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ON REPRESENTATION OF ONE CLASS OF SCHMIDT OPERATORS

In this paper, unitary symmetrizers are considered. It is well known that using Newton operator algorithm, similar to the usual Newton algorithm, for extracting the square root, one can prove that for every Hermitian operator $T \geq 0$, there exists a unique Hermitian operator $S \geq 0$ such that $T = S^2$. Moreover, S commutes with every bounded operator R with which commutes T. The operator S is called a square root of the operator T and is denoted by $T^{1/2}$. The existence of the square root allows one to determine the absolute value $|T| = (T^*T)^{1/2}$ of the bounded operator T. For every bounded linear operator $T : H \to H$ there exists a unique partially isometric operator $U : H \to H$ such that $T = U|T|$, $KerU = KerT$. Such an equality is called a polar expansion of the operator T. The Schmidt operator is understood as the unitary multiplier of the polar expansion of a compact inverse operator, with the help of which E. Schmidt was the first to obtain the expansion of a compact and not-self-adjoint operator and introduced so-called s-numbers. This paper shows that the unitary symmetrizer of an operator differs only in sign from the adjoint Schmidt operator. The main result of the paper: if A is an invertible and compact operator, and S is a unitary operator such that the operator SA is self-adjoint, then the operator AS is also self-adjoint and the formula $S = \pm U^*$ holds, where U is the Schmidt operator.
Key words: Unitary operator, symmetrizer, normal operator, Schmidt expansion, Schmidt operator, compact operator, polar representation of operator, square root of positive self-adjoint operator.

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Шмидт операторларының бір класы туралы түсінік

Бұл жұмыста біртұтас симметризаторлар қарастырылады. Ньютон операторлық алгоритмін қолданып, сәйкес қарапайым Ньютон алгоритмінде квадрат түбірді алу үшін кез келген $T \geq 0$ эрмиттік операторы үшін $T = S^2$ шарты орындалатындай жалғыз $S \geq 0$ эрмиттік оператордың табылатынын дәлелдеуге болатыны белгілі. Сонымен қатар, S - T операторымен алмасатын әрбір шектелген R операторымен алмасады. S операторы T операторының квадрат түбірі деп аталады және $T^{1/2}$ арқылы белгіленеді. Квадрат түбірдің бар болуы шектелген Т операторының $|T| = (T^*T)^{1/2}$ абсолют шамасын анықтауға мүмкіндік береді. Кез келген шектелген сызықты $T : H \to H$ операторы үшін $T = U|T|$, $KerU = KerT$ болатындай жалғыз жартылай изометриялық $U : H \to H$ оператоты бар. Бұл теңдік Т операторының полярлы жіктеуі деп аталады. Шмидт операторы ретінде жеткілікті үзіліссіз кеpiленетін оператордың полярлы жіктеуінің біртұтас көбейткішін түсінеміз, оның көмегімен Э.Шмидт бірінші болып жеткілікті үзіліссіз және өз-өзіне түйіндес емес оператордың жіктеуін алды және s-санын енгізді. Бұл жұмыста, оператордың біртұтас симмертизаторы Шмидт операторының түйіндесімен тек таңбасымен өзгешеленеді. Жұмыстың негізгі нәтижесі: егер A керіленетін және компакты оператор, SA операторы өз-өзіне түйіндес болатындай S – біртұтас оператор болса, онда AS операторы да өз-өзіне түйіндес және $S = \pm U^*$ формуласы орынды, мұндағы U – Шмидт операторы.

Түйін сөздер: Біртұтас оператор, симметриялаушы, қалыпты оператор, Шмидттің кеңеюі, Шмидт операторы, толық үздіксіз оператор, оператордың полярлық бейнесі, оң байланысқан оператордың квадрат түбірі.

