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STUDY OF THE INITIAL BOUNDARY VALUE PROBLEM FOR A TWO-DIMENSIONAL CONVECTION-DIFFUSION EQUATION WITH A FRACTIONAL TIME DERIVATIVE IN THE SENSE OF CAPUTO-FABRIZIO

In this paper, we study an initial boundary value problem for a differential equation with a fractional order derivative in time in the Caputo-Fabrizio sense. This equation is of great practical importance in modeling the processes of fluid motion in porous media and anomalous dispersion. The uniqueness and continuous dependence of the solution of the problem on the input data in differential form is proved. A computationally efficient implicit difference scheme with weights is proposed. A priori estimates are obtained for the solution of the problem under the assumption that the solution exists in the class of sufficiently smooth functions. The uniqueness of the solution and the stability of the difference scheme with respect to the initial data and the right-hand side of the equation follows from the obtained estimates. The convergence of the difference problem solution to the differential problem solution with the second order in time and space variables is proved. The results of computational experiments confirming the reliability of the theoretical analysis are presented.

Key words: Fractional differential equation, fractional derivative in the sense of Caputo-Fabrizio, finite difference method, energy inequality method, stability, convergence, a priori estimate.

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Уақыт бойынша Капуто-Фабрицио мағынасындағы бөлшек туындысы бар екі өлшемді конвекция-диффузия теңдеуі үшін қойылған бастапқы шекаралық есепті зерттеу

Бұл жұмыста Капуто-Фабрицио мағынасындағы уақыт бойынша бөлшек ретті туындысы бар дифференциалдық теңдеу үшін қойылған бастапқы шекаралық есеп зерттелді. Бұл теңдеу фильтрация үрдістерін және аномалды дисперсияны модельдеуде үлкен қолданбалы мәнге ие. Есеп шешімінің жалғыздығы мен бастапқы берілген мәндерден тәуелділігі дифференциалдық формада дәлелденді. Есептеуге тиімді салмағы бар айқын емес айырымдық сұлба ұсынылды. Жеткілікті тегіс функциялар класында шешімі бар деген болжаммен есептің шешімі үшін априорлық бағалаулар алынды. Осы бағалаулардан шешімнің жалғыздығы және бастапқы берілген мәндер мен теңдеудің оң жағы бойынша айырымдық сұлбаның орнықтылығы шығады. Айырымдық есептің шешімінің дифференциалдық есептің шешіміне уақыт және кеңістіктік айнымалылары бойынша екінші ретпен жинақталуы дәлелденді. Теориялық талдаудың дұрыстығын растайтын есептеу тәжірибелерінің нәтижелері ұсынылды.

Түйін сөздер: Бөлшек ретті дифференциалдық теңдеу, Капуто-Фабрицио мағынасындағы бөлшек туынды, ақырлы айырымдар әдісі, энергиялық теңсіздіктер әдісі, орнықтылық, жинақтылық, априорлы бағалау.

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Исследование начально-краевой задачи для двумерного уравнения конвекции-диффузии с дробной производной по времени в смысле Капуто-Фабрицио

В настоящей работе исследуется начально-краевая задача для дифференциального уравнения с производной дробного порядка по времени в смысле Капуто-Фабрицио. Данное уравнение имеет большую прикладную значимость при моделировании процессов фильтрации и аномальной дисперсии. Доказаны единственность и непрерывная зависимость решения задачи от входных данных в дифференциальной форме. Предложена вычислительно эффективная неявная разностная схема с весами. Получены априорные оценки для решения задачи в предположении существования решения в классе достаточно гладких функций. Из этих оценок следуют единственность решения и устойчивость разностной схемы по начальным данным и правой части уравнения. Доказана сходимость решения разностной задачи к решению дифференциальной задачи со вторым порядком по временной и пространственной переменным. Представлены результаты вычислительных экспериментов, подтверждающие достоверность теоретического анализа.

Ключевые слова: Дифференциальное уравнение дробного порядка, дробная производная в смысле Капуто-Фабрицио, метод конечных разностей, метод энергетических неравенств, устойчивость, сходимость, априорная оценка.

1 Introduction

Differential equations containing fractional derivatives have become popular because they are more suitable for modeling specific real-world problems than ordinary differential equations. Therefore, the development of analytical and numerical methods for the theory of fractional differential equations is an urgent and important problem. One of the important examples of applying this type of equations is the equations describing flows of a multiphase fluid in highly porous fractured formations with fractal well geometry.

In this paper, we obtain a priori estimates in differential form for this problem, which implies the uniqueness of the solution and its continuity from the input data. An implicit finite difference scheme of the second order of approximation in time and in a spatial variable is proposed. The stability of the proposed scheme as well as convergence with a speed equal to the approximation order is proved. The results obtained are confirmed by numerical calculations performed for two test problems.

2 Literature review

In recent years, the use of fractional order derivatives to construct mathematical models of various physical processes involving electrical circuits [1], thermal and diffusion processes [2,3], medicine [4,5], and other processes [6,7], as well as the development of numerical or analytical solutions for these fractional mathematical models are very relevant. Among them, the problems of fluid flows in porous media are of great interest, where their dynamics are significantly affected by memory effects, which are described by the theory of fractional-order integro-differentiation [8,9]. In the fluid flow problems, whose state and observation processes are controlled by the time-varying Brownian motion or the Levy process, the Riemann-Liouville fractional derivative was used for the Zakai equation [10]. In [9], several models were proposed to describe fluid flow processes in complex fractured porous media containing fractional Riemann–Liouville derivatives in time and space. For single-phase fluid flow, a nonlinear pressure equation containing fractional Riemann-Liouville derivatives with respect to time is obtained, a fractional differential modification of Darcy's law is proposed, and a fractional differential equation for anisotropic fluid flow is obtained. A fractional differential modification of the Barenblatt-Gilman model for nonequilibrium two-phase countercurrent capillary impregnation is also proposed, taking into account the effects of power memory when the system relaxes to a local equilibrium state. For the two-phase flow of an incompressible and immiscible fluid in porous media, a memory formalism using the fractional Caputo derivative was introduced and a two-level discrete time method was developed that uses a large time step for pressure and a small time step for saturation [11]. In [12], a nonlinear two-dimensional orthotropic fluid flow equation with a fractional Riemann–Liouville derivative in time is considered. In [13], a fractional model was presented for two immissible fluids flowing through a porous medium with an average capillary pressure, and the solution was obtained using the Mittag-Leffler function, the Sumudu transform, the sinusoidal Fourier transform and their inversions after obtaining the corresponding formulas for fractional integrals and derivatives. In [14], the laminar flow of a fluid in an axisymmetric porous cylindrical channel exposed to a magnetic field was investigated. The governing equations consisted of fractional partial differential equations based on Caputo-Fabrizio fractional derivatives in time.

