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THE TWO-SIDED ESTIMATES OF THE FREDHOLM RADIUS AND COMPACTNESS CONDITIONS FOR THE OPERATOR ASSOCIATED WITH A SECOND-ORDER DIFFERENTIAL EQUATION

In this paper we consider the properties of the resolvent of a linear operator corresponding to a degenerate singular second-order differential equation with variable coefficients, considered in the Lebesgue space. The singularity of the specified differential equation means that it is defined in a noncompact domain — on the whole set of real numbers, and its coefficients are unbounded functions. The conditions for the compactness of the resolvent were obtained, as well as a double-sided estimate of its fredgolm radius. The previously known compactness conditions of the resolvent were obtained under the assumption that the intermediate term of the differential operator either is missing or, in the operator sense, is subordinate to the sum of the extreme terms. In the current paper these conditions are not met due to the rapid growth at infinity of the intermediate coefficient of the differential equation, and the minor coefficient can change sign. The property of compactness of the resolvent allows, in particular, to justify the process of finding an approximate solution of the associated equation. The Fredholm radius of a bounded operator characterizes its closeness to the Fredholm operator. The operator coefficients are assumed to be smooth functions, but we do not impose any constraints on their derivatives. The result on the invertibility of the operator and the estimation of its maximum regularity obtained by the authors earlier is essentially used in this paper.

Key words: second-order differential operator, Fredholm radius, resolvent, compactness, differential equation in an unbounded domain, differential operator with unbounded coefficients.

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Фредгольм радиусының екі жақты бағалаулары және екінші ретті дифференциалдық теңдеумен байланысты оператордың компактылығының шарттары

Бұл жұмыста Лебег кеңістігінде берілген айнымалы коэффициентті нұқсанды екінші ретті сингулярлы дифференциалдық теңдеуге сәйкес келетін сызықты оператордың резольвентасының қасиеттері зерттелген. Аталған дифференциалдық теңдеудің сингулярлы болуын оның шексіз облыста — бүкіл сан осінде — берілуі мен оның коэффициенттерінің шенелмегендігі білдіреді. Резолютентаның компактылығының шарттары, сондай-ақ оның Фредгольм радиусының екі жақты бағасы алынды. Резольвентаның компактылығының бізге бұрыннан белгілі шарттары дифференциалдық оператордың аралық мүшесі жоқ немесе ол оператор мағынасында шеткі мүшелердің қосындысына бағынады деген болжамда алынған. Ал бұл жұмыста дифференциалдық теңдеудің аралық коэффициентінің шексіз алыс нүкте аймағында жылдам өсуі мен төменгі коэффициенттің таңбасы өзгермелі болуына байланысты аталған шарттар орындалмайды. Резольвентаның компактылық қасиетінің болуы, мысалы, онымен байланысты теңдеудің жуықталған шешімін табу процесін негіздеуге мүмкіндік береді. Шенелген оператордың Фредгольм радиусы оның фредгольмдік операторларға жақындығын сипаттайды. Оператордың коэффициенттері тегіс функциялар деп есептеледі, бірақ біз олардың туындыларына ешқандай шектеу қоймаймыз. Жұмыста авторлар өздері осыған дейін алған оператордың қайтарымдылығы жайлы нәтиже мен оның максималды регулярлығының бағасына сүйенеді.

Түйін сөздер: екінші ретті дифференциалдық оператор, Фредгольм радиусы, резольвента, компактылық, шенелмеген облыстағы дифференциалдық теңдеу, коэффициенттері шенелмеген дифференциалдық теңдеу.

