1-бөлім **Раздел 1** **Section 1**

Математика **Математика** **Mathematics**

IRSTI 27.29.19; 27.29.23 DOI: https://doi.org/10.26577/JMMCS.2021.v110.i2.01

S.I. Mitrokhin
Russian Academy of Natural Sciences, Lomonosov Moscow State University,
Research Computing Center, Russia, Moscow
E-mail: mitrokhin-sergey@yandex.ru

ASYMPTOTICS OF THE EIGENVALUES OF A PERIODIC BOUNDARY VALUE PROBLEM FOR A DIFFERENTIAL OPERATOR OF ODD ORDER WITH SUMMABLE OPERATOR

The paper is devoted to the study of spectral properties of differential operators of arbitrary odd order with a summable potential and periodic boundary conditions. For large values of the spectral parameter the asymptotics of the solutions of the differential equation that defines the differential operator is obtained. Differential equation that defines the differential operator is reduced to the Volterra integral equation. The integral equation is solved by Picards method of successive approximations. The method of studing of operators with a summable potential is an extension of the method of studing operators with piecewise smooth coefficients. The study of periodic boundary conditions leads to the study of the roots of the entire function represented in the form of an arbitrary odd order determinant. To obtain the roots of this function, the indicator diagram has been examined. The roots of this equation are in the sectors of an infinitesimal angle, determined by the indicator diagram. In the paper the asymptotics of eigenvalues of the differential operator under consideration is found. The obtained formulas make it impossible to study the spectral properties of the eigenfunctions and to derive the formula for the first regularized trace of the differential operator under study.
Key words: Differential operator of odd order, spectral parameter, summable potential, periodic boundary conditions, indicator diagram, asymptotics of solutions, asymptotics of eigenvalues.

С.И. Митрохин
Ресей жаратылыстану ғылымдары академиясы, М.В. Ломоносов атындағы Мәскеу мемлекеттік университетінің ғылыми-зерттеу орталығы, Ресей, Мәскеу қ.
E-mail: mitrokhin-sergey@yandex.ru

Қосылуыш операторы бар тақ ретті дифференциалдық оператор үшін периодтық шекара есебінің меншік мәндерінің асимптотикасы

Бұл жұмыста дифференциалдық операторлардың спектрлік қасиеттерін зерттеуге арналған. Спектрлік параметрдің үлкен мәндері үшін дифференциалдық операторды анықтайтын дифференциалдық теңдеу шешімдерінің асимптотикасы алынады. Дифференциалдық операторды анықтайтын дифференциалдық теңдеу Вольтерраның интегралдық теңдеуіне келтірілген. Интегралдық теңдеу Пикард әдісімен шешіледі. Жиынтық әлеуеті бар операторларды оқыту әдісі біртектес коэффициенттері бар операторларды оқыту әдістемесін кеңейту болып табылады. Периодтық шекаралық шарттарды зерттеу тақ тәрізді ерікті детерминант ретінде ұсынылған бүтін функцияның түбірлерін зерттеуге алып келеді. Бұл функцияның тамырларын білу үшін индикаторлық диаграмма зерттелді. Бұл теңдеудің түбірлері индикаторлық диаграммамен анықталған шексіз бұрыштың секторларында жатыр. Бұл мақалада дифференциалдық оператордың меншікті мәндерінің асимптотикасы келтірілген. Нәтижеде алынған формулалар меншікті функциялардың спектрлік қасиеттерін зерттеуге және зерттелетін дифференциалдық оператордың алғашқы регулирленген ізінің формуласын шығаруға мүмкіндік береді.
Түйін сөздер: Тақ ретті дифференциалдық оператор, спектрлік параметр, жиынтық потенциал, периодтық шекаралық шарттар, индикаторлық диаграмма, ерітінділердің асимптотикасы, меншікті мәндердің асимптотикасы.

С.И. Митрохин
Российская академия естественных наук, Научно-исследовательский вычислительный центр
Московского государственного университета имени М.В. Ломоносова, Россия, г. Москва
E-mail: mitrokhin-sergey@yandex.ru

Асимптотика собственных значений периодической краевой задачи для дифференциального оператора нечетного порядка с суммируемым оператором

Работа посвящена изучению спектральных свойств дифференциальных операторов произвольного нечетного порядка с суммируемым потенциалом и периодическими граничными условиями. При больших значениях спектрального параметра получена асимптотика решений дифференциального уравнения, определяющего дифференциальный оператор. Дифференциальное уравнение, определяющее дифференциальный оператор, сводится к интегральному уравнению Вольтерра. Интегральное уравнение решается методом последовательных приближений Пикара. Метод обучения операторов с суммируемым потенциалом является расширением метода обучения операторов с кусочно гладкими коэффициентами. Изучение периодических граничных условий приводит к изучению корней целой функции, представленной в виде произвольного определителя нечетного порядка. Для получения корней этой функции была исследована индикаторная диаграмма. Корни этого уравнения лежат в секторах бесконечно малого угла, определяемого диаграммой индикатора. В статье найдена асимптотика собственных значений рассматриваемого дифференциального оператора. Полученные формулы не позволяют исследовать спектральные свойства собственных функций и вывести формулу для первого регуляризованного следа исследуемого дифференциального оператора.

Ключевые слова: Дифференциальный оператор нечетного порядка, спектральный параметр, суммируемый потенциал, периодические граничные условия, индикаторная диаграмма, асимптотика решений, асимптотика собственные значения.

1 Statement of the problem

Let us investigate the spectrum of a differential operator arbitrary odd order defined on an interval $[0;\pi]$ by a differential equation of the form

$$y^{(2N+1)}(x) + q(x)y(x) = \lambda a^{2N+12} y(x), \quad 0 \le x \le \pi, \quad a > 0, \quad N = 1, 2, 3, \dots, \tag{1}$$

with periodic boundary conditions

$$y(0) = y(\pi), \quad y^{(m)}(0) = y^{(m)}(\pi), \quad m = 1, 2, \dots, 2N, \tag{2}$$

where the number λ is called the spectral parameter, the function $q(x)$ is called the potential and we assume that the potential $q(x)$ is a summable function on the segment $[0;\pi]$

$$q(x) \in L_1[0;\pi] \Leftrightarrow \left(\int_0^x q(t)dt \right)'_x = q(x) \tag{3}$$

for almost all values x from the segment $[0;\pi]$.

The spectral properties of differential operators were first studied in the case when the coefficients of the differential equations defining these operators were sufficiently smooth functions. The asymptotic formulas for the roots of quasipolynomials, which are obtained in the study of higher-order operators with regular boundary conditions with smooth coefficients, were obtained in paper [1].

In paper [2], the traces of higher-order ordinary differential operators with sufficiently smooth coefficients were calculated.

In work [3], the author studied the spectral properties of differential operators with piecewise smooth coefficients. In paper [4], we studied a differential operator in which not only the potential is a discontinuous function, but the weight function was also piecewise smooth.

In paper [5], a second-order operator with a summable potential was studied, the asymptotics of the eigenvalues and eigenfunctions of the Sturm—Liouville boundary value problem on a segment were calculated. A new method for studying differential operators with summable coefficients, whose order is higher than the second, was developed by the author in papers [6–8]. In all these works, the boundary conditions were separated. The periodic boundary conditions that we study in this paper are a classic example of nonseparated boundary conditions.