As we can see, many papers have been devoted to the theoretical development and application of fractional derivatives in various branches of science, but in this paper we want to use the recently introduced fractional derivative in the sense of Caputo-Fabrizio without a singular core [15]. The properties of the Caputo-Fabrizio fractional derivative are studied in [16], and various boundary value problems for the fractional heat equation involving this fractional derivative are studied in [17].

The use of the Caputo-Fabrizio fractional derivative has been studied in many papers. For example, in [18], the equation of groundwater flow within an unlimited aquifer is modified using the concept of the Caputo-Fabrizio fractional derivative without the singular core. In [19], the model of groundwater motion through a geological formation was extended using the Caputo-Fabrizio fractional order derivative and the equation was solved analytically using some integral transformations.

The main contribution of [20] is the construction and analysis of stable schemes based on the third-order finite difference method in time and spectral methods in space for the effective solution of the two-dimensional diffusion equation containing a fractional Caputo-Fabrizio time derivative. In [21], the Caputo-Fabrizio fractional derivative is used to introduce two new types of high-order derivatives and the existence of solutions for two such fractional high-order integro-differential equations is studied. The article [22] presents a parallel algorithm for solving a two-dimensional fractional differential equation. For this algorithm, a distribution model and a data layout with a virtual boundary are developed. In addition, in [23] application to a nonlinear Fischer-type reaction-diffusion equation was investigated, in [24] application to a stationary heat flow, in [25] application to a groundwater flow, and in [26] application to the study of chaos on the Wallis model for El Nino, in [27] the fractional Nagumo equation with nonlinear diffusion and convection was studied using the Caputo-Fabrizio fractional derivative.

3 Material and methods

3.1 Formulation of the problem

Let $\Omega = (0,1) \times (0,1)$ and $Q_T = \Omega \times (0,T)$ for $T > 0$. We consider the following initial boundary value problem: find $u \in \bar{Q}_T$ such that

$$\partial_{0t}^{\alpha} u = Ku + Du + f(x,t), \quad (x,t) \in Q_T, \tag{1}$$

$$u(x,0) = \rho(x), \quad x \in \bar{\Omega}, \tag{2}$$

$$u(x,t) = 0, \quad x \in \partial\Omega \times (0,T), \tag{3}$$

where $0 < \alpha < 1$, $x = (x_1, x_2)$; $K = K_1 + K_2$, $D = D_1 + D_2$,

$$K_m u = q_m(x,t) \frac{\partial u}{\partial x_m}, \quad D_m u = \frac{\partial}{\partial x_m}\left(k_m(x,t)\frac{\partial u}{\partial x_m}\right), \quad m = 1,2.$$

The fractional derivative is defined in the sense of the Caputo-Fabrizio definition:

$$\partial_{0t}^{\alpha} u(x,t) = \frac{1}{1-\alpha}\int\limits_0^t \exp\left(-\gamma(t-\tau)\right)\frac{\partial u}{\partial \tau}(x,\tau)\,d\tau, \quad \gamma = \frac{\alpha}{1-\alpha}. \tag{4}$$

Assume that the following conditions hold for the coefficients and the right-hand side of (1):

$$k_m(x,t) \in C^{1,0}\left(\bar{Q}_T\right), \ q_m(x,t), \ f(x,t) \in C\left(\bar{Q}_T\right), \tag{5}$$

$$0 < c_1 \le k_m(x,t) \le c_2, \quad |q_m(x,t)| \le c_2, \quad 2c_1 > c_2^2. \tag{6}$$

Assume that there exists a solution to the problem (1)-(3) in a class of sufficiently smooth functions.

3.2 Uniqueness of the solution and its continuous dependence on input data

Introduce the following scalar products and norms:

$$\|u\|_{0,\bar{Q}_T}^2 = \int_0^T \int_\Omega u^2 dx\, dt, \quad \|u\|_{0,\Omega}^2 = \int\limits_\Omega u^2 dx,$$

$$\|\nabla u\|_{0,\bar{Q}_T}^2 = \left\|\frac{\partial u}{\partial x_1}\right\|_{0,\bar{Q}_T}^2 + \left\|\frac{\partial u}{\partial x_2}\right\|_{0,\bar{Q}_T}^2, \quad (u,v) = \int_\Omega uv\, dx,$$

$$\|u\|_*^2 = \int_0^T \int_\Omega D_{0t}^{-\alpha} \|u\|_{0,\Omega}^2\, dx\, dt,$$

where u and v are functions defined in $\bar{Q}_T$; $D_{0t}^{-\alpha}u$ is the Caputo-Fabrizio fractional integration operator [28]:

$$D_{0t}^{-\alpha}u = \frac{2(1-\alpha)}{2-\alpha}u(t) + \frac{2\alpha}{2-\alpha}\int_0^t u(\tau)\,d\tau, \quad t \geq 0.$$

The following two lemmas are proved similarly to [29].