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Двухсторонние оценки радиуса фредгольмовости и условия компактности оператора, связанного с дифференциальным уравнением второго порядка

В настоящей работе изучаются свойства резольвенты линейного оператора, соответствующего вырожденному сингулярному дифференциальному уравнению второго порядка с переменными коэффициентами, рассматриваемому в пространстве Лебега. Сингулярность указанного дифференциального уравнения означает, что он задан в некомпактной области — на всей числовой оси, а его коэффициенты являются неограниченными функциями. Получены условия компактности резольвенты, а также двусторонняя оценка его радиуса фредгольмовости. Ранее известные условия компактности резольвенты были получены в предположении, что промежуточный член дифференциального оператора либо отсутствует, либо в операторном смысле подчиняется сумме крайних членов. В настоящей работе эти условия не выполняются вследствие быстрого роста на бесконечности промежуточного коэффициента дифференциального уравнения, а младший коэффициент может менять знак. Наличие свойства компактности резольвенты позволяет, в частности, обосновать процесс нахождения приближенного решения связанного с нею уравнения. Радиус фредгольмовости ограниченного оператора характеризует его близость к фредгольмовому оператору. Коэффициенты оператора предполагаются гладкими функциями, однако мы не накладываем какие-либо ограничения на их производные. В работе существенно использован результат об обратимости этого оператора и оценка его максимальной регулярности, полученные авторами ранее.

Ключевые слова: дифференциальный оператор второго порядка, радиус фредгольмовости, резольвента, компактность, дифференциальное уравнение в неограниченной области, дифференциальный оператор с неограниченными коэффициентами.

1 Introduction

For the Sturm-Liouville operator

$$Ly = -y'' + q(x)y,\ q \geqslant 1,\ x \in \mathbb{R} = (-\infty, +\infty)$$

the following fact proven by Molchanov [1] is known: for the inverse operator L^{-1} to be compact in the space $L_2(\mathbb{R})$ it is necessary and sufficient that for each $d > 0$ the coefficient $q(x)$ satisfies the condition

$$\lim_{|x|\to+\infty} \int\limits_{x-d}^{x+d} q(t)dt = +\infty.$$

This result is generalized in [2, 3] for rather general classes of elliptic operators with non-smooth and oscillating coefficients. In the case of non self-adjoint operators, such results were obtained using the properties that are inherent to semi-bounded self-adjoint operators. However, these results have been established in cases when the intermediate term of a second-order differential operator is either equal to zero or in the operator sense is subordinate to the sum of the extreme terms. The mentioned requirements are not satisfied when the intermediate coefficient grows strongly (see [4,5]), this case is considered in the current paper.

Fredholm operators are extensively studied and have a rich theory similar to the theory of second kind integral operators. One of the characteristics of the adjacency of a given bounded operator to a Fredholm operator is its so-called Fredholm radius [6]. In this paper we give an estimate of the Fredholm radius as well as a criterion for compactness of the resolvent of one class of second-order differential equations with a fast-growing intermediate coefficient. We will essentially use the maximum regularity estimate of the solution of this equation, which was established by us in [7].

Consider the equation

$$l_0 y = -y'' + ry' + sy = f(x), \tag{1}$$

where $x \in \mathbb{R}$, r is a continuously differentiable function, and s is a continuous function, $f \in L_p = L_p(\mathbb{R})$. Let $\widetilde{Q} \subseteq \mathbb{R}$. We denote $C_0^{(k)}(\widetilde{Q})$ $(k = 1, 2, \dots)$ as a set of k-times continuously differentiable functions with compact support in $\widetilde{Q}$. Let $D(l_0) = C_0^{(2)}(\mathbb{R})$ and l is a closure of the operator l_0 in the norm of L_p. We call a function $y \in D(l)$ such that $ly = f$ as a solution of the equation (1). It follows that the unique solvability of the equation (1) is equivalent to the bounded invertibility of the operator l.

2 Material and methods

We base on the maximum regularity estimates (2) below obtained in the Lemma 2. The embedding and compactness theorems of weighed functional classes of the Sobolev type are used. The properties of the average M. Otelbaev function are also used in obtaining the necessary and sufficient condition for the discreteness of the spectrum of the operator l. In addition, Hardy-type integral inequalities on the real axis and semi-axis are used to estimate the norm of the element from the domain of the operator l with singular weight.

