omega$ the Green's function that satisfies the conditions (22)-(29) exists and unique. The theorem is proved.

6 Construction of the Green's function of the problem (1)-(3)

As can be seen from the proof of Theorem (2), the Green's function $G(\xi,\eta;\xi_1,\eta_1)=0$ in the domains Ω_1, Ω_2, Ω_3, Ω_4. And in the domain Ω_5 it coincides with the Riemann function (56).

Let us find a representation of the Green's function in the domain Ω_6. To construct the Green's functions, we will continue the coefficients of equation (57) in $\Omega_6^*=\{(\xi,\eta):\eta_1\le\xi\le 1,\ \eta_1\le\eta\le\xi\}$ such a way that the following conditions

$$A(\xi,\eta)=\begin{cases}a(\xi,\eta), & (\xi,\eta)\in\Omega_6,\\ b(\eta,\xi), & (\xi,\eta)\in\Omega_6^*,\end{cases}$$

$$B(\xi,\eta) = \begin{cases} b(\xi,\eta), & (\xi,\eta) \in \Omega_6, \\ a(\eta,\xi), & (\xi,\eta) \in \Omega_6^*, \end{cases}$$

$$C(\xi,\eta) = \begin{cases} c(\xi,\eta), & (\xi,\eta) \in \Omega_6, \\ c(\eta,\xi). & (\xi,\eta) \in \Omega_6^* \end{cases}$$

are met. Actually, show that coefficients of (57) have the following symmetry:

$$A(\xi,\eta) = B(\eta,\xi),\ C(\xi,\eta) = C(\eta,\xi),\ (\xi,\eta) \in \Omega_6. \tag{63}$$

From (63) we have

$$A(\eta,\xi) = \begin{cases} a(\eta,\xi), & (\eta,\xi) \in \Omega_6, \\ b(\xi,\eta), & (\eta,\xi) \in \Omega_6^*, \end{cases} = \begin{cases} b(\xi,\eta), & (\xi,\eta) \in \Omega_6, \\ a(\eta,\xi), & (\xi,\eta) \in \Omega_6^*, \end{cases} = B(\xi,\eta).$$

If we have chosen (ξ,η) from Ω_6, then (η,ξ) will be from Ω_6^*.

From (4) and (5) we get

$$A(\xi,\xi) = B(\xi,\xi),\ A_\xi(\xi,\xi) = B_\eta(\xi,\xi),\ \eta_1 \le \xi \le 1.$$

If the coefficients $a, b, a_\xi, b_\eta, c \in C\left(\overline{\Omega}\right)$ then in virtue of (63) coefficients $A(\xi,\eta),\ B(\xi,\eta),\ C(\xi,\eta)$ in the domain $\widetilde{\Omega_6} = \Omega_6 \cup \Omega_6^* = \{(\xi,\eta) : \eta_1 \le \xi \le 1,\ \eta_1 \le \eta \le 1\}$ have the following smoothness

$$A, B, A_\xi, B_\eta, C \in C\left(\overline{\widetilde{\Omega_6}}\right). \tag{64}$$

Let (ξ_1,η_1) be an arbitrary point of the domain Ω. In order to construct the Green function in the domain Ω_6, consider the problem:

$$\frac{\partial^2 G_1}{\partial\xi\partial\eta} + A(\xi,\eta)\frac{\partial G_1}{\partial\xi} + B(\xi,\eta)\frac{\partial G_1}{\partial\eta} + C(\xi,\eta)G_1 = 0,\ (\xi,\eta) \in \widetilde{\Omega}_6; \tag{65}$$

$$G(\eta_1+0,\eta;\xi_1,\eta_1) = R(\eta_1,\eta;\xi_1,\eta_1) - \exp\left(-\int_{\xi_1}^{\eta} A(\eta_1,t)dt\right),\ \eta_1 \le \eta \le 1; \tag{66}$$

$$G(\xi,\eta_1+0;\xi_1,\eta_1) = -R(\eta_1,\xi;\xi_1,\eta_1) + \exp\left(-\int_{\xi_1}^{\xi} B(t,\eta_1)dt\right),\ \eta_1 \le \xi \le 1. \tag{67}$$

The problem (65)-(67) is a Goursat problem. Its solution exists and unique. We are interested in the representation of the function $G_1(\xi,\eta;\xi_1,\eta_1)$.