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О представлении одного класса операторов Шмидта

В настоящей работе рассматриваются унитарные симметризаторы. Хорошо известно, что, используя операторный алгоритм Ньютона, аналогичный обычному алгоритму Ньютона для извлечения квадратного корня, можно доказать, что для каждого эрмитова оператора $T \geq 0$ существует единственный эрмитов оператор $S \geq 0$ такой, что $T = S^2$. При этом S перестановочен с каждым ограниченным оператором R, с которым перестановочен T. Оператор S называется квадратным корнем оператора T и обозначается $T^{1/2}$. Существование квадратного корня позволяет определить абсолютную величину $|T| = (T^*T)^{1/2}$ ограниченного оператора T. Для каждого ограниченного линейного оператора $T : H \to H$ существует единственный частично изометрический оператор $U : H \to H$ такой, что $T = U|T|$, $KerU = KerT$. Такое равенство называется полярным разложением оператора T. Под оператором Шмидта понимается унитарный сомножитель полярного разложения вполне непрерывного обратимого оператора, с помощью которого Э. Шмид впервые получил разложение вполне непрерывного и несамосопряженного оператора и ввел так называемых s-чисел. В данной работе показано, что унитарный симметризатор оператора отличается, лишь знаком от сопряжения оператора Шмидта. Основной результат работы: если A – обратимый и компактный оператор, а S – унитарный оператор такие, что оператор SA самосопряжен, то оператор AS также самосопряжен и имеет место формула $S = \pm U^*$, где U – оператор Шмидта.

Ключевые слова: Унитарный оператор, симметризатор, нормальный оператор, разложение Шмидта, оператора Шмидта, вполне непрерывный оператор, полярное представление оператора, квадратный корень положительного самосопряженного оператора.

1 Introduction

As is known, many problems in mathematical physics lead to the need to study linear equations with symmetrizable operators. An example of such equations is the integral equation with a kernel symmetrizable in sense of Marty [5], which is led, for example, by boundary value problems for differential Sturm – Liouville equations as well as boundary value problems for linear differential equations of higher orders, both ordinary and with partial derivatives. A number of problems in mechanics (see, for example, [6]) lead to the study of boundary value problems for differential equations with coefficients non-linearly depending on a parameter (the case of a polynomial of the second degree and higher degrees).

Let X be some Hilbert space (separable,generally speaking, complex). An operator A mapping X into itself is called symmetrizable, if there exists a positive operator H mapping X into itself and such that $P = HA$ is a self-adjoint (Hermitian) operator.

Linear equations of the form

$$x - \lambda Ax = y,\ x \in X,\ y \in X\ (\lambda \text{ is a parameter})$$

with the symmetrizable compact operator A were studied by Saanen [7] and Ryde [8]. The results obtained in this direction were systemized in the monographs of Saanen [9]. In the present paper, we consider unitary symmetrizers.

1.1 Formulation of the problem

Let $B(H)$ be the algebra of all bounded linear operators A on the Hilbert space $H \neq \{0\}$ with the norm

$$||A|| = \sup\{||Ax|| : x \in H,\ ||x|| \leq 1\}. \tag{1}$$

Suppose that $A, S \in B(H),\ S^* = S$ exists $S^{-1} \in B(H)$ and the equality

$$A = A^*S,\ AS = SA^* \tag{2}$$

holds. The question is, what properties does the operator A have?

Denote, as usual, by $D(A),\ R(A),\ N(A)$ the domain, the range and the kernel of the operator A, respectively.

If $N(A) = \{0\}$ and the operator A is compact, then the so-called Schmidt expansion takes place

$$Ax = \sum_{1}^{\infty} s_i(x, \varphi_n) \cup \varphi_n, \tag{3}$$

where s_i are the s-numbers of the operator A, $\{\varphi_n\},\ (n = 1, 2, \dots)$ is an orthonormal basis of the space H composed of eigenvectors of the operator A^*A, U is a unitary operator from the polar representation of the operator A, that is,

$$A = UP, \tag{4}$$

where P is a square root of the positive self-adjoint operator A^*A. In this case, what is the relationship between the operators S and U.

Definition 1 *If a compact operator A with the kernel $N(A) = \{0\}$ admits the polar representation* (4)*, then the unitary operator U is called the Schmidt operator.*

Note that the unitary operators play an important role in the theory of operators, suffice it to mention scattering operators, wave operators, Fourier operators.

In solving this problem, we will use some general and less general theories of functional analysis. For accuracy and convenience, we present their formulation.