3 Auxiliary statements

Let $g(x)$ and $h(x) \neq 0$ be some real continuous functions, $q = \frac{p}{p-1}$ and $\|\cdot\|_p$ is the norm of L_p. Let us introduce the following notations

$$\alpha_{g,h}(t) = \|g\|_{L_p(0,t)} \left\|h^{-1}\right\|_{L_q(t,+\infty)} \ (t > 0),$$

$$\beta_{g,h}(\tau) = \|g\|_{L_p(\tau,0)} \left\|h^{-1}\right\|_{L_q(-\infty,\tau)} \ (\tau < 0),$$

$$\alpha_{g,h} = \sup_{t>0} \alpha_{g,h}(t), \ \beta_{g,h} = \sup_{\tau<0} \beta_{g,h}(\tau), \ \gamma_{g,h} = \max\left(\alpha_{g,h}, \beta_{g,h}\right).$$

In [7] the following statements were proved.

Lemma 1 *If $g(x)$ and $h(x) \neq 0$ are continuous functions with $\gamma_{g,h} < +\infty$, then*

$$\int_{-\infty}^{+\infty} |g(x)y(x)|^p \, dx \leqslant C \int_{-\infty}^{+\infty} |h(x)y'(x)|^p \, dx, \quad \forall y \in C_0^{(1)}(\mathbb{R}).$$

Moreover, if C is the smallest constant for which this inequality holds, then

$$\left(\min\left(\alpha_{g,h}, \beta_{g,h}\right)\right)^p \leqslant C \leqslant \left(p^{\frac{1}{p}} q^{\frac{1}{q}} \gamma_{g,h}\right)^p.$$

Lemma 2 *Let $1 < p < +\infty$, $r(x) \geqslant 1$ be continuously differentiable function, $s(x)$ be continuous function such that $\gamma_{1,r^{1/p}} < +\infty$ and $\gamma_{s,r} < +\infty$. If there exist $C_1 > 1$ such that*

$$C_1^{-1} \leqslant \frac{r(x)}{r(\nu)} \leqslant C_1, \ x, \nu \in \mathbb{R} : |x - \nu| \leqslant 1,$$

then for any $f \in L_p$, the equation (1) has a unique solution $y(x)$, which satisfies the following inequality:

$$\|y''\|_p + \|ry'\|_p + \|sy\|_p \leqslant C_2 \|f\|_p, \tag{2}$$

where C_2 depends only on $\gamma_{s,r}$ and p.

4 Main results

4.1 Estimation of the Fredholm radius of a degenerate operator in L_p

Definition 1 *We call the following value as the Fredholm radius of the bounded operator $A : L_p \to L_p$*

$$\rho_A = \left[\inf_{T \in \sigma_\infty(L_p)} \|A - T\|_{L_p \to L_p} \right]^{-1},$$

where σ_∞ is the set of all compact operators in L_p.

Let $\mathbb{R}^+ = (0, +\infty)$, $\mathbb{R}^- = (-\infty, 0)$. If $\gamma_{1,r} < +\infty$ then according to the Theorem 2 [10] $\|ry'\|_{L_p(\mathbb{R}^+)}$ and $\|ry'\|_{L_p(\mathbb{R}^-)}$ are the norms.

Let X and Y are normed spaces. The transformation $E : X \to Y$ that matches each element $a \in X$ with the same element from space Y is called an embedding operator. By $H_p(r, \mathbb{R}^+)$ we denote the completion of the set $C_0^{(2)}(\mathbb{R}^+)$ by a norm $\|u\|_{H_p(r,\mathbb{R}^+)} = \|u''\|_{L_p(\mathbb{R}^+)} + \|ru'\|_{L_p(\mathbb{R}^+)}$.

The following lemma is a special case of the Theorem 3 [8].

Lemma 3 *Let the function $Q(x) \geqslant \delta > 0$ be continuous on $\mathbb{R}^+$ and there exists a positive constant C such that $C^{-1} \leqslant \frac{Q(x)}{Q(\nu)} \leqslant C, x, \nu \in \mathbb{R}^+ : |x - \nu| \leqslant 1$. Suppose that the embedding operator $E_+ : H_p(Q, \mathbb{R}^+) \to L_p(\mathbb{R}^+)$ is bounded. Then for the Fredholm radius ρ_{E_+} of the operator E_+ the following estimates hold*

$$C_1^{-1} \leqslant \rho_{E_+} \gamma_+ \leqslant C_1,$$

where $\gamma_+ = \lim\limits_{t \to +\infty} \alpha_{1,Q}(t)$, and $C_1 > 1$ does not depend on $Q(x)$.