Lemma 1 *For any absolutely continuous function on* $[0,T]$, $y(t)$, *the following inequality holds:*

$$y\partial_{0t}^{\alpha}y \geq \frac{1}{2}\partial_{0t}^{\alpha}y^2, \quad 0 < \alpha < 1.$$

Lemma 2 *Let* $y(t)$ *be a non-negative absolutely continuous function satisfying the inequality*

$$\partial_{0t}^{\alpha}y \leq \gamma_1 y(t) + \gamma_2(t), \ 0 \leq \alpha \leq 1$$

for almost every $t \in [0,T]$, *where* $\gamma_1 > 0$, $\gamma_2(t)$ *are nonnegative summable functions on* $[0,T]$. *Then*

$$y(t) \leq y(0)\,E_{\alpha}(\gamma_1 t^{\alpha}) + \Gamma(\alpha)\,E_{\alpha,\alpha}(\gamma_1 t^{\alpha})\,D_{0t}^{-\alpha}\gamma_2(t),$$

where $E_{\alpha}(z)$, $E_{\alpha,\mu}(z)$ *are Mittag-Leffler functions:*

$$E_{\alpha}(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)}, \quad E_{\alpha,\mu}(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\mu)}.$$

Boundedness of the functions $E_{\alpha}(t^{\alpha})$ and $E_{\alpha,\alpha}(t^{\alpha})$ for $0 \leq t \leq T$ yields the following inequality for a non-negative absolutely continuous function $y(x,t)$ under the conditions of Lemma 2:

$$\|y\|_{0,\bar{Q}_T}^2 \leq M_1\,\|y(x,0)\|_{0,\bar{Q}_T}^2 + M_2\,\|\gamma_2\|_*^2. \tag{7}$$

Theorem 1 *If* $u(x,t) \in C^{2,0}(Q_T) \cap C^{1,0}\left(\bar{Q}_T\right)$, $\partial_{0t}^{\alpha}u(x,t) \in C\left(\bar{Q}_T\right)$, *then under the conditions (5)-(6) the following inequality holds for the solution of the problem (1)-(3):*

$$\|u(x,t)\|_{0,\bar{Q}_T}^2 \leq M_1\,\|u(x,0)\|_{0,\bar{Q}_T}^2 + M_2\,\|f(x,t)\|_*^2, \quad M_1, M_2 > 0,$$

which yields the uniqueness and continuous dependence of the solution on input data.

Proof. Using (1), we get

$$\int_0^T\int_{\Omega} u\partial_{0t}^{\alpha}u\,dx\,dt = \int_0^T\int_{\Omega} Ku\cdot u\,dx\,dt + \int_0^T\int_{\Omega} Du\cdot u\,dx\,dt + \int_0^T\int_{\Omega} f(x,t)\,u\,dx\,dt. \tag{8}$$

Estimate the integral on the left-hand side of (8) using Lemma 1:

$$\int_0^T\int_{\Omega} u\partial_{0t}^{\alpha}u\,dx\,dt \geq \frac{1}{2}\int_0^T \partial_{0t}^{\alpha}\,\|u\|_{0,\Omega}^2\,dt. \tag{9}$$

The integrals on the right-hand side of (8) are estimated as follows:

$$\int_0^T \int\limits_\Omega Ku \cdot u \, dx \, dt \le 2c_2\varepsilon_1 \|u\|^2_{0,\bar{Q}_T} + \frac{c_2}{4\varepsilon_1} \|\nabla u\|^2_{0,\bar{Q}_T}, \tag{10}$$

$$-\int_0^T \int\limits_\Omega Du \cdot u \, dx \, dt \ge c_1 \|\nabla u\|^2_{0,\bar{Q}_T}, \tag{11}$$

$$\int_0^T \int\limits_\Omega fu \, dx \, dt \le \frac{\varepsilon_2}{2} \|f\|^2_{0,\bar{Q}_T} + \frac{1}{2\varepsilon_2} \|u\|^2_{0,\bar{Q}_T}. \tag{12}$$

Choosing $\varepsilon_1 = (2c_2)^{-1}$, $\varepsilon_2 = \left(2c_1 - c_2^2\right)^{-1}$, it follows from (8) that

$$\int_0^T \partial^\alpha_{0t} \|u\|^2_{0,\Omega} \, dt + \left(c_1 - \frac{c_2^2}{2}\right) \|\nabla u\|^2_{0,\bar{Q}_T} \le \|u\|^2_{0,\bar{Q}_T} + \frac{1}{2c_1 - c_2^2} \|f\|^2_{0,\bar{Q}_T}.$$

Using Lemma 2 implies

$$\int_0^T \|u(x,t)\|^2_{0,\Omega} \, dt \le \int_0^T \|u(x,0)\|^2_{0,\Omega} E_\alpha(t^\alpha) \, dt +$$

$$+\frac{\Gamma(\alpha)}{2c_1 - c_2^2} \int_0^T E_{\alpha,\alpha}(t^\alpha) D^{-\alpha}_{0t} \|f(x,t)\|^2_{0,\Omega} \, dt. \tag{13}$$

Using the inequality (7), we obtain the statement of the theorem from (13).

3.3 Construction of the numerical method

For the numerical solution of the problem (1)-(3) we apply the finite difference method. In $\bar{Q}_T$, we introduce a uniform finite difference grid $\overline{\omega}_{h\tau} = \overline{\omega}_h \times \overline{\omega}_\tau$, where

$$\overline{\omega}_h = \{x_{ij} = (ih, jh): \quad i = 0, 1, ..., N, \ \ j = 0, 1, ..., N, \ \ Nh = 1\},$$

$$\overline{\omega}_\tau = \{t_n = n\tau, \ \ n = 0, 1, ..., M; \ \ T = \tau M\}.$$

First let us derive a discrete analog of the fractional derivative in the sense of Caputo-Fabrizio. For this purpose we use the technique applied in [30] for the derivation of the discrete analog of the fractional derivate in the sense of Caputo. In the following lemma we assume $u(t) = u(\cdot, t)$.