The spectral properties of differential operators with periodic boundary conditions with smooth potentials were studied in papers [9–11]. Interest in the study of such operators is caused by physical applications: in the case of fourth-order operators, they describe a model of a beam or plate with a hinged joint or with fixed ends. Operators with periodic boundary conditions were also studied in papers [12, 13].

In papers [14, 15] the author studied model examples of differential operators with summable potential with periodic boundary conditions. The differential operators of odd order with periodic boundary conditions have not actually been studied. A third-order operator on the real axis with periodic boundary conditions was studied in paper [16].

2 Asymptotics of solutions of differential equation (1) for large values of the spectral parameter λ

Let us introduce the following notation: $\lambda = s^{2N+1}$, $s = \sqrt[2N+1]{\lambda}$, while fixing that branch of the arithmetic root for which $\sqrt[2N+1]{1} = +1$. Let us denote by ω_k $(k = 1, 2, \ldots, 2N+1)$ the various roots of the $(2N+1)$-th degree from the unity:

$$\begin{aligned} &\omega_k^{2N+1} = 1,\ \omega_k = e^{\frac{2\pi i(k-1)}{2N+1}}\ (k = 1, 2, \ldots, 2N+1); \\ &\omega_1 = 1;\ \omega_2 = e^{\frac{2\pi i}{2N+1}} = \cos\left(\frac{2\pi}{2N+1}\right) + i\sin\left(\frac{2\pi}{2N+1}\right) = z^2; \ldots; \\ &\omega_m = z^{m-1},\ m = 1, 2, \ldots, 2N+1. \end{aligned} \tag{4}$$

The numbers ω_k $(k = 1, 2, \ldots, 2N+1)$ from (4) decide the unit circle into $(2N+1)$ an equal part, and they satisfy the following relations:

$$\sum_{k=1}^{2N+1} \omega_k^m = 0,\ m = 1, 2, \ldots, 2N; \quad \sum_{k=1}^{2N+1} \omega_k^m = 2N+1,\ m = 0,\ m = 2N+1. \tag{5}$$

The following statement is established by the method of variation of arbitrary constants under the condition (3) of the summability of the potential $q(x)$.

Theorem 1 *The solution $y(x,s)$ of the differential equation* (1) *is the solution to the following integral equation:*

$$y(x,s)=\sum_{k=1}^{2N+1} C_k e^{a\omega_k sx}-\frac{1}{(2N+1)a^{2N}s^{2N}}\sum_{k=1}^{2N+1}\omega_k e^{a\omega_k sx}\int\limits_0^x q(t)e^{-a\omega_k st}y(t,s)dt, \qquad (6)$$

where C_k $(k=1,2,\ldots,2N+1)$ are arbitrary constants.

Proof. Let us prove that the function $y(x,s)$ from (6) is indeed a solution of the differential equation (1). Since, in view of condition (3) $[q(x)\in L_1[0;\pi]$, the function $e^{-a\omega_k sx}$ is infinitely differentiable with respect to the variable x, the function $y(x,s)$ must satisfy the equation (1), which means that it must be $(2N+1)$ times differentiable with respect to variable x, then the function $G(x,s)=q(x)e^{-a\omega_k sx}y(x,s)\in L_1[0;\pi]$ and then the relation

$$\left(\int\limits_0^x G(t,s)dt\right)'_x=\left(\int\limits_0^x a(t)e^{-a\omega_k st}y(t,s)dt\right)'_x=q(x)e^{-a\omega_k sx}y(x,s) \qquad (7)$$

holds for almost all x from segment $[0;\pi]$.

Differentiating the function $y(x,s)$ from (6) with respect to the variable x, using the properties (3) and (7), we get:

$$y'(x,s)=\sum_{k=1}^{2N+1} C_k(a\omega_k s)e^{a\omega_k sx}-\frac{1}{M_N}\sum_{k=1}^{2N+1}\omega_k(a\omega_k s)e^{a\omega_k sx}\int\limits_0^x G(t,s)dt-\frac{1}{M_N}\phi_1(x,s), \quad (8)$$

$$M_N=(2N+1)a^{2N}s^{2N},\quad \phi_1(x,s)=-\frac{1}{M_N}\sum_{k=1}^{2N+1}\omega_k e^{a\omega_k sx}q(x)e^{-a\omega_k sx}y(x,s)\overset{(2.2)}{=}0. \qquad (9)$$

From formulas (8), (9), using the properties (3) and (7), we have:

$$y''(x,s)=\sum_{k=1}^{2N+1} C_k(a\omega_k s)^2e^{a\omega_k sx}-\frac{1}{M_N}\sum_{k=1}^{2N+1}\omega_k(a\omega_k s)^2e^{a\omega_k sx}\int\limits_0^x G(t,s)dt-\frac{1}{M_N}\phi_2(x,s), \quad (10)$$

$$\phi_2(x,s)=-\frac{1}{M_N}\sum_{k=1}^{2N+1}\omega_k(a\omega_k s)e^{a\omega_k sx}q(x)e^{-a\omega_k sx}y(x,s)=-\frac{asq(x)y(x,s)}{M_N}\sum_{k=1}^{2N+1}\omega_k^2\overset{(2.2)}{=}0. \quad (11)$$

Similarly, the following formulas are derived:

$$\begin{aligned}y^{(n)}(x,s)&=\sum_{k=1}^{2N+1} C_k(a\omega_k s)^n e^{a\omega_k sx}-\\&-\frac{1}{M_N}\sum_{k=1}^{2N+1}\omega_k(a\omega_k s)^n e^{a\omega_k sx}\int\limits_0^x G(t,s)dt-\frac{1}{M_N}\phi_n(x,s), n=3,4,\ldots,2N,\end{aligned} \qquad (12)$$

$$\phi_n(x,s) = -\frac{1}{M_N}\sum_{k=1}^{2N+1}\omega_k(a\omega_k s)^{n-1}e^{a\omega_k sx}q(x)e^{-a\omega_k sx}y(x,s) =$$
$$= -\frac{(as)^{n-1}q(x)y(x,s)}{M_N}\sum_{k=1}^{2N+1}(\omega_k)^n \overset{(2.2)}{=} -\frac{(as)^{n-1}q(x)y(x,s)\cdot 0}{M_N} = 0, \quad n = 3,4,\ldots,2N. \tag{13}$$

Substituting the formulas (12), (13) at $n = 2N$ in the differential equation (1), we obtain;

$$y^{2N+1)}(x,s) + q(x)y(x,s) - \lambda^{2N+1}y(x,s) \overset{(2.3)}{=} \sum_{k=1}^{2N+1} C_k(a\omega_k s)^{2N+1}e^{a\omega_k sx} -$$
$$-\frac{1}{M_N}\sum_{k=1}^{2N+1}\omega_k(a\omega_k s)^{2N+1}e^{a\omega_k sx}\int_0^x G(t,s)dt -$$
$$-\frac{1}{M_N}\sum_{k=1}^{2N+1}\omega_k(a\omega_k s)^{2N}e^{a\omega_k sx}q(x)e^{-a\omega_k sx}y(x,s) + q(x)y(x,s) - \tag{14}$$
$$-\lambda a^{2N+1}\sum_{k=1}^{2N+1}C_k e^{a\omega_k sx} + \lambda a^{2N+1}\frac{1}{M_N}\sum_{k=1}^{2N+1}\omega_k e^{a\omega_k sx}\int_0^x G(t,s)dt =$$
$$= -q(x)y)x,s) + q(x)y(x,s) = 0$$

for almost all x from the interval $[0;\pi]$, which means that the function $y(x,s)$ from (6) is indeed a solution of the differential equation (1).