2 Research methods.

Definition 2 *An operator $A \in B(H)$ is called*

(*a*) *normal, if $AA^* = A^*A$;*

(*b*) *self-adjoint (Hermitian), if $A^* = A$;*

(*c*) *unitary, if $A^*A = I = AA^*$, where I is the unit operator in H;*

(*d*) *a projector, if $A^2 = A$.*

Lemma 1 ***[[10], p.354]***

(*a*) *If an operator $A \in B(H)$ is invertible, then it has a unique polar expansion $A = UP$;*

(*b*) *If an operator $A \in B(H)$ is normal, then it has a polar expansion $A = UP$, where the operators U and P commute with each other and with the operator A.*

Lemma 2 *Let $M, N, T \in B(H)$, where the operators M and N are normal. If*

$$MT = TN, \tag{5}$$

then

$$M^*T = TN^*. \tag{6}$$

Proof. See [11].

This lemma and the spectral theorem of normal operators imply the following Lemma 3, see [12].

Lemma 3 *Let $M, N, T \in B(H)$, where the operators M and N are normal, and the operator T is invertible. Suppose that*

$$M = TNT^{-1}. \tag{7}$$

If $T = UP$ is a polar expansion of the operator T, then

$$M = UNU^{-1}. \tag{8}$$

Two operators connected by relation (7) are called similar. If U is a unitary operator and relation (8) holds, then the operators M and N are called unitarily equivalent. Thus, it is established in Lemma 3 that the similar normal operators are equivalent.

Lemma 4 *Let S, T and ST are densely defined operators in H. Then*

$$T^*S^* \subset (ST)^*. \tag{9}$$

Moreover, if $S \in B(H)$, then

$$T^*S^* = (ST)^*. \tag{10}$$

Proof. The proofs of Lemmas 3 and 4 can be found in [10], on p. 355 and p. 370, respectively.

Remark 1 *If the equalities*

$$SA = A^*S, \; S = SA^*$$

hold, then

$$S^2A = AS^2, \; S^2A^* = A^*S^2.$$

Indeed,

$$S^2A = S(SA) = (SA^*)S = (AS)S = AS^2; \tag{11}$$

$$S^2A^* = SSA^* = S(AS) = (SA)S = (A^*S)S = A^*S^2. \tag{12}$$

Lemma 5 ***[[13], p.253].*** *If A is a compact and self-adjoint operator in H, then for any $x \in H$ the element Ax expands into a convergent Fourier series with respect to the orthonormal system of eigenvectors of the operator A.*

Corollary 1 *If the kernel of a compact self-adjoint operator A consists only of zero, then the orthonormal eigenvectors of this operator form a basis in the space H.*

Proof. By Lemma 5 for any $x \in H$ there is the expansion

$$Ax = \sum_1^\infty (Ax, \varphi_n)\varphi_n = \sum_1^\infty (x, A\varphi_n)\varphi_n = \sum_1^\infty \lambda_n (x, A\varphi_n)\varphi_n,$$

where $\{\varphi_n\}$ are orthonormal eigenvectors of this operator A. If $(x, \varphi_n) = 0$ $(n = 1, 2, ..)$, then $Ax = 0$ and by virtue of invertibility of the operator $x = 0$. Consequently, the system $\{\varphi_n\}$ is complete in H. Since it is orthonormal, then it forms an orthonormal basis of the space.

Lemma 6 ***[13.c.130].*** *Let X be normal, and Y a Banach space and A a linear operator with $D(A) \subseteq X$, $R(A) \subseteq Y$, moreover $\overline{D(A)} = X$ and the operator is bounded on $D(A)$. Then there exists a linear bounded operator $\widehat{A}$ such that*

(a) $\widehat{A}x = Ax$ for any $x \in D(A)$;

(b) $||\widehat{A}|| = ||A||$.

3 Research results.

Let $A,\ S,\ S^{-1} \in B(A),\ S = S^*,\ N(A) = \{0\}$ and the formulas hold:

$$A = A^*S,\ AS = SA^*. \tag{13}$$

Then the equality

$$SAA^* = A^*SA^* = A^*AS \tag{14}$$

holds. By virtue of Lemma 4 the operators AA^* and A^*A are self-adjoint, so they are normal operators. Consequently, by Lemma 3 the following formula holds:

$$UAA^* = A^*AU, \tag{15}$$

where $S = UP$ is the polar expansion of the operator S.