Lemma 3 *Let $u(t) \in C^3[0,T]$. The discrete analog of the derivative (4) with the approximation order $O(\tau^{3-\alpha})$ is given by*

$$\partial^\alpha_{0t_{n+\sigma}} u(t) \approx \Delta^\alpha_{0t_{n+\sigma}} u \equiv \frac{1}{\alpha\tau} \sum_{s=0}^n g_{n-s} \left(u(t_{s+1}) - u(t_s)\right), \tag{14}$$

where

$$g_s^{\alpha,\sigma}=\begin{cases} A_0^{\alpha,\sigma}, & s=0,\ n=0,\\ A_0^{\alpha,\sigma}+B_1^{\alpha,\sigma}, & s=0,\ n>0,\\ A_s^{\alpha,\sigma}+B_{s+1}^{\alpha,\sigma}-B_s^{\alpha,\sigma}, & 1\le s\le n-1,\ n>0,\\ A_n^{\alpha,\sigma}-B_n^{\alpha,\sigma}, & s=n,\ n>0,\end{cases}$$

$$A_0^{\alpha,\sigma}=\frac{e^{\gamma\tau\sigma}-1}{e^{\gamma\tau\sigma}};\quad A_s^{\alpha,\sigma}=\frac{e^{\gamma\tau}-1}{e^{\gamma\tau(\sigma+s)}},\quad B_s^{\alpha,\sigma}=\frac{e^{\gamma\tau}\left(\gamma\tau-2\right)+\gamma\tau+2}{2\gamma e^{\gamma\tau(\sigma+s)}},\quad s\ge 1. \tag{15}$$

Proof. Following [30], let $\sigma=1-\alpha/2$. Using the definition (4), construct the following approximation for the fractional derivative of the function $u(t)\in C^3[0,T]$ of order α, $0<\alpha<1$, in the sense of Caputo-Fabrizio at a fixed point $t_{n+\sigma}$, $n\in\{0,1,...,M-1\}$:

$$\partial_{0t_{n+\sigma}}^{\alpha}u(t)\approx\Delta_{0t_{n+\sigma}}^{\alpha}u=\frac{1}{1-\alpha}\sum_{s=1}^{n}\int_{t_{s-1}}^{t_s}\exp\left(-\gamma\left(t_{n+\sigma}-\eta\right)\right)\tilde{u}_s'(\eta)\,d\eta+$$

$$+\frac{1}{1-\alpha}\int_{t_n}^{t_{n+\sigma}}\exp\left(-\gamma\left(t_{n+\sigma}-\eta\right)\right)\tilde{u}_s'(\eta)\,d\eta, \tag{16}$$

where $\tilde{u}_s(\eta)$ is the approximation of $u(\eta)$ on $[t_{s-1},t_s]$, $s\in\{1,2,...,n\}$. Various approaches to approximate $\tilde{u}_s(\eta)$ result in different computational schemes which differ by the approximation error, the complexity of the calculations. Among them, approaches based on applying the trapezoidal rule, interpolation and predictor-corrector methods are known. In this paper, we utilize the quadratic interpolation polynomial of u using three nodes t_{s-1}, t_s and t_{s+1}:

$$\tilde{u}_s(t)=\frac{(t-t_s)(t-t_{s+1})}{2\tau^2}u(t_{s-1})-\frac{(t-t_{s-1})(t-t_{s+1})}{\tau^2}u(t_s)+\frac{(t-t_{s-1})(t-t_s)}{2\tau^2}u(t_{s+1}), \tag{17}$$

for which

$$u(t)-\tilde{u}_s(t)=\frac{u'''\left(\overline{\xi}_s\right)}{6}(t-t_{s-1})(t-t_s)(t-t_{s+1}) \tag{18}$$

holds, where $t\in[t_{s-1},t_{s+1}]$, $\overline{\xi}_s\in(t_{s-1},t_{s+1})$. Using (17) in (16), we obtain

$$\Delta_{0t_{n+\sigma}}^{\alpha}u=\frac{1}{1-\alpha}\sum_{s=1}^{n}\int_{t_{s-1}}^{t_s}\frac{u_{t,s-1}+u_{\bar{t}t,s}\left(\eta-t_{s-\frac{1}{2}}\right)}{\exp\left(\gamma\left(t_{n+\sigma}-\eta\right)\right)}d\eta+\frac{u_{t,n}}{1-\alpha}\int_{t_n}^{t_{n+\sigma}}\frac{d\eta}{\exp\left(\gamma\left(t_{n+\sigma}-\eta\right)\right)},$$

where $u_{t,s}=\left(u^{s+1}-u^s\right)\tau^{-1}$, $u_{\bar{t},s}=\left(u^s-u^{s-1}\right)\tau^{-1}$. Taking into account the equality

$$\int_{t_{s-1}}^{t_s}\frac{\eta-t_{s-\frac{1}{2}}}{\exp\left(\gamma\left(t_{n+\sigma}-\eta\right)\right)}d\eta=\frac{\exp(\gamma\tau)\left(\gamma\tau-2\right)+\gamma\tau+2}{2\gamma^2\exp\left(\gamma\tau\left(n+\sigma-s+1\right)\right)},\quad 1\le s\le n,$$

we arrive at

$$\Delta_{0t_{n+\sigma}}^{\alpha}u=\frac{1}{\alpha}\left(\sum_{s=1}^{n}\frac{\exp(\gamma\tau)-1}{\exp\left(\gamma\tau\left(n+\sigma-s+1\right)\right)}u_{t,s-1}+\right.$$

$$+\sum_{s=1}^{n}\frac{\exp(\gamma\tau)(\gamma\tau-2)+\gamma\tau+2}{2\gamma\exp(\gamma\tau(n+\sigma-s+1))}(u_{t,s}-u_{t,s-1})+\frac{\exp(\gamma\tau\sigma)-1}{\exp(\gamma\tau\sigma)}u_{t,n}\Bigg).$$

Finally, using the notations (15), we arrive at the assertion of the lemma.