(In equation (14)), the first and fifth terms cancel out, the second and sixth terms cancel out due the fact that $\lambda = s^{2N+1}$, $\omega_k^{2N+1} = 1$, $M_N = (2N+1)a^{2N}s^{2N}$.

Futher, the asymptotics of solution of the differential equation (1) will be found by the method of successive approximations of Picard: from the formula (6) we obtain the function $y(t,s)$ and substitute it into equation (6):

$$y(x,s) = \sum_{k=1}^{2N+1}C_k e^{a\omega_k sx} - \frac{1}{(2N+1)a^{2N}s^{2N}}\sum_{k=1}^{2N+1}\omega_k e^{a\omega_k sx}\int_0^x q(t)e^{-a\omega_k st}\times$$
$$\times\left[\sum_{n=1}^{2N+1}C_n e^{a\omega_n st} - \frac{1}{(2N+1)a^{2N}s^{2N}}\sum_{k=1}^{2N+1}\omega_n e^{a\omega_n st}\int_0^t q(\xi)e^{-a\omega_n s\xi}y(\xi,s)d\xi\right]dt. \tag{15}$$

Changing the order of summation in formula (15), performing the necessary transformations and estimates similar to the monograph [1, chapter 2], we come to the conclusion that the following statement is true.

Theorem 2 *The general solution of the differential equation* (1) *is represented in the following form:*

$$
\begin{aligned}
&y(x,s)=\sum_{k=1}^{2N+1}C_k y_k(x,s);\\
&y^{(m)}(x,s)=\sum_{k=1}^{2N+1}C_k y_k^{(m)}(x,s),\ \ m=1,2,\dots,2N,
\end{aligned}
\tag{16}
$$

where C_k $(k=1,2,\dots,2N+1)$ *are arbitrary constants, and the following asymptotic expansions and estimates are valid for the fundamental system of solutions* $\{y_k(x,s)\}_{k=1}^{2N+1}$*:*

$$
\begin{aligned}
&y_k(x,s)=e^{a\omega_k sx}-\frac{1}{(2N+1)a^{2N}s^{2N}}\sum_{n=1}^{2N+1}\omega_n e^{a\omega_n sx}\int\limits_0^x q(t)e^{a(\omega_k-\omega_n)st}dt_{akn}+\\
&++\underline{O}\left(\frac{e^{|\Im s|ax}}{s^{4N}}\right),\quad k=1,2,\dots,2N+1,\quad y_k(0,s)=1;
\end{aligned}
\tag{17}
$$

$$
\begin{aligned}
&y_k^{(m)}(x,s)=(as)^m\Bigg\{\omega_k^m e^{a\omega_k sx}-\frac{1}{(2N+1)a^{2N}s^{2N}}\sum_{n=1}^{2N+1}\omega_n^{m+1}e^{a\omega_n sx}\times\\
&\times\int\limits_0^x q(t)e^{a(\omega_k-\omega_n)st}dt_{akn}+\underline{O}\left(\frac{e^{|\Im s|ax}}{s^{4N}}\right)\Bigg\},\\
&k=1,2,\dots,2N+1,\quad m=1,2,\dots,2N;\quad y_k^{(m)}(0,s)=(as)^m\omega_k^m.
\end{aligned}
\tag{18}
$$

3 The study of boundary conditions (3)

Using the formulas (16), from the boundary conditions (2) we get:

$$
\begin{cases}
y(\pi,s)\overset{(1.2)}{=}y(0,s)\Leftrightarrow\sum\limits_{k=1}^{2N+1}C_k y_k(\pi,s)=\sum\limits_{k=1}^{2N+1}C_k y_k(0,s)\Leftrightarrow\\
\Leftrightarrow\sum\limits_{k=1}^{2N+1}C_k y_k[(\pi,s)-y_k(0,s)]=0\overset{(2.7)}{\Leftrightarrow}\sum\limits_{k=1}^{2N+1}C_k[y_k(\pi,s)-1]=0;
\end{cases}
\tag{19}
$$

$$
\begin{cases}
\dfrac{y^{(m)}(\pi,s)}{(as)^m}\overset{(1.2)}{=}\dfrac{y^{(m)}(0,s)}{(as)^m}\Leftrightarrow\sum\limits_{k=1}^{2N+1}C_k\left[\dfrac{y_k^{(m)}(\pi,s)}{(as)^m}-\dfrac{y_k^{(m)}(0,s)}{(as)^m}\right]=0,\\
m=1,2,\dots,2N.
\end{cases}
\tag{20}
$$

Theorem 3 *The eigenvalue equation of the differential operator ((1))–((3)) has the following form:*

$$\begin{vmatrix} y_1(\pi,s)-y_1(0,s) & y_2(\pi,s)-y_2(0,s) & \dots & y_{2N+1}(\pi,s)-y_{2N+1}(0,s) \\ \frac{y_1'(\pi,s)}{as}-\dfrac{y_1'(0,s)}{as} & \frac{y_2'(\pi,s)}{as}-\dfrac{y_2'(0,s)}{as} & \dots & \frac{y_{2N+1}'(\pi,s)}{as}-\dfrac{y_{2N+1}'(0,s)}{as} \\ \dots & \dots & \dots & \dots \\ \frac{y_1^{(2N)}(\pi,s)}{(as)^{2N}}-\dfrac{y_1^{(2N)}(0,s)}{(as)^{2N}} & \frac{y_2^{(2N)}(\pi,s)}{(as)^{2N}}-\dfrac{y_2^{(2N)}(0,s)}{(as)^{2N}} & \dots & \frac{y_{2N+1}^{(2N)}(\pi,s)}{(as)^{2N}}-\dfrac{y_{2N+1}^{(2N)}(0,s)}{(as)^{2N}} \end{vmatrix}=0. \quad (21)$$