Further, assuming

$$A_R = \frac{A + A^*}{2},\ A_J = \frac{A - A^*}{2i},$$

we have

$$SA_R = S\frac{A + A^*}{2} = \frac{A^*S + AS}{2} = A_R S;$$

$$SA_J = \frac{S(A - A^*)}{2i} = \frac{(A^*S - AS)}{2i} = \frac{(A - A^*)S}{2i} = -A_J S.$$

Again, by Lemma 3, we have

$$UA_R = A_R U,\ UA_J = -A_J U, \tag{16}$$

therefore

$$UA = U(A_R + iA_J) = A_R U - iA_J U = (A_R - iA_J)U = A^* U; \tag{17}$$

$$AU = (A_R + iA_J)U = UA_R - iUA_J = U(A_R - iA_J) = UA^*. \tag{18}$$

By formula (15), we get

$$(AU)^* AU = U^* A^* AU = U^* U AA^* = AA^* = AUU^* A^* = AU(AU)^*.$$

By virtue of (17), (18), we have

$$(UA)^* UA = A^* U^* UA = A^* A = A^* UU^* A = UAU^* A^* = UAA^* U^* = UA(UA)^*.$$

We have proved the following theorem.

Theorem 1 *If*

$$A,\ S,\ S^{-1} \in B(H); \tag{19}$$

$$SA = A^* S,\ AS = SA^*, \tag{20}$$

then the operators AU and UA are the normal operators, where $S = UP$ is the polar expansion of the similarity operator.

From the self-adjointness of the operator S and item (b) of Lemma 1 it follows that

$$S = UP = PU,$$

then $S^* = PU^* = PU = S$. Consequently, $P(U^* - U) = 0$. Therefore, by virtue of the condition $N(A) = \{0\}$ it follows that $U^* = U$, that is, the operator U is self-adjoint. Then

$$(AU)^* = U^* A^* = UA^* = AU,$$

$$(UA)^* = A^* U^* = A^* U = UA,$$

that is, the operators AU and UA are self-adjoint.

Thus, the following Theorem 2 holds.

Theorem 2 *If*

$$S,\ S^{-1} \in B(H),\ S^* = S; \tag{21}$$

$$A \in B(H),\ N(A) = \{0\},\ SA = A^* S,\ AS = SA^*, \tag{22}$$

then the operators AU and UA are self-adjoint, where $S = UP$ is the polar expansion of the similarity operator S.

Suppose that $N(A)=\{0\}$, A is compact, the operator AS is self-adjoint, and S is unitary, then the operator SA is self-adjoint and compact. In fact, from $(AS)^*=AS$ and $S^*=S^{-1}$ we have $S^*A^*=AS,\ A^*=SAS,\ A^*S^*=SA$, therefore, $(SA)^*=A^*S^*=S$.

From $N(A)=\{0\}$ it follows that $N(SA)=\{0\}$. Indeed, if $SAX=0$, then, since the operator $S$is invertible, we have $Ax=0,\ x=0$.

By the Hilbert-Schmidt theorem

$$SAx=\sum_1^\infty \lambda_n(x,\varphi_n)\varphi_n, \tag{23}$$

where φ_n $(n=1,2,..)$ are the orthonormal eigenvectors of the operator SA.

If all $(x,\varphi_n)=0$, then according to the formula $SAx=0$. Therefore, $Ax=0,\ x=0$. Consequently, the eigenvectors of the operator SA form a complete system in H. Since this system is orthonormal, it forms the basis in H. Acting by the operator S^* on both sides of (23), we obtain

$$Ax=\sum_1^\infty \lambda_n(x,\varphi_n)S^*\varphi_n, \tag{24}$$

where the system $\{S^*\varphi_n\},\ (n=1,2,\dots)$ also forms the orthonormal basis in H.