In $\overline{\omega}_{h\tau}$ we introduce the following difference scheme with weights of the approximation order $O(h^2+\tau^2)$:

$$\Delta^{\alpha}_{0t_{n+\sigma}}y_{ij}=\Theta y_{ij}+\Psi y_{ij}+\varphi^{n}_{ij}, \tag{19}$$

$$y^{0}_{ij}=\rho_{ij}, \tag{20}$$

$$y^{(\sigma)}\big|_{\gamma_h}=0,\;\; n>0, \tag{21}$$

where $\Theta=\Theta_1+\Theta_2$, $\Psi=\Psi_1+\Psi_2$,

$$\Theta_m y_{ij}=0.5\left(\eta_{m,ij}-|\eta_{m,ij}|\right)\xi_{m,ij}y^{(\sigma)}_{\bar{x}_m,ij}+0.5\left(\eta^{+1_m}_{m,ij}+\left|\eta^{+1_m}_{m,ij}\right|\right)\xi^{+1_m}_{m,ij}y^{(\sigma)}_{x_m,ij},$$

$$\Psi_m y_{ij}=\left(\xi_m y^{(\sigma)}_{\bar{x}_m}\right)_{x_m,ij},\quad \varphi^{n}_{ij}=f\left(x_{ij},t_{n+\sigma}\right),$$

$$\xi^{n}_{1,ij}=k\left(x_{i-\frac{1}{2},j},t_{n+\sigma}\right),\quad \xi^{n}_{2,ij}=k\left(x_{i,j-\frac{1}{2}},t_{n+\sigma}\right),\quad \eta^{n}_{m,ij}=\frac{q_m\left(x_{ij},t_{n+\sigma}\right)}{k_m\left(x_{ij},t_{n+\sigma}\right)},$$

$$y^{(\sigma)}=\sigma y^{n+1}+(1-\sigma)y^{n}, \tag{22}$$

γ_h is the set of boundary nodes of $\overline{\omega}_h$. Here we use standard notations from the theory of difference schemes.

3.4 Stability and convergence of the difference scheme

We introduce scalar products and norms:

$$(u,v)=\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}u_{ij}v_{ij}h^2,\quad (u,v]=\sum_{i=1}^{N}\sum_{j=1}^{N}u_{ij}v_{ij}h^2,$$

$$\|u\|^2=\sum_{i=0}^{N}\sum_{j=0}^{N}u_{ij}v_{ij}h^2,\quad \|\nabla u\|^2=\|u_{\bar{x}_1}\|^2+\|u_{\bar{x}_2}\|^2,$$

$$\|u_{\bar{x}_1}\|^2=\sum_{i=1}^{N}\sum_{j=0}^{N}u_{\bar{x}_1,ij}h^2,\quad \|u_{\bar{x}_2}\|^2=\sum_{i=0}^{N}\sum_{j=1}^{N}u_{\bar{x}_2,ij}h^2.$$

We prove several auxiliary lemmas.

Lemma 4 *For any function $y(t)$ defined on the grid $\overline{\omega}_{h\tau}$, the following inequality holds:*

$$y^{(\sigma)}\Delta^{\alpha}_{0t_{n+\sigma}}y\geq\frac{1}{2}\Delta^{\alpha}_{0t_{n+\sigma}}y^2.$$

This Lemma is proved similarly to Lemma 1 from [29]. Below, the letters μ with indices denote positive numbers that do not depend on h and τ.

Lemma 5 *Under the conditions (6), the following inequality holds for the solution of the difference problem (19)-(21):*

$$\Delta^{\alpha}_{0t_{n+\sigma}} \|y\|^2 + c_1 \left\|\nabla y^{(\sigma)}\right\|^2 \leq \mu_1 \left\|y^{(\sigma)}\right\|^2 + \|\varphi\|^2 .$$

Proof. Multiply the equation (19) scalarly by $y^{(\sigma)}$:

$$\left(\Delta^{\alpha}_{0t_{n+\sigma}} y,\, y^{(\sigma)}\right) = \left(\Theta y,\, y^{(\sigma)}\right) + \left(\Psi y,\, y^{(\sigma)}\right) + \left(\varphi^n,\, y^{(\sigma)}\right). \tag{23}$$

Estimate the scalar products on the left-hand side of (23) using Lemma 4:

$$\left(\Delta^{\alpha}_{0t_{n+\sigma}} y, y^{(\sigma)}\right) \geq \frac{1}{2}\Delta^{\alpha}_{0t_{n+\sigma}} \|y\|^2 . \tag{24}$$

Estimate the terms on the right-hand side of (23) as follows:

$$\left(\Theta y,\, y^{(\sigma)}\right) = \sum_{m=1}^{2} \left(\Theta_m y,\, y^{(\sigma)}\right) \leq \frac{c_2^2}{4\varepsilon c_1} \left\|\nabla y^{(\sigma)}\right\|^2 + \frac{\varepsilon c_2^2}{2c_1} \left\|y^{(\sigma)}\right\|^2 , \tag{25}$$

$$-\left(\Psi y,\, y^{(\sigma)}\right) \geq c_1 \sum_{m=1}^{2} \left(1,\, \left(y^{(\sigma)}_{\bar{x}_m}\right)^2\right] = c_1 \left\|\nabla y^{(\sigma)}\right\|^2 , \tag{26}$$

$$\left(\varphi, y^{(\sigma)}\right) \leq \frac{1}{2}\left(\|\varphi\|^2 + \left\|y^{(\sigma)}\right\|^2\right). \tag{27}$$

Taking into account (24)-(27) and choosing $\varepsilon = \dfrac{c_2^2}{2c_1^2}$, we obtain from (23):

$$\frac{1}{2}\Delta^{\alpha}_{0t_{n+\sigma}} \|y\|^2 + \frac{c_1}{2} \left\|\nabla y^{(\sigma)}\right\|^2 \leq \frac{2c_1^3 + c_2^4}{4c_1^3} \left\|y^{(\sigma)}\right\|^2 + \frac{1}{2}\|\varphi\|^2 . \tag{28}$$

Using the definition $y^{(\sigma)}$, from (28) we arrive at the statement of the lemma.