The following statement holds.

Lemma 4 *Let the function $Q(x) \geqslant \delta > 0$ be continuous on $\mathbb{R}^-$ and there exists a positive constant C such that $C^{-1} \leqslant \frac{Q(x)}{Q(\nu)} \leqslant C, x, \nu \in \mathbb{R}^- : |x - \nu| \leqslant 1$. Suppose that the embedding operator $E_- : H_p(Q, \mathbb{R}^-) \to L_p(\mathbb{R}^-)$ is bounded. Then for the Fredholm radius ρ_{E_-} of the operator E_- the following estimates hold*

$$C_2^{-1} \leqslant \rho_{E_-} \gamma_- \leqslant C_2,$$

where $\gamma_- = \lim\limits_{\tau \to -\infty} \beta_{1,Q}(\tau)$, and $C_2 > 1$ does not depend on $Q(x)$.

Proof. Let $u \in H_p(Q, \mathbb{R}^-)$ thus $E_- u \in L_p(\mathbb{R}^-)$. Suppose $u_1(x) = u(-x) \in H_p(Q, \mathbb{R}^+)$ then $E_- u = E_+ u_1$. Let $T_1 \in \sigma_\infty(L_p(\mathbb{R}^-))$, $T \in \sigma_\infty(L_p(\mathbb{R}^+))$ and $T_1 u = T u_1$. Then $(E_- - T_1)u = (E_+ - T)u_1$. Further

$$\|E_- - T_1\|_{L_p(\mathbb{R}^-)} = \sup_{u\neq 0} \frac{\|(E_- - T_1)\, u\|_{L_p(\mathbb{R}^-)}}{\|u\|_{L_p(\mathbb{R}^-)}} = \sup_{u_1\neq 0} \frac{\left[\int\limits_{-\infty}^{0} |(E_- - T_1)\, u_1(-x)|^p\, dx\right]^{\frac{1}{p}}}{\left[\int\limits_{-\infty}^{0} |u_1(-x)|^p\, dx\right]^{\frac{1}{p}}} =$$

$$= \sup_{u_1\neq 0} \frac{\left[\int\limits_{0}^{+\infty} |(E_+ - T)\, u_1(x)|^p\, dx\right]^{\frac{1}{p}}}{\left[\int\limits_{0}^{+\infty} |u_1(x)|^p\, dx\right]^{\frac{1}{p}}} = \sup_{u_1\neq 0} \frac{\|(E_+ - T)\, u_1\|_{L_p(\mathbb{R}^+)}}{\|u_1\|_{L_p(\mathbb{R}^+)}} = \|E_+ - T\|_{L_p(\mathbb{R}^+)} .$$

That means $\rho_{E_+} = \rho_{E_-}$, it follows by the Lemma 3 that $C_2^{-1} \leqslant \rho_{E_-} \lim\limits_{t\to+\infty} \alpha_{1,Q(-x)}(t) \leqslant C_2$. Moreover

$$\lim_{t\to+\infty} \alpha_{1,Q(-x)}(t) = \lim_{t\to+\infty} t^{\frac{1}{p}} \left(\int\limits_{t}^{+\infty} \frac{dx}{|Q(x)|^q}\right)^{\frac{1}{q}} = \lim_{\tau\to-\infty} (-\tau)^{\frac{1}{p}} \left(\int\limits_{-\infty}^{\tau} \frac{dx}{|Q(x)|^q}\right)^{\frac{1}{q}} = \gamma_- .$$

The lemma is proved.

Here is another lemma.