Further, we have the formulas

$$(SA)^2=SASA=SS^*A^*A=A^*A, \tag{25}$$

$$A=UP, \tag{26}$$

where U is a unitary operator which we have called the Schmidt operator and P is the square root of the positive operator A^*A, see [[12], p.22].

From the formula $SA\varphi_n=\lambda_n\varphi_n$ it follows that $(SA)^2\varphi_n,=\lambda_n^2\varphi_n=A^*A\varphi_n=P^2\varphi_n$. Consequently, $P\varphi_n=s_n\varphi_n,\ s_n=|\lambda_n|,\ n=1,2,\dots,$ these numbers are called s-numbers of the operator A, see [[14], p.46]. Since the system $\varphi_n,\ (n=1,2,\dots)$ forms the orthonormal basis of the space, then the following expansion holds

$$Px=\sum_1^\infty (Px,\varphi_n)\varphi_n=\sum_1^\infty (x,P\varphi_n)\varphi_n=\sum_1^\infty s_n(x,\varphi_n)\varphi_n. \tag{27}$$

Then by virtue of formula (26), we obtain the Schmidt expansion of the compact operator

$$Ax=\sum_1^\infty s_n(x,\varphi_n)\cup\varphi_n. \tag{28}$$

For $x=\varphi_n,\ (n=1,2,\dots)$ from formulas (24) and (28) we obtain

$$A\varphi_n=\lambda_nS^*\varphi_n=s_nU\varphi_n=|\lambda_n|U\varphi_n,$$

$$S^*\varphi_n=\frac{|\lambda_n|U\varphi_n}{\lambda_n}=sign\lambda_nU\varphi_n,$$

$$U^*S^*\varphi_n = sign\lambda_n\varphi_n = SU\varphi_n,\ (n = 1, 2, \dots). \tag{29}$$

Due to the basicity of the system $\{\varphi_n\}$ and extension theorem from (29), we deduce that $U^*S^* = SU$, in other words, the operator SU is self-adjoint. The unitary operators form a group, therefore the operator SU is also unitary. Indeed:

$$(SU)^*SU = U^*S^*SU = U^*U = I,$$

$$(SU)(SU)^* = SUU^*S^* = SS^* = I.$$

Thus,

$$(SU)^* = (SU)^{-1} = SU, \rightarrow (SU)^2 = I,\ SU = \pm I,\ S = \pm U^*.$$

Let us formulate the obtained result.

Theorem 3 *Let A be an invertible and compact operator, and S a unitary operator.*

If the operator SA is self-adjoint, then the operator AS is also self-adjoint and the formula holds

$$S = \pm U^*, \tag{30}$$

where U is the Schmidt operator.

Remark 2 *This theorem says that the stock of S-operators is not large.*

Remark 3 *To construct the Schmidt operator, it is necessary to study the operator A^*A. That is a more complicated problem than the study of the operator itself. Therefore it is desirable to have an easier way of constructing the Schmidt operator. For some class of operators such a way exists, we will make sure of this in the next section, see [15]- [17].*

4 Discussions.

Lemma 1 was obtained in [11], Lemma 2 was obtained in [12], the initial proofs of these results were simplified in [18]. Our research is a continuation of these works. For clarity, we will give an example illustrating the results obtained.

Example 1 *In $L^2(0,1)$ consider an integration operator*

$$u(x) = \int_0^x u(t)dt,\ x \in [0, 1]. \tag{31}$$

Obviously that

$$A^*\nu(x) = \int_x^1 \nu(t)dt,\ x \in [0, 1]. \tag{32}$$

Moreover, the equality

$$SA = A^*S,\ \text{where } Su(x) = u(1 - x) \tag{33}$$

holds and S is a unitary self-adjoint operator.

Indeed,

$$SAu(x) = \int_0^{1-x} u(t)dt = \left| \xi = 1 - t \right| = -\int_1^x u(1-\xi)d\xi$$
$$= \int_x^1 u(1-\xi)d\xi = A^*Su(x)$$

Consequently,

$$(SA)^* = A^*S^* = A^*S = SA,$$

$$(AS)^* = S^*A^* = SA^* = AS,$$

that is, the operators SA and SA are self-adjoint. By virtue of the proved theorem, we have the formula

$$S = \pm U^*, \tag{34}$$

where U is the Schmidt operator, that is,

$$A = UP, \tag{35}$$

where $P = \sqrt{A^*A}$.