Applying the asymptotic formulas (17), (18), we rewrite the equation (21) in the following form:

$$f(s)=\begin{vmatrix} D_1-\frac{1}{M_N}\sum\limits_{n=1}^{2N+1}\omega_n e^{a\omega_n s\pi}\left(\int\limits_0^\pi\dots\right)_{a1n}+\underline{O}\left(\frac{1}{s^{4N}}\right) & \dots & B_{1,2N+1} \\ \omega_1 D_1-\frac{1}{M_N}\sum\limits_{n=1}^{2N+1}\omega_n^2 e^{a\omega_n s\pi}\left(\int\limits_0^\pi\dots\right)_{a1n}+\underline{O}\left(\frac{1}{s^{4N}}\right) & \dots & B_{2,2N+1} \\ \dots & \dots & \dots \\ \omega_1^{2N} D_1-\frac{1}{M_N}\sum\limits_{n=1}^{2N+1}\omega_n^{2N+1} e^{a\omega_n s\pi}\left(\int\limits_0^\pi\dots\right)_{a1n}+\underline{O}\left(\frac{1}{s^{4N}}\right) & \dots & B_{2N+1,2N+1} \end{vmatrix}=0, \quad (22)$$

$$D_m=e^{a\omega_n s\pi}-1; \quad M_n=(2N+1)a^{2N}s^{2N};$$

$$B_{m,2N+1}=\omega_{2N+1}^{m-1}D_{2N+1}-\frac{1}{M_N}\sum_{n=1}^{2N+1}\omega_n^m e^{a\omega_n s\pi}\left(\int\limits_0^\pi\dots\right)_{a,2N+1,n}+\underline{O}\left(\frac{1}{s^{4N}}\right);$$

$$m=1,2,\dots,2N+1.$$

Expanding the determinant $f(s)$ from (22) into columns into the sum of determinants, we obtain:

$$f(s)=f_0(s)-\frac{f_{2N}(s)}{(2N+1)a^{2N}s^{2N}}+\underline{O}\left(\frac{1}{s^{4N}}\right)=0, \quad (23)$$

$$f_0(s)=\Delta_{00}[e^{a\omega_1 s\pi}-1][e^{a\omega_2 s\pi}-1][e^{a\omega_3 s\pi}-1](\dots)[e^{a\omega_{2N+1} s\pi}-1], \quad (24)$$

$$f_0(s)=\begin{vmatrix} 1\cdot[e^{a\omega_1 s\pi}-1] & 1\cdot[e^{a\omega_2 s\pi}-1] & \dots & 1\cdot[e^{a\omega_{2N+1} s\pi}-1] \\ \omega_1[e^{a\omega_1 s\pi}-1] & \omega_2[e^{a\omega_2 s\pi}-1] & \dots & \omega_{2N+1}[e^{a\omega_{2N+1} s\pi}-1] \\ \dots & \dots & \dots & \dots \\ \omega_1^{2N}[e^{a\omega_1 s\pi}-1] & \omega_2^{2N}[e^{a\omega_2 s\pi}-1] & \dots & \omega_{2N+1}^{2N}[e^{a\omega_{2N+1} s\pi}-1] \end{vmatrix}; \quad (25)$$

Δ_{00} is the Wandermonde's determinant of the numbers $\omega_1, \omega_2, \dots, \omega_{2N+1}$:

$$\Delta_{00} = \begin{vmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ \omega_1 & \omega_2 & \omega_3 & \dots & \omega_{2N} & \omega_{2N+1} \\ \omega_1^2 & \omega_2^2 & \omega_3^2 & \dots & \omega_{2N}^2 & \omega_{2N+1}^2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \omega_1^{2N} & \omega_2^{2N} & \omega_3^{2N} & \dots & \omega_{2N}^{2N} & \omega_{2N+1}^{2N} \end{vmatrix} = \tag{26}$$

$$= \prod_{\substack{k>n \\ k,n=1,2,\dots,2N+1}} (\omega_k - \omega_n) \neq 0,$$

$$\begin{aligned} f_{2N}(s) = \Delta_{00} \Bigg\{ & \sum_{k=1}^{2N+1} \omega_k \left(\int_0^\pi \dots \right)_{a1k} e^{a\omega_k s\pi} \left(\prod_{\substack{n=1 \\ n\neq 1}}^{2N+1} (e^{a\omega_n s\pi} - 1) \right) + \\ + & \sum_{k=1}^{2N+1} \omega_k \left(\int_0^\pi \dots \right)_{a2k} e^{a\omega_k s\pi} \left(\prod_{\substack{n=1 \\ n\neq 2}}^{2N+1} (e^{a\omega_n s\pi} - 1) \right) + \dots + \\ + & \sum_{k=1}^{2N+1} \omega_k \left(\int_0^\pi \dots \right)_{a,2N+1,k} e^{a\omega_k s\pi} \left(\prod_{\substack{n=1 \\ n\neq 2N+1}}^{2N+1} (e^{a\omega_n s\pi} - 1) \right) \Bigg\}. \end{aligned} \tag{27}$$

For the determinant Δ_{00} from ((26)) the following property holds: if (δ_{mk}) $(m, k = 1, 2, \dots, 2N+1)$ is the matrix of algebraic minors to the elements $b_{m,k}$ $(m, k = 1, 2, \dots, 2N+1)$ of the determinant Δ_{00} of ((26)), then

$$(\delta_{mn}) = \begin{pmatrix} \delta_{11} & \delta_{12} & \dots & \delta_{1,2N+1} \\ \delta_{21} & \delta_{22} & \dots & \delta_{2,2N+1} \\ \dots & \dots & \dots & \dots \\ \delta_{2N+1,1} & \delta_{2N+1,2} & \dots & \delta_{2N+1,2N+1} \end{pmatrix} =$$

$$= \frac{\Delta_{00}}{2N+1} \begin{pmatrix} 1 & -1 & 1 & -1 & \dots & -1 & 1 \\ -\omega_1^{-1} & \omega_2^{-1} & -\omega_3^{-1} & \omega_4^{-1} & \dots & \omega_{2N}^{-1} & -\omega_{2N+1}^{-1} \\ \omega_1^{-2} & -\omega_2^{-2} & \omega_3^{-2} & -\omega_4^{-2} & \dots & -\omega_{2N}^{-2} & \omega_{2N+1}^{-2} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ -\omega_1^{-2N-1} & \omega_2^{-2N-1} & -\omega_3^{-2N-1} & \omega_4^{-2N-1} & \dots & \omega_{2N}^{-2N-1} & -\omega_{2N+1}^{-2N-1} \\ \omega_1^{-2N} & -\omega_2^{-2N} & \omega_3^{-2N} & -\omega_4^{-2N} & \dots & -\omega_{2N}^{-2N} & \omega_{2N+1}^{-2N} \end{pmatrix}. \tag{28}$$

The proof of property (28) can be found in the work of the author [2]. The formula (27) is derived using the property (28). To find the roots of the equation $f_0(s)$ from (24), it is necessary to study the indicator diagram of this equation (see [3, chapter 12]).