Lemma 5 *[9] Let* $U = \bigcup\limits_{k=1}^{n} U_k$, $V = \bigcup\limits_{k=1}^{n} V_k$ *are unions of mutually disjoint intervals and an operator* $T = \sum\limits_{k=1}^{n} T_k$ *is such that* $T_k : L_p(U_k) \to L_q(V_k)$, $k = \overline{1,n}$ *and* $T : L_p(U) \to L_q(V)$, $0 < p \leqslant q < +\infty$. *Then*

$$\|T\|_{L_p(U)\to L_q(V)} = \sup_{1\leqslant k\leqslant n} \|T_k\|_{L_p(U_k)\to L_q(V_k)}.$$

The main result of this section is the following statement.

Theorem 1 *Let the functions* r, s *satisfy the conditions of Lemma 2. Then there exists a constant* C_3 *that for the Fredholm radius* $\rho_{l^{-1}}$ *of the inverse to* l *operator* l^{-1} *the following estimates hold*

$$C_3^{-1} \leqslant \rho_{l^{-1}} \gamma_{1,r} \leqslant C_3. \tag{3}$$

Proof. By $H_p(r, \mathbb{R})$ we denote the completion of the set $C_0^{(2)}(\mathbb{R})$ by a norm $\|u\|_{H_p(r,\mathbb{R})} = \|u''\|_{L_p(\mathbb{R})} + \|ru'\|_{L_p(\mathbb{R})}$. Let $f \in L_p(\mathbb{R})$, we denote

$$f_+(x) = \begin{cases} 0, & x \in (-\infty, 0), \\ f(x), & x \in [0, +\infty), \end{cases} \qquad f_-(x) = \begin{cases} f(x), & x \in (-\infty, 0), \\ 0, & x \in [0, +\infty). \end{cases}$$

Then obviously $f = f_- + f_+$. Let E be an embedding operator of the space $H_p(r,\mathbb{R})$ in $L_p(\mathbb{R})$, $T \in \sigma_\infty(L_p(\mathbb{R}))$, $(E-T)_-$ and $(E-T)_+$ be restrictions of the operator $E-T$ in the spaces $L_p(\mathbb{R}^+)$ and $L_p(\mathbb{R}^-)$ respectively. According to Lemma 5

$$\|E-T\|_{L_p(\mathbb{R})} = \max\left(\|(E-T)_-\|_{L_p(\mathbb{R}^-)}, \|(E-T)_+\|_{L_p(\mathbb{R}^+)}\right). \tag{4}$$

Hence

$$\|E-T\|_{L_p(\mathbb{R})} \leqslant \|(E-T)_-\|_{L_p(\mathbb{R}^-)} + \|(E-T)_+\|_{L_p(\mathbb{R}^+)},$$

which implies

$$\rho_E^{-1} = \inf_{T\in\sigma_\infty(L_p(\mathbb{R}))} \|E-T\|_{L_p(\mathbb{R})} \leqslant \inf_{T_-\in\sigma_\infty(L_p(\mathbb{R}^-))} \|E_- - T_-\|_{L_p(\mathbb{R}^-)} +$$

$$+ \inf_{T_+\in\sigma_\infty(L_p(\mathbb{R}^+))} \|E_+ - T_+\|_{L_p(\mathbb{R}^+)} = \rho_{E_-}^{-1} + \rho_{E_+}^{-1} \leqslant C_2\gamma_- + C_1\gamma_+. \tag{5}$$

It also follows from (4)

$$\|(E-T)_-\|_{L_p(\mathbb{R}^-)} + \|(E-T)_+\|_{L_p(\mathbb{R}^+)} \leqslant 2\|E-T\|_{L_p(\mathbb{R})},$$

which means

$$2\rho_E^{-1} \geqslant \rho_{E_-}^{-1} + \rho_{E_+}^{-1} \geqslant C_2^{-1}\gamma_- + C_1^{-1}\gamma_+.$$

An estimate follows from the last inequality and (5)

$$\widetilde{C_3}^{-1} \leqslant \rho_E\gamma_{1,r} \leqslant \widetilde{C_3}$$

According to Lemma 2, the operator l^{-1} is bounded from L_p into the space W with the norm $\|u\|_W = \|u''\|_p + \|ru'\|_p + \|su\|_p$, which coincides with $H_p(r,\mathbb{R})$ by virtue of the condition $\gamma_{s,r} < +\infty$. It is clear that l is bounded from $H_p(r,\mathbb{R})$ to L_p operator. So, l is a one-to-one relationship between $H_p(r,\mathbb{R})$ and $L_p(\mathbb{R})$. Therefore the last inequalities lead to estimates (3). The theorem is proved.