Lemma 1 *If the function $G_1(\xi,\eta;\xi_1,\eta_1)$ is the solution to the problem* (65)-(67), *then for any $(\xi,\eta) \in \widetilde{\Omega}_6$ we have $G_1(\xi,\eta;\xi_1,\eta_1) = -G_1(\eta,\xi;\xi_1,\eta_1)$.*

To show that the function $-G_1(\eta,\xi;\xi_1,\eta_1)$ satisfies the equation (65), in (62) replace $\xi = \eta_2,\ \eta = \xi_2,\ (\eta_2,\xi_2) \in \Omega_6^*$ and after using the symmetry conditions of coefficients, we get that $-G_1(\eta,\xi;\xi_1,\eta_1)$ satisfies the equation (65).

Also doing the substitution of $\xi = \eta_2$ in (66) and using the symmetry conditions of coefficients, we get the condition (67). Similarly, by replacing $\eta = \xi_2$ in (67) and using the symmetry conditions of coefficients, we get the condition (66).

Thus, we have shown that the function $-G_1(\eta, \xi; \xi_1, \eta_1)$ is also a solution to the problem (65)-(67). Since the solution to problem (65)-(67) is unique, then

$$G_1(\xi, \eta; \xi_1, \eta_1) = -G_1(\eta, \xi; \xi_1, \eta_1),\ (\xi, \eta) \in \widetilde{\Omega}_6.$$

Solution of the problem (65)-(67) we search in the following form

$$G_1(\xi, \eta; \xi_1, \eta_1) = R(\xi, \eta; \xi_1, \eta_1) - g(\xi, \eta; \xi_1, \eta_1), (\xi, \eta) \in \widetilde{\Omega}_6.$$

Then we get the following problem

$$\frac{\partial^2 g}{\partial \xi \partial \eta} + A(\xi, \eta)\frac{\partial g}{\partial \xi} + B(\xi, \eta)\frac{\partial g}{\partial \eta} + C(\xi, \eta)g = 0,\ (\xi, \eta) \in \widetilde{\Omega}_6; \tag{68}$$

$$g(\eta_1, \eta; \xi_1, \eta_1) - R(\eta, \eta_1; \xi_1, \eta_1) = 0,\ \eta_1 \le \eta \le 1; \tag{69}$$

$$g(\xi, \eta_1; \xi_1, \eta_1) - R(\eta_1, \xi; \xi_1, \eta_1) = 0,\ \eta_1 \le \xi \le 1. \tag{70}$$

It is easy to see that the solution to the problem (68)-(70) has the form

$$g(\xi, \eta; \xi_1, \eta_1) = R(\eta, \xi; \xi_1, \eta_1),\ (\xi, \eta) \in \widetilde{\Omega}_6.$$

Lemma 2 *Let (ξ, η) be an arbitrary point of the domain Ω. By internal variables (ξ_1, η_1) the Green's function of the problem (1)-(3) has the following properties:*

$$L^*_{(\xi_1,\eta_1)} G(\xi, \eta; \xi_1, \eta_1) = 0,\ (\xi_1, \eta_1) \in \Omega,\ at\ \xi_1 \ne \xi,\ \ \xi_1 \ne \eta,\ \ \eta_1 \ne \xi; \tag{71}$$

$$G(\xi_1, \xi_1; \xi_1, \eta_1) = 0,\ \ 0 \le \xi_1 \le 1; \tag{72}$$

$$\frac{\partial G(\xi, \eta; \xi - 0, \eta_1)}{\partial \eta_1} - a(\xi, \eta_1)G(\xi, \eta; \xi - 0, \eta_1) = 0,\ at\ \eta_1 \ne \eta,\ \ \eta_1 \ne \xi; \tag{73}$$