For the equation $f_0(s)=0$ from (24) – (25) the following relation is valid:

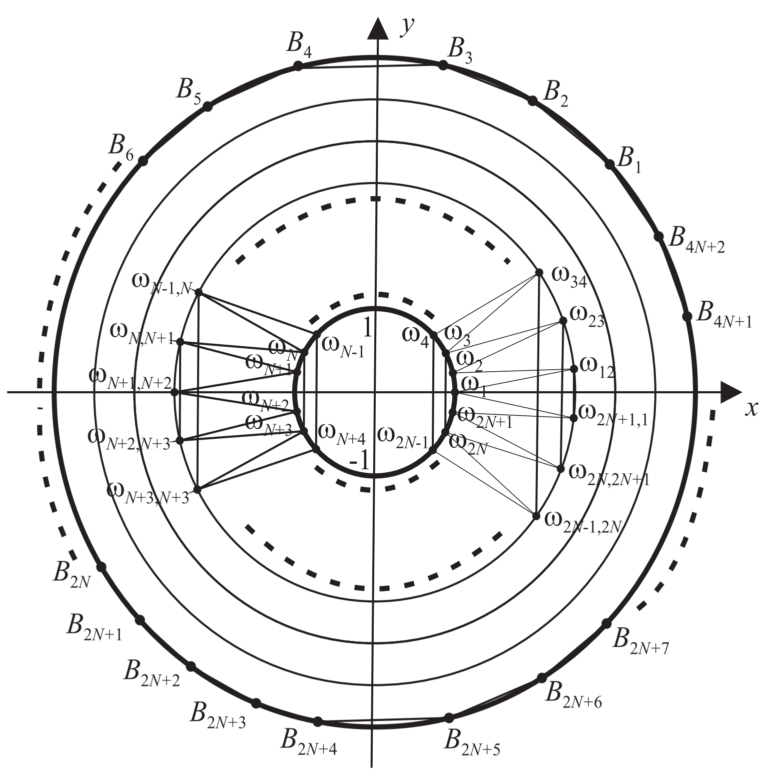
$$f_0(s)=\Delta_{00}\Big\{-1+\sum_{n_1=1}^{2N+1}e^{a\omega_{n_1}s\pi}-\sum_{\substack{n_1,n_2=1\\ n_1\neq n_2}}^{2N+1}e^{a(\omega_{n_1}+\omega_{n_2})s\pi}+ \\ +\sum_{\substack{n_1,n_2,n_3=1\\ n_1\neq n_2,n_1\neq n_3,n_2\neq n_3}}^{2N+1}e^{a(\omega_{n_1}+\omega_{n_2}+\omega_{n_3})s\pi}-\sum_{\substack{n_1,n_2,n_3,n_4=1\\ n_k\neq n_m,(k\neq m)}}^{2N+1}e^{a(\omega_{n_1}+\omega_{n_2}+\omega_{n_3}+\omega_{n_4})s\pi}+\ldots\Big\}=0. \tag{29}$$

To construct the indicator diagram of the equation (29), it is necessary to study the convex hulls of the sets

$$\{\omega_{n_1}\}_{n_1=1}^{2N+1},\ \{\omega_{n_1}+\omega_{n_2}\}_{\substack{n_1,n_2=1\\ n_1\neq n_2}}^{2N+1},\ \{\omega_{n_1}+\omega_{n_2}+\omega_{n_3}\}_{n_1,n_2,n_3=1}^{2N+1},\ \Big\{\sum_{m=1}^{4}\omega_{m_n}\Big\}_{\substack{n_m=1\\ m=1,2,3,4}}^{2N+1},\ldots$$

The indicator diagram has the following form:

$$\omega_{k,m}=\omega_k+\omega_n,\quad \omega_{n_1,n_2,\ldots,n_k}=\omega_{n_1}+\omega_{n_2}+\cdots+\omega_{n_k} \tag{30}$$



In figure (30) the following designations are introduce: the points $B_1,B_2,B_3,B_4,B_5,\ldots,$ $B_{2N},B_{2N+1},\ldots,B_{2N+5},B_{2N+6},\ldots,B_{4N+1},B_{4N+2}$ correspond to exponents with exponents $\omega_{1,2,\ldots,N}$; $\omega_{1,2,\ldots,N,N+1}$; $\omega_{2,3,\ldots,N,N+1}$; $\omega_{2,3,\ldots,N+1,N+2}$; $\omega_{3,4,\ldots,N+1,N+2}$; $\ldots$; $\omega_{N,N+1,\ldots,2N}$; $\omega_{N+1,N+2,\ldots,2N}$; $\omega_{N+1,N+2,\ldots,2N,2N+1}$; $\omega_{N+2,N+3,\ldots,2N,2N+1}$; $\omega_{N+2,N+3,\ldots,2N+1,1}$; $\omega_{N+3,N+4,\ldots,2N+1,1}$; $\omega_{N+3,N+4,\ldots,2N+1,1,2}$; $\ldots$; $\omega_{2N+1,1,2,3,\ldots N-1}$; $\omega_{N+1,1,2,3,\ldots,2N-1,N}$, where

$\omega_{n_1,n_2,\ldots,n_k}$ corresponds to the sum $\omega_{n_1}+\omega_{n_2}+\cdots+\omega_{n_k}$ indices n_k, n_m do not coincide in pairs at $k\neq m$.

In figure (30), the circle of the smallest radius $R_1=1$ is the set of points $\{\omega_k\}_{k=1}^{2N+1}$ from (4) that divide the unit circle $(2N+1)$ equal parts. The circle of the second largest radius $R_2=|\omega_1+\omega_2|>1$ is a set of points $\{\omega_k+\omega_m\}_{\substack{k,m\neq 1\\ k\neq m}}^{2N+1}$ that are constructed according to the parallelogram rule, while only points $\omega_1+\omega_2$, $\omega_2+\omega_3$, $\omega_3+\omega_4,\ldots$, $\omega_{2N}+\omega_{2N+1}$, $\omega_{2N+1}+\omega_1$ appear on the circle, the point $\omega_{n_1}+\omega_{n_2}$ under the condition $|n_1-n_2|\geqslant 2$ fall inside the circle of radius R_2 and do not affect the asymptotics of the roots of equation (23) – (27) (see [3, chapter 12]).

The third largest circle of radius $R_3=|\omega_1+\omega_2+\omega_3|>R_2$ is a set of the points $\{\omega_k+\omega_m+\omega_n\}_{k,m,n=1}^{2N+1}$ only points $\omega_1+\omega_2+\omega_3$,$\omega_2+\omega_3+\omega_4$, $\omega_3+\omega_4+\omega_5,\ldots$, $\omega_{2N-2}+\omega_{2N-1}+\omega_{2N}$, $\omega_{2N-1}+\omega_{2N}+\omega_{2N+1}$, $\omega_{2N}+\omega_{2N+1}+\omega_1$, $\omega_{2N+1}+\omega_1+\omega_2$ are located on the boundary of the circle, the remaining points are inside this circle, and the asymptotics of the roots of equation (23) – (27) are not affected. Next come the circles of the radius $R_4=|\omega_1+\omega_2+\omega_3+\omega_4|>R_3$ (this is a set of points $\{\omega_{k_1}+\omega_{k_2}+\omega_{k_3}+\omega_{k_4}\}_{\substack{k_m=1\\ m=1,2,3,4}}^{2N+1}$), the circle of radius $R_5=|\sum\limits_{k=1}^{5}\omega_k|>R_4$ (this is the set of points $\{\omega_{k_1}+\omega_{k_2}+\cdots+\omega_{k_5}\}_{\substack{k_m=1\\ m=1,2,\ldots,5}}^{2N+1}$),..., the circle of radius $R_{N-1}=|\sum\limits_{k=1}^{N-1}\omega_k|>R_{N-2}$ (this is the set of points $\{\omega_{k_1}+\omega_{k_2}+\cdots+\omega_{k_{N-1}}\}_{\substack{k_m=1\\ m=1,2,\ldots,N-1}}^{2N+1}$), and finally, the circle of the largest radius $R_N=|\sum\limits_{k=1}^{N}\omega_k|>R_{N+1}=|\sum\limits_{k=1}^{N}\omega_k|$ due to equality $\omega_1+\omega_2+\omega_3+\cdots+\omega_{2N}+\omega_{2N+1}\overset{(2.2)}{=}0$ (these are the sets of points $\{\omega_{k_1}+\omega_{k_2}+\cdots+\omega_{k_N}\}_{\substack{k_m=1\\ m=1,2,\ldots,N}}^{2N+1}$ and $\{\omega_{k_1}+\omega_{k_2}+\cdots+\omega_{k_N}+\omega_{k_{N-1}}\}_{\substack{k_m=1\\ m=1,2,\ldots,N+1}}^{2N+1}$).