4.2 Degenerate operator resolvent compactness criterion in L_p

For a continuous function $r(x) \geqslant 1$ we denote the following notation (see [8]):

$$r^*(x) = \inf_{d>0}\left\{ d^{-1} : d^{1-p} \geqslant \int_{x-d}^{x+d} r^p(t)dt \right\}, \qquad x \in \mathbb{R}.$$

Lemma 6 *Let $r(x) \geqslant 1$ be a continuous function, and there exists a $C > 1$ such that*

$$C^{-1} \leqslant \frac{r(x)}{r(\nu)} \leqslant C \text{ for } x,\nu \in \mathbb{R} : |x-\nu| < 1. \tag{6}$$

Then there exists $C_1 > 1$ such that

$$C_1^{-1}r(x) \leqslant r^*(x) \leqslant C_1 r(x), \quad x \in \mathbb{R}.$$

Proof. Let $d_x = (r^*(x))^{-1}$. The continuity of the function $r(x)$ and the condition $r(x) \geqslant 1$ imply that

$$d_x^{-1} = r^*(x) = \left(\int\limits_{x-d_x/2}^{x+d_x/2} r^p(t)dt \right)^{\frac{1}{p-1}} \geqslant \left(\int\limits_{x-d_x/2}^{x+d_x/2} dt \right)^{\frac{1}{p-1}} = d_x^{\frac{1}{p-1}}.$$

Since $\frac{1}{p-1} > 0$, then $d_x \leqslant 1$, so the condition (6) is satisfied when $|x-\nu| < d_x$. Hence, taking into account the previous equality, we have

$$r^*(x) \geqslant \left(C^{-1} r^p(x) \int\limits_{x-d_x/2}^{x+d_x/2} dx \right)^{\frac{1}{p-1}} = C^{-\frac{1}{p-1}} r^{\frac{p}{p-1}}(x) d_x^{\frac{1}{p-1}} = C^{-\frac{1}{p-1}} r^{\frac{p}{p-1}}(x)(r^*(x))^{-\frac{1}{p-1}},$$

$$r^*(x) \leqslant \left(C r^p(x) \int\limits_{x-d_x/2}^{x+d_x/2} dx \right)^{\frac{1}{p-1}} = C^{\frac{1}{p-1}} r^{\frac{p}{p-1}}(x) d_x^{\frac{1}{p-1}} = C^{\frac{1}{p-1}} r^{\frac{p}{p-1}}(x)(r^*(x))^{-\frac{1}{p-1}}.$$

By putting $C_1 = C^{\frac{1}{p}}$, we obtain the required estimates. The lemma is proved.

Lemma 7 *Let the function $r(x) \geqslant 1$ be continuous and satisfy the condition*

$$C^{-1} \leqslant \frac{r(x)}{r(\nu)} \leqslant C \textit{ for } x, \nu \in \mathbb{R}^+ : |x-\nu| \leqslant 1,$$

and $E_+ : H_p(r, \mathbb{R}^+) \to L_p(r, \mathbb{R}^+)$ is an embedding operator. Then the operator E_+ is compact if and only if

$$\lim_{t \to +\infty} \alpha_{1,r}(t) = 0.$$

Proof. Due to the Lemma 6, the equality $\lim\limits_{t \to +\infty} \alpha_{1,r}(t) = 0$ holds if and only if $\lim\limits_{t \to +\infty} \alpha_{1,r^*}(t) = 0$. Therefore, taking into account Theorem 2 [8], we obtain the statement of the lemma. The lemma is proved.