$$\frac{\partial G(\xi, \eta; \xi_1, \eta - 0)}{\partial \xi_1} - b(\xi_1, \eta)G(\xi, \eta; \xi_1, \eta - 0) = 0,\ at\ \xi_1 \ne \xi; \tag{74}$$

$$\frac{\partial G(\xi, \eta; \xi_1, \xi - 0)}{\partial \xi_1} - b(\xi_1, \xi)\ G(\xi, \eta; \xi_1, \xi - 0) =$$

$$= \frac{\partial G(\xi, \eta; \xi_1, \xi + 0)}{\partial \xi_1} - b(\xi_1, \xi)G(\xi, \eta; \xi_1, \xi + 0); \tag{75}$$

$$G(\xi, \eta; \xi - 0, \eta - 0) - G(\xi, \eta; \xi + 0, \eta - 0) +$$

$$+ G(\xi, \eta; \xi + 0, \eta + 0) - G(\xi, \eta; \xi - 0, \eta + 0) = 1. \tag{76}$$

Properties (71)-(76) are easy to get out of the construction of the Green's function of problem (1)-(3). Under these conditions (71)-(76) it is possible to uniquely restore the Green's function of problem (1)-(3).

Using properties (71)-(76) we can use it to write the integral representation of the solution to problem (1)-(3). To do this, we consider the following integral

$$\iint_{\Omega_{(\xi\eta)}} G(\xi,\eta;\xi_1,\eta_1) f(\xi_1,\eta_1) d\xi_1 d\eta_1 =$$

$$= \iint_{\Omega_{(\xi\eta)}} G(\xi,\eta;\xi_1,\eta_1) \left(\frac{\partial^2 u}{\partial\xi_1 \partial\eta_1} + a\frac{\partial u}{\partial\xi_1} + b\frac{\partial u}{\partial\eta_1} + cu \right) d\xi_1 d\eta_1. \tag{77}$$

Applying Green's theorem in a plane [12] and using the initial conditions (2), properties of Green's function (71)-(76), from (77) we get the following representation of the solution to problem (28)-(30) in the domain $\Omega_{(\xi\eta)} = \Omega_5 \cup \Omega_6$:

$$u(\xi,\eta) = \frac{1}{2} G(\xi,\eta;0,\eta-0)\tau_1(\eta) - \frac{1}{2}\int_0^{\xi} \frac{\partial G}{\partial N_1}(\xi,\eta;\xi_1,\xi_1)\tau_0(\xi_1) d\xi_1 +$$

$$+\frac{1}{2}\int_0^{\xi} \left(-\frac{\partial G}{\partial\eta_1}(\xi,\eta;0,\eta_1)\tau_0(\eta_1) + G(\xi,\eta;0,\eta_1)\tau_1'(\eta_1) + a(0,\eta_1)G(\xi,\eta;0,\eta_1)\tau_1(\eta_1) \right) d\eta_1$$

$$+ \iint_{\Omega_{(\xi\eta)}} G(\xi,\eta;\xi_1,\eta_1) f(\xi_1,\eta_1) d\xi_1 d\eta_1. \tag{78}$$

7 Acknowledgement

The work was supported by grant funding of scientific and technical programs and projects of the Ministry of Science and Education of the Republic of Kazakhstan (Grant No. AP09561656).

References

[1] Wang Y. Triharmonic boundary value problems in a sector // Complex Variables and Elliptic Equations. - 2014. V. 59, №5. -P. 732-749.

[2] Wang Y., Ye L. Biharmonic Green function and biharmonic Neumann function in a sector // Complex Variables and Elliptic Equations. - 2013. V. 58, №1. -P. 7-22.

[3] Constantin E., Pavel N. Green function of the Laplacian for the Neumann problem in R_n^+ // Libertas Mathematica. - 2010. V. 30, №16. -P. 57-69.

[4] Begehr H., Vaitekhovich T. Modified harmonic Robin function // Complex Variables and Elliptic Equations. - 2013. V. 58, №4. -P. 483-496.