The circle of he radius $R_{N+2}=|\sum\limits_{k=1}^{N+2}\omega_k|$ coincides with the circle of the radius R_{N-1}, the circle of the radius $R_{N+m}=|\sum\limits_{k=1}^{N+m}\omega_k|$ coincide with the circles of the radius $R_{N-m+1}=|\sum\limits_{k=1}^{N-m+1}\omega_k|$ $(m=2,3,\ldots,N)$ due to equality $\sum\limits_{k=1}^{2N+1}\omega_k\overset{(2.2)}{=}0$ they are located inside the indicator diagram (30) and such exponentials do not affect the asymptotics of the roots of equation (23) – (27).

The roots of equation (23) – (27) are located in the sectors $1), 2), \ldots, (4N+1)), (4N+2))$ of an infinitesimal opening, the bisectors of which are perpendicular to the segment $[B_m; B_{m+1}]$ $(m=1,2,\ldots,4N+2)$ and pass through the midpoints of the segments.

4 The asymptotics of the eigenvalues of the differential operator (1) – (2) in the sector 1) of the indicator diagram (30)

To find the roots of the roots of the equation $f(s)=0$ from (23) – (27) in the sector 1) of the indicator diagram (1), only the exponents with the exponents $\omega_{1,2,3,\ldots,N}=\sum\limits_{k=1}^{N}\omega_k$ and

$\omega_{1,2,3,\dots,N,N+1} = \sum\limits_{k=1}^{N+1} \omega_k$ should be left in this equation.

Theorem 4 *The equation for the eigenvalues of the differential operator* (1) – (3) *in the sector 1) of the indicator diagram* (30) *has the following form:*

$$v_1(s) = v_{1,0}(s) - \frac{v_{1,2N}(s)}{(2N+1)a^{2N}s^{2N}} + \underline{O}\left(\frac{1}{s^{4N}}\right) = 0, \tag{31}$$

$$v_{1,0}(s) \overset{(3.6)}{=} \Delta_{00}[e^{a(\omega_1+\omega_2+\dots+\omega_N)s\pi} - e^{a(\omega_1+\omega_2+\dots+\omega_N+\omega_{N+1})s\pi}], \tag{32}$$

while the main approximation has the form $v_{1,0}(s)$

$$\begin{gathered}\frac{v_{1,2N}(s)}{\Delta_{00}} = \omega_1\left(\int\limits_0^\pi \dots\right)_{a11}\frac{v_{1,0}(s)}{\Delta_{00}} + \omega_2\left(\int\limits_0^\pi \dots\right)_{a22}\frac{v_{1,0}(s)}{\Delta_{00}} + \dots + \\ +\omega_N\left(\int\limits_0^\pi \dots\right)_{aNN}\frac{v_{1,0}(s)}{\Delta_{00}} + \omega_{N+1}\left(\int\limits_0^\pi \dots\right)_{a,N+1,N+1}(-1)h_{N+1}(s) + \\ +\sum_{k=1}^N \omega_k h_N(s)\left(\int\limits_0^\pi \dots\right)_{a,N+1,k} + \sum_{k=1}^N \omega_k\left(\int\limits_0^\pi \dots\right)_{a,N+2,k}\frac{v_{1,0}(s)}{\Delta_{00}} + \\ +\omega_{N+1}\left(\int\limits_0^\pi \dots\right)_{a,N+2,N+1}(-1)h_{N+1}(s) + \dots + \sum_{k=1}^N \omega_k\left(\int\limits_0^\pi \dots\right)_{a,2N+1,k}\frac{v_{1,0}(s)}{\Delta_{00}} + \\ +\omega_{N+1}\left(\int\limits_0^\pi \dots\right)_{a,2N+1,N+1}(-1)h_{N+1}(s),\end{gathered} \tag{33}$$

Where the notation $h_N(s) = e^{a(\omega_1+\omega_2+\dots+\omega_N)s\pi}$ and $h_{N+1}(s) = e^{a(\omega_1+\omega_2+\dots+\omega_N+\omega_{N+1})s\pi}$ are introduced.

Dividing in the equation (31) – (33) by $(-1)\Delta_{00}h_N(s) \neq 0$ we obtain:

$$\begin{gathered}v_1(s) = [e^{a\omega_{N+1}s\pi} - 1] - \frac{1}{(2N+1)a^{2N}s^{2N}}\left\{\int\limits_0^\pi q(t)dt_{a11}\sum_{k=1}^N \omega_k[e^{a\omega_{N+1}s\pi} - 1] + \right. \\ +\omega_{N+1}\left(\int\limits_0^\pi \dots\right)_{a,N+1,N+1} e^{a\omega_{N+1}s\pi} - \sum_{k=1}^N \omega_k\left(\int\limits_0^\pi \dots\right)_{a,N+1,k} + \\ +\sum_{k=1}^N \omega_k\left(\int\limits_0^\pi \dots\right)_{a,N+2,k}[e^{a\omega_{N+1}s\pi} - 1] + \omega_{N+1}\left(\int\limits_0^\pi \dots\right)_{a,N+2,N+1} e^{a\omega_{N+1}s\pi} +\end{gathered}$$

$$+\sum_{k=1}^{N}\omega_k\left(\int_0^{\pi}\dots\right)_{a,N+3,k}[e^{a\omega_{N+1}s\pi}-1]+\omega_{N+1}\left(\int_0^{\pi}\dots\right)_{a,N+3,N+1}e^{a\omega_{N+1}s\pi}+\dots+$$
$$+\sum_{k=1}^{N}\omega_k\left(\int_0^{\pi}\dots\right)_{a,2N+1,k}[e^{a\omega_{N+1}s\pi}-1]+\omega_{N+1}\left(\int_0^{\pi}\dots\right)_{a,2N+1,N+1}e^{a\omega_{N+1}s\pi}\Bigg\}+ \tag{34}$$
$$+\underline{O}\left(\frac{1}{s^{4N}}\right)=0,$$

at the same time

$$\left(\int_0^{\pi}\dots\right)_{a11}=\left(\int_0^{\pi}\dots\right)_{a22}=\dots=\left(\int_0^{\pi}\dots\right)_{akk}\overset{(2.14)}{=}\int_0^{\pi}q(t)dt_{a11},$$
$$k=1,2,\dots,2N+1.$$

The basic approximation of equation ((34)) has the form:

$$e^{a\omega_{N+1}s\pi}-1=0\Leftrightarrow e^{a\omega_{N+1}s\pi}=1=e^{2\pi ik}\Leftrightarrow s_{k,1,\text{bas}}=\frac{2ik}{a\omega_{N+1}},\quad k\in\mathbb{N}. \tag{35}$$

The following statement follows from the formula (35) and the general theory of finding the roots of quasipolynomes of the form (34) (see [4], [5]).