Let us find eigenvalues and eigenvectors of the operator SA. Let

$$SA\varphi_n(x) = \lambda_n^{-1}\varphi_n(x), \tag{36}$$

then

$$S\int_0^x \varphi_n(t)dt = \lambda_n^{-1}\varphi_n(x),$$
$$\int_0^x \varphi_n(t)dt = \lambda_n^{-1}S\varphi_n(x) = \lambda_n^{-1}\varphi_n(1-x),$$
$$\varphi_n(x) = -\lambda_n^{-1}\varphi_n'(1-x),\ \varphi_n'(1-x) = -\lambda_n\varphi_n(x),$$

$$\varphi_n'(x) = -\lambda_n\varphi_n(1-x),\ \varphi_n(1) = 0. \tag{37}$$

Differentiating both sides of this equation, we have

$$\varphi_n''(x) = \lambda_n\varphi_n'(1-x) = -\lambda_n^2\varphi_n(x),$$
$$-\varphi_n''(x) = \lambda_n^2\varphi_n(x).$$

It is easy to see that $\varphi_n(1) = 0$, $\varphi_n'(0) = 0$. Therefore, we seek solutions of the boundary value problem

$$\varphi_n''(x) = \lambda_n^2\varphi_n(x),\ \varphi_n(1) = 0,\ \varphi_n'(0) = 0 \tag{38}$$

in the form

$$\varphi(x,\lambda) = A\cos\lambda x.$$

Then $\varphi'(x,\lambda) = -A\cos\lambda x = 0$ *for* $x = 0$.

If $\lambda = 0$, *then* $\varphi'(x) = 0$, *therefore,* $\varphi = const$, $\varphi(1) = 0$, *consequently,* $\varphi(x) \equiv 0$. *Therefore the value* $\lambda = 0$ *is not an eigenvalue. From the equation* $\varphi(1) = 0$ *we have* $\cos\lambda = 0$, *consequently,* $\lambda_n = n\pi + \frac{\pi}{2}$, $n = 0, \pm 1, \pm 2, ...$ *Therefore the eigenfunctions are* $\varphi_n(x) = A_n \cos\big(\pi n + \frac{\pi}{2}\big)x$. *We find the unknown coefficients* A_n *from the normalization condition:* $||\varphi_n|| = 1$.

$$\begin{aligned}||\varphi_n(x)||^2 &= A_n^2 \int_0^1 \cos^2\left(\pi n + \frac{\pi}{2}\right) x dx = \frac{A_n^2}{2}\int_0^1 \big[1 + \cos\left(2\pi n + \pi\right)x\big]dx \\ &= \frac{A_n^2}{2} = 1,\ A_n = \sqrt{2}.\end{aligned}$$

Thus, the eigenfunctions of the Sturm-Liouville problem (38) *have the form:*

$$\varphi_n(x) = \sqrt{2}\cos\left(\pi n + \frac{\pi}{2}\right)x,\ n = 0, \pm 1, \pm 2, ... \tag{39}$$

Substituting $\varphi_n(x)$ *into the initial equation* (37), *we find the eigenvalues of the spectral problem:*

$$\varphi_n'(x) = -\lambda_n \varphi_n(1-x),\ \varphi_n(1) = 0;$$

$$\varphi_n'(x) = -\sqrt{2}\sin\left(\pi n + \frac{\pi}{2}\right)x \cdot \left(\pi n + \frac{\pi}{2}\right),$$

$$\begin{aligned}\varphi_n(1-x) &= \sqrt{2}\cos\left(\pi n + \frac{\pi}{2}\right)(1-x) = \sqrt{2}\cos\left[\pi n + \frac{\pi}{2} - \left(\pi n + \frac{\pi}{2}\right)x\right] \\ &= \sqrt{2}\left[\cos\left(\pi n + \frac{\pi}{2}\right)\cos\left(\pi n + \frac{\pi}{2}\right)x + \sin\left(\pi n + \frac{\pi}{2}\right)\sin\left(\pi n + \frac{\pi}{2}\right)x\right] \\ &= \cos\pi n \sin\left(\pi n + \frac{\pi}{2}\right)x = \sqrt{2}(-1)^n \sin\left(\pi n + \frac{\pi}{2}\right)x,\end{aligned}$$