Lemma 6 *[30] Let the non-negative sequences y^n and φ^n, $n = 0, 1, 2,\ldots$ satisfy the inequality*

$$\Delta^{\alpha}_{0t_{n+\sigma}} y \leq \lambda_1 y^{n+1} + \lambda_2 y^n + \varphi^n, \quad n \geq 1,$$

where $\lambda_1 \geq 0$, $\lambda_2 \geq 0$ are some constants. Then there exists t_0 such that for $\tau \leq t_0$, the inequality holds

$$y^{n+1} \leq 2\left(y^0 + \frac{(t_n)^{\alpha}}{\Gamma(1+\alpha)} \max_{0\leq m\leq n} \varphi^m\right) E_{\alpha}\left(2\lambda (t_n)^{\alpha}\right),$$

where $\lambda = \lambda_1 + \dfrac{\lambda_2}{2 + 2^{1-\alpha}}$.

Based on Lemma 5 and Lemma 6, the following theorem is proved.

Theorem 2 *Under the conditions (6), there exists t_0 such that for $\tau \leq t_0$, the following inequality holds for the solution of the difference problem (19)-(21):*

$$\left\|y^{n+1}\right\|^2 \leq \mu_2 \left(\left\|y^0\right\|^2 + \frac{(t_n)^\alpha}{\Gamma(1+\alpha)} \max_{0\leq m\leq n} \|\varphi^m\|^2\right),$$

which implies the uniqueness of the solution and stability of the difference scheme (19)-(21) with respect to the initial data and the right-hand side.

Theorem 3 *Under the the conditions of Theorem 2, the solution of the difference problem (19)-(21) converges to the solution of the differential problem (1)-(3) and the following inequality holds:*

$$\left\|y^{n+1} - u(\cdot, t_{n+1})\right\| \leq \mu_3 \left(h^2 + \tau^2\right).$$

Proof. Consider the problem for the difference $z = y - u$:

$$\Delta^\alpha_{0t_{n+\sigma}} z_{ij} = \Theta z_{ij} + \Psi z_{ij} + \psi^n_{ij}, \quad (x,t) \in \omega_{h\tau}, \tag{29}$$

$$z^0_{ij} = 0, \tag{30}$$

$$z^{(\sigma)}\Big|_{\gamma_h} = 0, \quad n > 0, \tag{31}$$

where $\psi^n_{ij} = \varphi^n_{ij} - \Delta^\alpha_{0t_{n+\sigma}} u^n_i + \Theta u^{(\sigma)}_{ij} + \Psi u^{(\sigma)}_{ij}$. The following inequality holds for the solution of the difference problem (29)-(31):

$$\left\|z^{n+1}\right\|^2 \leq \frac{\mu_4 t^\alpha_n}{\Gamma(1+\alpha)} \max_{0\leq m\leq n} \|\psi^m\|^2, \tag{32}$$

where $\|\psi^m\| = O\left(h^2 + \tau^2\right)$. (32) yields the convergence of the solution of the difference problem (19)-(21) to the solution of the differential problem (1)-(3).

3.5 Implementation of the difference scheme

To solve the problem (19)-(21), we use the alternating directions method, which consists of two stages [31]:

$$\frac{g^{\alpha,\sigma}_0}{\tau\alpha}\left(y^{n+\frac{1}{2}}_{i,j} - y^n_{i,j}\right) + \frac{1}{\tau\alpha}\sum_{s=0}^{n-1} g^{\alpha,\sigma}_{n-s}\left(y^{s+1}_{i,j} - y^s_{i,j}\right) =$$

$$= \sigma\left(\Theta_1 + \Psi_1\right) y^{n+\frac{1}{2}}_{i,j} + (1-\sigma)\left(\Theta_2 + \Psi_2\right) y^n_{i,j}, \tag{33}$$

$$\frac{g^{\alpha,\sigma}_0}{\tau\alpha}\left(y^{n+1}_{i,j} - y^{n+\frac{1}{2}}_{i,j}\right) + \frac{1}{\tau\alpha}\sum_{s=0}^{n-1} g^{\alpha,\sigma}_{n-s}\left(y^{s+1}_{i,j} - y^s_{i,j}\right) =$$

$$= \sigma\left(\Theta_1 + \Psi_1\right) y^{n+\frac{1}{2}}_{i,j} + (1-\sigma)\left(\Theta_2 + \Psi_2\right) y^{n+1}_{i,j}. \tag{34}$$

On each time layer, the solution of the problem (19)-(21) is reduced to a sequential solution of tridiagonal systems of equations, which are solved by the Thomas algorithm. By checking directly, one can make sure that the stability condition of the Thomas algorithm holds. To check the accuracy of the difference scheme (19)-(21), a number of computational experiments were performed on the example of two test problems.

4 Results and discussion

Problem 1. Consider the equation

$$\partial_{0t}^{\alpha} u = \frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} + \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + f(x_1, x_2, t) \tag{35}$$

with the right-hand side

$$f(x_1, x_2, t) = -\frac{3}{\alpha^3} \sin^2(\pi x_1) \sin^2(\pi x_2) \left(2\alpha^2 \exp\left(\frac{t\alpha}{\alpha - 1}\right) - 4\alpha \exp\left(\frac{t\alpha}{\alpha - 1}\right) + \right.$$
$$\left. +2\alpha^2 \exp\left(\frac{t\alpha}{\alpha - 1}\right) - t^2\alpha^2 - 2t\alpha^2 - 2\alpha^2 + 2ta + 4\alpha - 2 \right) +$$
$$+4\pi^2 t^3 \sin^2(\pi x_1) \sin^2(\pi x_2) - 2\pi t^3 \cos(\pi x_1) \sin(\pi x_1) \sin^2(\pi x_2) -$$
$$-2\pi^2 t^3 \cos^2(\pi x_1) \sin^2(\pi x_2) - 2\pi t^3 \sin^2(\pi x_1) \cos(\pi x_2) \sin(\pi x_2) -$$
$$-2\pi^2 t^3 \sin^2(\pi x_1) \cos^2(\pi x_2)$$

and homogeneous initial and boundary conditions.