Theorem 5 *The asymptotics of the eigenvalues of the differential operator* (1) – (3) *in the sector 1) of the indicator diagram* (30) *has the following form:*

$$s_{k,1}=\frac{2i}{a\omega_{N+1}}\left[k+\frac{d_{2N,k,1}}{k^{2N}}+\underline{O}\left(\frac{1}{k^{2N}}\right)\right],\quad k\in\mathbb{N}. \tag{36}$$

Proof. Applying the Maclaurin's formulas, we have:

$$e^{a\omega_{N+1}s\pi}\big|_{s_{k,1}}=\exp\left[a\omega_{N+1}\pi\frac{2i}{a\omega_{N+1}}\left(k+\frac{d_{2N,k,1}}{k^{2N}}+\underline{O}\left(\frac{1}{k^{4N}}\right)\right)\right]=$$
$$=e^{2\pi ik}\exp\Big[2\pi i\Big(\frac{d_{2N,k,1}}{k^{2N}}+\underline{O}\left(\frac{1}{k^{4N}}\right)\Big]=1+\frac{2\pi id_{2N,k,1}}{k^{2N}}+\underline{O}\left(\frac{1}{k^{4N}}\right)\Big); \tag{37}$$

$$\frac{1}{s^{2N}}\bigg|_{s_{k,1}}=\frac{a^{2N}\omega_{N+1}^{2N}}{2^{2N}i^{2N}}\frac{1}{k^{2N}}\left(1-\frac{2Nd_{2N,k,1}}{k^{2N+1}}\underline{O}\left(\frac{1}{k^{4N+1}}\right)\right). \tag{38}$$

Substituting formulas (36) – (38) into equation (34), we obtain:

$$\left[1+\frac{2\pi id_{2N,k,1}}{k^{2N}}+\underline{O}\left(\frac{1}{k^{4N}}\right)-1\right]-\frac{a^{2N}\omega_{N+1}^{2N}}{(2N+1)a^{2N}2^{2N}i^{2N}}\frac{1}{k^{2N}}\left(1+\underline{O}\left(\frac{1}{k^{2N+1}}\right)\right)\times$$
$$\times\Bigg\{\int_0^{\pi}q(t)dt_{a11}\sum_{k=1}^{N}\omega_k\underline{O}\left(\frac{1}{k^{2N}}\right)+\omega_{N+1}\int_0^{\pi}q(t)dt_{a11}\left(1+\underline{O}\left(\frac{1}{k^{2N}}\right)\right)-$$
$$-\sum_{k=1}^{N}\omega_k\left(\int_0^{\pi}\dots\right)_{a,N+1,k}+\sum_{k=1}^{N}\omega_k\left(\sum_{m=2}^{N+1}\left(\int_0^{\pi}\dots\right)_{a,N+m,k}\right)+\underline{O}\left(\frac{1}{k^{2N}}\right)+ \tag{39}$$
$$+\omega_{N+1}\sum_{m=2}^{N+1}\left(\int_0^{\pi}\dots\right)_{a,N+m,N+1}\left(1+\underline{O}\left(\frac{1}{k^{2N}}\right)\right)\Bigg\}+\underline{O}\left(\frac{1}{k^{4N}}\right)=0.$$

For k^0 in (39) we obtain the correct equality $1 - 1 = 0$, which means that the form of asymptotic formula (36) is chosen correctly. For k^{-2N} in (39) we have:

$$d_{2N,k,1} = \frac{1}{2\pi i}\frac{\omega_{N+1}^{2N}}{(2N+1)2^{2N}(-1)^N}\Big[\omega_{N+1}\int_0^\pi q(t)dt_{a11} - \sum_{k=1}^{N}\omega_k\left(\int_0^\pi \ldots\right)_{a,N+1,k} + \\ +\omega_{N+1}\sum_{k=N+2}^{2N+1}\left(\int_0^\pi \ldots\right)_{a,k,N+1}\Big], \quad k\in\mathbb{N}. \tag{40}$$

In the formula (40) the first term is transformed to the following form:

$$\frac{1}{2\pi i}\frac{\omega_{N+1}^{2N}}{(2N+1)2^{2N}(-1)^N}\omega_{N+1}\int_0^\pi q(t)dt \overset{(2.1)}{=} \frac{(-1)^{N+1}}{(2N+1)\pi 2^{2N+1}}\int_0^\pi q(t)dt. \tag{41}$$

The second and the third terms in formula (40) will be calculated as follows:

$$H_N = -\sum_{m=1}^{N}\omega_m\left(\int_0^\pi \ldots\right)_{a,N+1,m} + \omega_{N+1}\sum_{m=N+2}^{2N+1}\left(\int_0^\pi \ldots\right)_{a,m,N+1} =$$

$$= \sum_{k=1}^{N}\left[\omega_{N+1}\left(\int_0^\pi \ldots\right)_{a,2N+2-m,N+1} - \omega_m\left(\int_0^\pi \ldots\right)_{a,N+1,m}\right] \overset{(2.14)}{=}$$

$$= \sum_{m=1}^{N}\omega_{N+1}\int_0^\pi q(t)e^{a(\omega_{2N+2-m}-\omega_{N+1})st}dt_{a,2N+2-m,N+1} -$$

$$-\omega_m\int_0^\pi q(t)e^{a(\omega_{N+1}-\omega_m)st}dt_{a,N+1,m} \overset{(2.1),(4.5)}{=}$$

$$= \sum_{m=1}^{N}\Big[e^{\dfrac{2\pi i N}{2N+1}}\int_0^\pi q(t)\exp\left[at\frac{2ik}{a\omega_{N+1}}\left(e^{\dfrac{2\pi i(2N+1-m)}{2N+1}} - e^{\dfrac{2\pi i N}{2N+1}}\right)\right]dt_{a,2N+2-m,N+1} -$$

$$-e^{\dfrac{2\pi i(m-1)}{2N+1}}\int_0^\pi q(t)\exp\left[at\frac{2ik}{a\omega_{N+1}}(\omega_{N+1}-\omega_m)\right]dt_{a,N+1,m}\Big]dt_{a,N+1,m},$$

this expression will be transformed and simplified as follows:

$$H_N = \sum_{m=1}^{N} e^{\dfrac{\pi i(N+m-1)}{2N+1}}\Big[e^{\dfrac{\pi i(N-m+1)}{2N+1}}\int_0^\pi q(t)e^{-2kti}\exp\left[2kti\exp\left(\frac{2\pi i(1-m+N)}{2N+1}\right)\right]dt -$$

$$-e^{-\dfrac{\pi i(N-m+1)}{2N+1}}\int\limits_0^\pi q(t)e^{2kti}\exp\left[(-2kti)\exp\left(\frac{2\pi i(m-1-N)}{2N+1}\right)\right]dt\Bigg]=$$