$$\begin{aligned}-\lambda_n\varphi_n(1-x) &= \lambda_n(-1)^n\left[-\sqrt{2}\sin\left(\pi n + \frac{\pi}{2}\right)x\right] \\ &= -\sqrt{2}\left(\pi n + \frac{\pi}{2}\right)\sin\left(\pi n + \frac{\pi}{2}\right)x,\end{aligned}$$

$$\lambda_n(-1)^n = \pi n + \frac{\pi}{2},\ \lambda_n = (-1)^n\left[\pi n + \frac{\pi}{2}\right],\ n = 0, \pm 1, \pm 2, ...$$

$$\varphi_n'(x) = -(-1)^n\left(\pi n + \frac{\pi}{2}\right)\varphi_n(1-x).$$

Consequently,

$$SA\varphi_n = \frac{(-1)^n}{\pi n + \frac{\pi}{2}}\varphi_n(x),\ n = 0, 1, 2, ..., \tag{40}$$

where

$$\varphi_n(x) = \sqrt{2}\cos\left(\pi n + \frac{\pi}{2}\right)x. \tag{41}$$

Formula (24) *takes the form:*

$$\int_0^x u(t)dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{\pi n + \frac{\pi}{2}}(u, \varphi_n(x))\varphi_n(1-x). \tag{42}$$

The Schmidt expansion (27) *has the form:*

$$\int_0^x u(t)dt = \sum_{n=0}^{\infty} \frac{1}{\pi n + \frac{\pi}{2}}(u, \varphi_n(x))U\varphi_n(x).$$

where U is the Schmidt operator. For $u = \varphi_n$, we have

$$\frac{(-1)^n}{\pi n + \frac{\pi}{2}}\varphi_n(1-x) = \frac{1}{\pi n + \frac{\pi}{2}}U\varphi_n(x),$$

consequently, $U\varphi_n = (-1)^n\varphi_n(1-x) = (-1)^n S\varphi_n$, that is, $U = \pm S$, which is consistent with the overall result.

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References

[1] Kharazov D. F. "On a class of linear equations with symmetrizable operators" , *Dokl. Akad. Nauk SSSR, n. Ser.* 91 (1953) 1023-1026 [in Russian]

[2] Kharazov D. F. "On the theory of symmetrizable operators with polynomial dependence on a parameter" , *Dokl. Akad. Nauk SSSR, n. Ser.* 91 (1953) 1285-1287 [in Russian]

[3] Kharazov D. F. "On boundary value problems in the theory of ordinary differential equations" , *Dokl. Akad. Nauk SSSR* 100(1955) 217-220 [in Russian]

[4] Kharazov D. F. "On the investigation of boundary problems for elliptic differential equations" , *Dokl. Akad. Nauk SSSR* 100 (1955) 421-424 [in Russian]

[5] Marty I. "Sur ime equation integrate" , *C. R. Acad. Sc, Paris* 150 (1910) 1499–1502

[6] Collatz L. *Eigenwertprobleme und ihre numerische Behandlung* (Leipzig, 1945)

[7] Zaanen A. "Uber vollstetige symmetrische und syrnmetrisierbare Operatoren" , *Nieuw Arch. v. Wisk.* 2:22 (1943) 255–260

[8] Reid W. "Symmetrizable completely continuous linear transformations in Hilbert space" , *Duke Math. Journ.* 18:1 (1951) 41–56

[9] Zaanen A. *Linear Analysis* (New York–Amsterdam, 1953)

[10] Rudin U. emph Functional analysis (M., Mir, 1975) 443 (Russian)

[11] Fuglide B. *Proc.Nat.Acad.Sci.USA* 36 (1950) 5–40

[12] Putnam C. R. "On normal operators in Hilbert space" , *Amer. J. Math.* 73 (1951) 357–362