The exact solution to this problem is as follows:

$$u(x_1, x_2, t) = t^3 \sin^2(\pi x_1) \sin^2(\pi x_2).$$

When analyzing the dependence of the error order on the spatial step, the value of the time step is selected as $\tau = 10^{-5}$. The step value for the spatial variable h varied between $h = 10^{-2}$ and $h = 10^{-5}$.

The error value was determined by the formula

$$E = \max_{0 \le n \le M} \max_{0 \le i \le N} \max_{0 \le j \le N} \left| y_{ij}^n - u(ih, jh, t_n) \right|.$$

When analyzing the dependence of the error order on the time step, the value of the spatial step is selected as $h = 10^{-4}$. The value of the time step varied between $\tau = 10^{-5}$ and $\tau = 10^{-8}$. The order of the fractional derivative is set to $\alpha = 0.3$, $\alpha = 0.45$ and $\alpha = 0.85$.

Tables 1 and 2 show error values for various values of the parameters σ, h and τ.

Table 1: Error analysis for Problem 1

	$\sigma = 0.85$ ($\alpha = 0.3$)	$\sigma = 0.775$ ($\alpha = 0.45$)	$\sigma = 0.575$ ($\alpha = 0.85$)
$h = 1/100$	$1.582643 \cdot 10^{-7}$	$1.535625 \cdot 10^{-7}$	$5.953420 \cdot 10^{-8}$
$h = 1/500$	$4.543282 \cdot 10^{-9}$	$6.625594 \cdot 10^{-9}$	$2.655208 \cdot 10^{-9}$
$h = 1/1000$	$1.683145 \cdot 10^{-9}$	$1.659836 \cdot 10^{-9}$	$8.956221 \cdot 10^{-10}$
$h = 1/2000$	$5.325643 \cdot 10^{-10}$	$4.859264 \cdot 10^{-10}$	$6.958645 \cdot 10^{-10}$
$h = 1/5000$	$2.546234 \cdot 10^{-10}$	$2.654822 \cdot 10^{-10}$	$5.659750 \cdot 10^{-10}$
$h = 1/10000$	$8.203144 \cdot 10^{-11}$	$2.659820 \cdot 10^{-10}$	$3.659504 \cdot 10^{-10}$

Table 2: Error analysis for Problem 1

	$\sigma = 0.85$ ($\alpha = 0.3$)	$\sigma = 0.775$ ($\alpha = 0.45$)	$\sigma = 0.575$ ($\alpha = 0.85$)
$\tau = 10^{-5}$	$1.625822 \cdot 10^{-9}$	$1.956430 \cdot 10^{-9}$	$8.659832 \cdot 10^{-10}$
$\tau = 10^{-6}$	$1.659832 \cdot 10^{-11}$	$8.956268 \cdot 10^{-12}$	$2.956354 \cdot 10^{-12}$
$\tau = 10^{-7}$	$8.659825 \cdot 10^{-14}$	$4.986372 \cdot 10^{-14}$	$4.356320 \cdot 10^{-15}$
$\tau = 10^{-8}$	$5.953167 \cdot 10^{-16}$	$2.956363 \cdot 10^{-16}$	$7.923544 \cdot 10^{-18}$

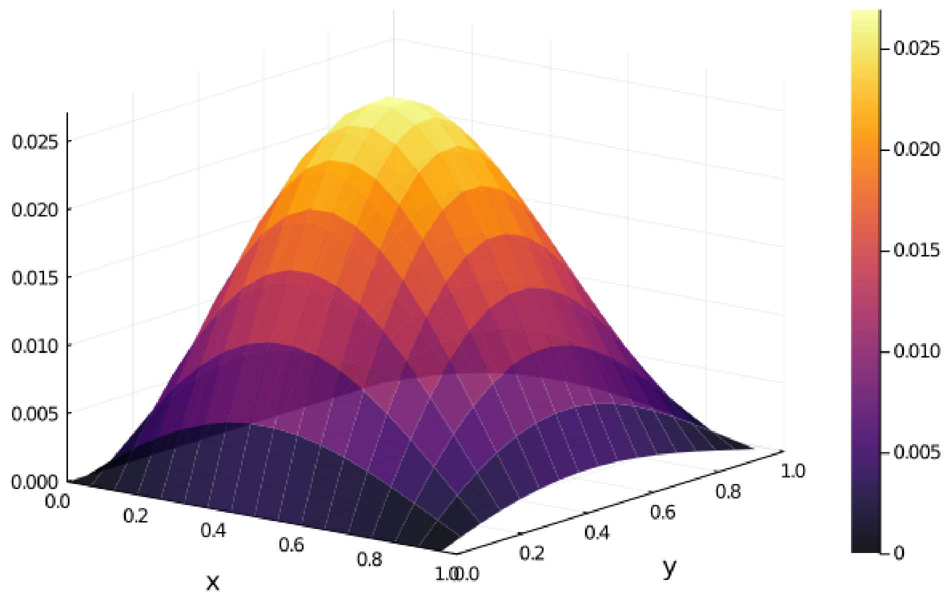

Figure 1: Solution of Problem 1, $\alpha = 0.85$, $n = 1000$

Figure 1 shows a graph of the approximate solution of the problem on time layer $n = 1000$ at $\alpha = 0.85$.