$$=\sum_{m=1}^{N}e^{\dfrac{\pi i(N+m-1)}{2N+1}}\int\limits_0^\pi q(t)e^{-2kti}\exp\Big[2kti\Big(\cos\left(\frac{2\pi(N-m+1)}{2N+1}\right)+$$

$$+i\sin\left(\frac{2\pi(N-m+1)}{2N+1}\right)\Big)\Big]e^{\dfrac{\pi i(N-m+1)}{2N+1}}dt-$$

$$-\int\limits_0^\pi q(t)e^{2kti}e^{-\dfrac{\pi i(N-m+1)}{2N+1}}\exp\Big[(-2kti)\Big[\cos\left(\frac{2\pi(N-m+1)}{2N+1}\right)-$$

$$-i\sin\left(\frac{2\pi(N-m+1)}{2N+1}\right)\Big]dt\Big],$$

as a result of which we will receive:

$$H_N=(-2i)\sum_{m=1}^{N}\exp\left(\frac{\pi i(N+m-1)}{2N+1}\right)\int\limits_0^\pi q(t)\sin\Big[2kt-2kt\cos\left(\frac{2\pi(N-m+1)}{2N+1}\right)-$$
$$-\frac{\pi(N-m+1)}{2N+1}\Big]dt\exp\left(-2kti\sin\left(\frac{2\pi(N-m+1)}{2N+1}\right)\right)dt_{bNm}. \tag{42}$$

Substituting the formulas (41), (42) into formula (40), we find:

$$d_{2N,k,1}=\frac{\omega_{N+1}^{2N}}{(2\pi i)(2N+1)2^{2N}(-1)^N}\left[\omega_{N+1}\int\limits_0^\pi q(t)dt_{a11}+H_N\right]=$$
$$=\frac{(-1)^{N+1}i}{(2N+1)\pi 2^{2N+1}}\int\limits_0^\pi q(t)dt-2ie^{-\dfrac{2\pi iN}{2N+1}}\sum_{m=1}^{N}\exp\left(\frac{\pi i(N+m-1)}{2N+1}\right)\int\limits_0^\pi q(t)\times$$
$$\times\sin\left[2kt-2kt\cos\left(\frac{2\pi(N-m+1)}{2N+1}\right)-\frac{\pi(N-m+1)}{2N+1}\right]\times$$
$$\times\exp\left(-2kt\sin\left(\frac{2\pi(N-m+1)}{2N+1}\right)\right)dt_{bNm},\quad k=1,2,3,\ldots;\quad N=1,2,3,\ldots. \tag{43}$$

We have proved that all the coefficients $d_{2N,k,1}$ ($k=1,2,3,\dots$) of formula (36) are found in a unique way, in formula (43) we have given explicit formulas for calculating them, so theorem 5 is completely proved. Studying in a similar way sectors $2),3),\ldots,(4N+2))$ of the indicator diagram (30), we come to the following statement.

Theorem 6 *1) The asymptotics of the eigenvalues of the differential operator* (1) – (3) *in the sectors* $2),3),\ldots,(4N+2))$ *of the indicator diagram* (30) *satisfies the following law:*

$$s_{k,2}=s_{k,1}e^{-\dfrac{2\pi i}{4N+2}};\quad s_{k,3}=s_{k,2}e^{-\dfrac{2\pi i}{4N+2}}=s_{k,1}e^{-\dfrac{4\pi i}{4N+2}};\ldots;$$

$$s_{k,m} = s_{k,m-1} e^{-\frac{2\pi i}{4N+2}} = s_{k,1} e^{-\frac{2\pi i(m-1)}{4N+2}},$$
$$m = 1, 2, 3, \dots, 4N+2,$$

where $s_{k,1}$ *satisfies formulas* (36), (43).

2) Wherein $\lambda_{k,m} = s_{k,m}^{2N+1}$, $m = 1, 2, 3, \dots, 4N+2$; $k = 1, 2, 3, \dots$.

References

[1] Lidskii V. B., Sadovnichii V. A., "Asymptotic formulas for the zeros of a class of entire functions", *Math. USSR-izv* 2 (3) (1968): 585–600.

[2] Sadovnichii V. A., "The trace of ordinary differential operators of higher orders", *Math. USSR-Sb* 1 (2) (1967): 263–288.

[3] Mitrokhin S. I., "On spectral properties of differential operators with discontinuous coefficients", *Differential Equations* 28 (3) (1992): 530–532.

[4] Mitrokhin S. I., "On some spectral properties of second-order differential operators with a discontinuous weight function", *Reports of the Russian Academy of Sciences* 356 (1) (1997): 13–15.

[5] Vinokurov V. A., Sadovnichy V. A., "Asymptotics of any order for the eigenvalues and eigenfunctions of the Sturm-Liouville boundary-value problem on a segment with a summable potential", *Izv. Math.* 64 (4) (2000): 695–754.

[6] Mitrokhin S. I., "Spectral properties of a fourth-order differential operator with integrable coefficients", *Proc. Steklov Inst. Math.* 270 (2010): 184–193.

[7] Mitrokhin S. I., "On spectral properties of odd order differential operators with summable potential", *Differential Equations* 47 (2) (2011): 1808–1811.

[8] Mitrokhin S. I., "Spectral properties of the family of even order differential operators with a summable potential", *Moscow University Mathematics Bulletin* 72 (4) (2017): 137–148.

[9] Badanin A. V., Korotyaev E. L., "Spectral estimates for a periodic fourth-order operator", *St. Petersburg Math. J.* 22 (5) (2011): 703–736.

[10] Dzhakov P., Mityagin B. S., "Instability zones of one-dimensional periodic Schrodinger and Dirac operators", *Russian Math. Surveys* 61 (4) (2006): 663–766.

[11] Polyakov D. M., "Spectral analysis of a fourth-order differential operator with periodic and antiperiodic boundary conditions", *St. Petersburg Math. J.* 27 (5) (2016): 789–811.

[12] Veliev O. A., "On the nonself-adjoint ordinary differential operator with periodic boundary conditions", *Israel J. Math.* 27 (2010): 195–207.

[13] Mikhailets V., Moliboga V., "Uniform estimates for the semi-periodic eigenvalues of the singular differential operators", *Methods Funct. Anal. Topology* 10 (4) (2004): 30–57.

[14] Mitrokhin S. I., "The asymptotics of the spectrum of a periodic boundary value problem for a differential operator with a summable potential", *Proceedings of IMM Uro RAS* 25 (1) (2019): 136–149.

[15] Mitrokhin S. I., "Periodic boundary value problem for a fourth-order differential operator with a summable potential", *Vladikavkaz mathematical journal* 19 (4) (2017): 35–49.

[16] Badanin A. V., Korotyaev E. L., "Third-order operator with periodic coefficients on the real axis", *St. Petersburg Math. J.* 25 (5) (2014): 713–734.

[17] Naimark M. A., *Linear differential operators* (Moscow: Nauka, 1969): 528.

[18] Bellman R., Cook K. L., *Differential-difference equations* (Moscow: Mir, 1967): 548.