[13] Trenogin V. A. emph Functional analysis (M., Nauka, 1980) 496 [in Russian]

[14] Gokhberg I. Ts., Kerin M. G. emph Introduction to the theory of linear of non-self-adjoint operators in a Hilbert space (M, Science, 1965) 448 [in Russian]

[15] Kalmenov T. Sh., Akhmetova S. T., Shaldanbaev A. Sh. "On spectral theory equations with deviating arguments" , emph Mathematical journal 13:4 (2014) 41-48 [in Russian]

[16] Sadybekov M. A., Sarsenbi A. M. "Basic solutions spectral questions of all boundary value problems for one differential equation of the first order with deviating argument" , emph Uzbek Mathematical Journal 3 (2007) 49-53 [in Russian]

[17] Akylbayev M. I., Beysebayeva A., Shaldanbayev A. Sh. "On the periodic solution of the Goursat problem for a wave equation of a special form with variable coefficients" , *News of the National Academy of Sciences of the Republic of Kazakhstan Physico-mathematicaL Series* 1 (2018) 34–50

[18] Rosenblum M. "On a theorem of Fuglede and Putnam" , *J. London Math. Soc.* 33 (1958) 376–377

Список литературы

[1] Харазов Д. Ф. Об одном классе линейных уравнений с симметризуемыми операторами // ДАН СССР. - 1953. - Т.91, N5. - С.1023–1026.

[2] Харазов Д. Ф. К теории симметризуемых операторов, полиномиально зависящих от параметра // ДАН СССР. - 1953. - Т.91, N6. - С.1285–1287.

[3] Харазов Д. Ф. О граничных задачах в теории обыкновенных дифференциальных уравнений // ДАН СССР. - 1955. - Т.100, N2. - С.217–220.

[4] Харазов Д. Ф. К исследованию граничных задач для эллиптических дифференциальных уравнений // ДАН СССР. - 1955. - Т.100, N3. - С.421–424.

[5] Marty I. Sur ime equation integrate // C. R. Acad. Sc, Paris. - 1910. - N.150. - P.1499–1502.

[6] Collatz L. Eigenwertprobleme und ihre numerische Behandlung. - Leipzig.–1945.

[7] Zaanen A. Uber vollstetige symmetrische und syrnmetrisierbare Operatoren // Nieuw Arch. v. Wisk. - 1943. - V.2, N22. - P.255–260.

[8] Reid W. Symmetrizable completely continuous linear transformations in Hilbert space // Duke Math. Journ. - 1951. - V.18, N1. - P.41–56.

[9] Zaanen A. Linear Analysis. - New York–Amsterdam.–1953.

[10] Рудин У. Функциональный анализ. –М.: Мир. 1975. – 443 с.

[11] Fuglide B. // Proc.Nat.Acad.Sci.USA. - 1950. - N36. - P.35–40.

[12] Putnam C. R. On normal operators in Hilbert space // Amer. J. Math. -1953. - 73. - 357–362.

[13] Треногин В. А. Функциональный анализ. –М.: Наука. 1980. – 496.

[14] Гохберг И. Ц., Крейн М. Г. Введение в теорию линейных несамосопряженных операторов в гильбертовом пространстве. –М.: Наука. 1965. - 448.

[15] Кальменов Т. Ш., Ахметова С. Т., Шалданбаев А. Ш. К спектральной теории уравнении с отклоняющимся аргументами // Математический журнал. - 2014. - Т.13, N3 - С.41–48.

[16] Садыбеков М. А., Сарсенби А. М. Решения основных спектральных вопросов всех краевых задач для одного дифференциального уравнения первого порядка с отклоняющимся аргументом // Узбекский математический журнал. - 2007. - N3. - С. 49–53.

[17] Akylbayev M. I., Beysebayeva A., Shaldanbayev A. Sh. On the periodic solution of the Goursat problem for a wave equation of a special form with variable coefficients // News of the National Academy of Sciences of the Republic of Kazakhstan Physico-mathematicaL Series. - 2018. - T.1, N 317. - C. 34–50.

[18] Rosenblum M. On a theorem of Fuglede and Putnam // J. London Math. Soc. - 1958. - 33. - P. 376-377