Problem 2. Consider the equation (35) with the right-hand side

$$f(x_1, x_2, t) = -\frac{2}{\alpha}\pi \sin(2\pi x_1)\sin(2\pi x_2)\left(\exp\left(\frac{t\alpha}{\alpha - 1}\right) - 1\right) +$$

$$+16\pi^3 t \sin(2\pi x_1)\sin(2\pi x_2) - 4\pi^2 t\cos(2\pi x_1)\sin(2\pi x_2) - 4\pi^2 t\sin(2\pi x_1)\cos(2\pi x_2)$$

and homogeneous initial and boundary conditions.

The exact solution to this problem is as follows:

$$u(x_1, x_2, t) = 2\pi t \sin(2\pi x_1)\sin(2\pi x_2).$$

Tables 3 and 4 show error values for various values of the parameters σ, h and τ. In Figure 2 the graph of the approximate solution of the problem on the layer $n = 1000$ at $\alpha = 0.85$ is given.

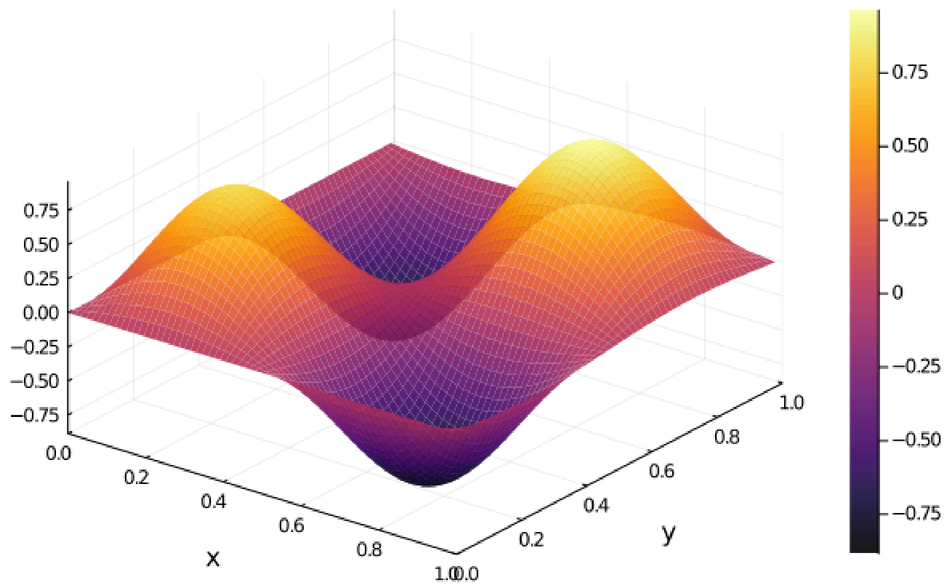

Figure 2: Solution of Problem 2, $\alpha = 0.85$, $n = 1000$

Table 3: Error analysis for Problem 2

	$\sigma = 0.85$ ($\alpha = 0.3$)	$\sigma = 0.775$ ($\alpha = 0.45$)	$\sigma = 0.575$ ($\alpha = 0.85$)
$h = 1/100$	$8.568827 \cdot 10^{-7}$	$6.884689 \cdot 10^{-7}$	$3.954957 \cdot 10^{-7}$
$h = 1/500$	$3.448455 \cdot 10^{-8}$	$2.817526 \cdot 10^{-8}$	$1.665078 \cdot 10^{-8}$
$h = 1/1000$	$8.754912 \cdot 10^{-9}$	$7.523579 \cdot 10^{-9}$	$4.796715 \cdot 10^{-9}$
$h = 1/2000$	$2.324212 \cdot 10^{-9}$	$2.378951 \cdot 10^{-9}$	$1.838244 \cdot 10^{-9}$
$h = 1/5000$	$5.345784 \cdot 10^{-10}$	$9.680494 \cdot 10^{-10}$	$1.015005 \cdot 10^{-9}$
$h = 1/10000$	$2.876746 \cdot 10^{-10}$	$7.735310 \cdot 10^{-10}$	$8.981407 \cdot 10^{-10}$

Table 4: Error analysis for Problem 2

	$\sigma = 0.85$ ($\alpha = 0.3$)	$\sigma = 0.775$ ($\alpha = 0.45$)	$\sigma = 0.575$ ($\alpha = 0.85$)
$\tau = 10^{-5}$	$2.523844 \cdot 10^{-9}$	$1.635420 \cdot 10^{-9}$	$6.623524 \cdot 10^{-10}$
$\tau = 10^{-6}$	$3.520531 \cdot 10^{-11}$	$7.435820 \cdot 10^{-12}$	$1.023465 \cdot 10^{-12}$
$\tau = 10^{-7}$	$7.025432 \cdot 10^{-14}$	$3.023564 \cdot 10^{-14}$	$3.342564 \cdot 10^{-15}$
$\tau = 10^{-8}$	$3.623501 \cdot 10^{-16}$	$3.623524 \cdot 10^{-16}$	$6.526534 \cdot 10^{-18}$

It follows from the results shown in Tables 1 and 3 that the error is a value of magnitude $O\left(h^2\right)$. Similarly, the results in Tables 2 and 4 show that the error is of magnitude $O\left(\tau^2\right)$.

Thus, computational experiments have confirmed that the difference scheme converges with the second order in both spatial and temporal variables.

5 Conclusion

Thus, an implicit finite difference scheme is constructed for a fractional differential equation with variable coefficients containing a fractional time derivative in the sense of Caputo-Fabrizio. The stability and error estimates of the difference scheme are established. The empirical convergence agrees well with the theoretical estimates.

The results obtained in this work are the basis for the construction of finite element methods for solving fluid flow problems in fractured porous media. In particular, the constructed discrete analogue of the fractional derivative in the sense of Caputo-Fabrizio will be used in subsequent works. Also, a comparison of solutions obtained using finite element and finite difference methods will be carried out. Moreover, the results obtained can be applied to the numerical solution of other equations containing a fractional time derivative.

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