[5] Sadybekov M. A., Torebek B. T., Turmetov B. Kh. On an explicit form of the Green function of the third boundary value problem for the Poisson equation in a circle // AIP Conference Proceedings. - 2014. V. 1611, -P. 255-260.

[6] Sadybekov M. A., Torebek B. T., Turmetov B. Kh. On an explicit form of the Green function of the Robin problem for the Laplace operator in a circle // Advances in Pure and Applied Mathematics. - 2015. V. 6, №3. -P. 163-172.

[7] Kal'menov T. Sh., Koshanov B. D., Nemchenko M. Y. Green function representation in the Dirichlet problem for polyharmonic equations in a ball // Doklady Mathematics. - 2008. V. 78, №1. -P. 528-530.

[8] Kal'menov T. Sh., Suragan D. On a new method for constructing the Green function of the Dirichlet problem for the polyharmonic equation // Differential Equations. - 2012. V. 48, №3. -P. 441-445.

[9] Sadybekov M. A., Torebek B. T., Turmetov B. Kh. Representation of the green function of an external Neumann problem for the Laplace operator // Siberian Mathematical Journal. - 2017. V. 58, №1. -P. 199-205.

[10] Sadybekov M. A., Torebek B. T., Turmetov B. Kh. Representation of Green's function of the Neumann problem for a multi-dimensional ball // Complex Variables and Elliptic Equations. - 2016. V. 61, №1. -P. 104-123.

[11] Sobolev S.L. Equations of Mathematical physics // Nauka. - 1966. [in Russian]

[12] Riley K. F., Hobson M. P., Bence S. J. Mathematical methods for physics and engineering // Cambridge University Press. - 2010.

Список литературы

[1] Wang Y. Triharmonic boundary value problems in a sector // Complex Variables and Elliptic Equations. - 2014. V. 59, №5. -P. 732-749.

[2] Wang Y., Ye L. Biharmonic Green function and biharmonic Neumann function in a sector // Complex Variables and Elliptic Equations. - 2013. V. 58, №1. -P. 7-22.

[3] Constantin E., Pavel N. Green function of the Laplacian for the Neumann problem in R_n^+ // Libertas Mathematica. - 2010. V. 30, №16. -P. 57-69.

[4] Begehr H., Vaitekhovich T. Modified harmonic Robin function // Complex Variables and Elliptic Equations. - 2013. V. 58, №4. -P. 483-496.

[5] Sadybekov M. A., Torebek B. T., Turmetov B. Kh. On an explicit form of the Green function of the third boundary value problem for the Poisson equation in a circle // AIP Conference Proceedings. - 2014. V. 1611, -P. 255-260.

[6] Sadybekov M. A., Torebek B. T., Turmetov B. Kh. On an explicit form of the Green function of the Robin problem for the Laplace operator in a circle // Advances in Pure and Applied Mathematics. - 2015. V. 6, №3. -P. 163-172.

[7] Kal'menov T. Sh., Koshanov B. D., Nemchenko M. Y. Green function representation in the Dirichlet problem for polyharmonic equations in a ball // Doklady Mathematics. - 2008. V. 78, №1. -P. 528-530.

[8] Kal'menov T. Sh., Suragan D. On a new method for constructing the Green function of the Dirichlet problem for the polyharmonic equation // Differential Equations. - 2012. V. 48, №3. -P. 441-445.

[9] Sadybekov M. A., Torebek B. T., Turmetov B. Kh. Representation of the green function of an external Neumann problem for the Laplace operator // Siberian Mathematical Journal. - 2017. V. 58, №1. -P. 199-205.

[10] Sadybekov M. A., Torebek B. T., Turmetov B. Kh. Representation of Green's function of the Neumann problem for a multi-dimensional ball // Complex Variables and Elliptic Equations. - 2016. V. 61, №1. -P. 104-123.

[11] Соболев С. Л. Уравнения математической физики // Наука. - 1966.

[12] Riley K. F., Hobson M. P., Bence S. J. Mathematical methods for physics and engineering // Cambridge University Press. - 2010.