Lemma 8 *Let the function $r(x) \geqslant 1$ be continuous and satisfy the condition*

$$C^{-1} \leqslant \frac{r(x)}{r(\nu)} \leqslant C \textit{ for } x, \nu \in \mathbb{R}^- : |x-\nu| \leqslant 1,$$

and $E_- : H_p(r, \mathbb{R}^-) \to L_p(r, \mathbb{R}^-)$ is an embedding operator. Then the operator E_- is compact if and only if

$$\lim_{\tau \to -\infty} \beta_{1,r}(\tau) = 0.$$

Proof. Let $u \in H_p(r, \mathbb{R}^-)$, and the functions $u_1(t) = u(-t)$ and $r_1(t) = r(-t)$ are defined on $\mathbb{R}^+$. Then the compactness of the embedding operator E_- is equivalent to the compactness of the operator $E_1 : H_p(r_1, \mathbb{R}^+) \to L_p(r_1, \mathbb{R}^+)$, where $E_1 u_1 = E_- u$. Due to the previous lemma, the operator E_1 is compact if and only if the following equality holds $\lim\limits_{t\to+\infty} \alpha_{1,r_1}(t) = 0$. On the other hand

$$\lim_{t\to+\infty} \alpha_{1,r_1(x)}(t) = \lim_{t\to+\infty} \beta_{1,r_1(-x)}(-t) = \lim_{\tau\to-\infty} \beta_{1,r(x)}(\tau),$$

where $\tau = -t$. The lemma is proved.

Lemma 9 *Let the function $r(x) \geqslant 1$ be continuous and satisfy the condition* (6) *on $\mathbb{R}$, and $E : H_p(r, \mathbb{R}) \to L_p(r, \mathbb{R})$ is an embedding operator. Then E is compact if and only if*

$$\lim_{t\to+\infty} \alpha_{1,r}(t) = 0, \quad \lim_{\tau\to-\infty} \beta_{1,r}(\tau) = 0. \tag{7}$$

Proof. We denote

$$r_+(x) = \begin{cases} 0, & x \in (-\infty, 0), \\ r(x), & x \in [0, +\infty), \end{cases} \qquad r_-(x) = \begin{cases} r(x), & x \in (-\infty, 0), \\ 0, & x \in [0, +\infty). \end{cases}$$

Then $r(x) = r_-(x) + r_+(x)$ and $\|r\|_{L_p(\mathbb{R})} = \|r_-\|_{L_p(\mathbb{R}^-)} + \|r_+\|_{L_p(\mathbb{R}^+)}$. Let us prove that embedding E is compact if and only if the following embedding operators are compact: $E_- : H_p(r, \mathbb{R}^-) \to L_p(\mathbb{R}^-)$ and $E_+ : H_p(r, \mathbb{R}^+) \to L_p(\mathbb{R}^+)$.

Let E_- and E_+ be compact. Consider the function $u_1 \in H_p(r, \mathbb{R}^+)$. According to the Lemma 1

$$\|u_1\|_{L_p(N,+\infty)} \leqslant pq^{\frac{p}{q}} \left(\sup_{t\geqslant N} \alpha_{1,r}(t)\right)^p \|ru_1'\|_{L_p(N,+\infty)} \leqslant C \left(\sup_{t\geqslant N} \alpha_{1,r}(t)\right)^p \|u_1\|_{H_p(r,\mathbb{R}^+)}.$$

Applying the first condition in (7), we obtain for any $u_1 \in H_p(r, \mathbb{R}^+)$

$$\lim_{N\to+\infty} \|u_1\|_{L_p(N,+\infty)} = 0.$$

Similarly, considering the function $u_2 \in H_p(r, \mathbb{R}^-)$ and using the second condition in (7), we obtain

$$\lim_{N\to-\infty} \|u_2\|_{L_p(-\infty,N)} = 0.$$

Therefore, due to the Frechet-Kolmogorov theorem the operator E is compact.

On the other hand, if E is a compact operator, let us prove that the embedding operator E_+ is compact. Consider a Cauchy sequence $\{u_n\}_{n=1}^{+\infty}$ from the unit ball of $H_p(r, \mathbb{R}^+)$: $\{u_n\}_{n=1}^{+\infty} \subset H_p(r, \mathbb{R}^+), \|u_n\|_{H_p(r,\mathbb{R}^+)} \leqslant 1$. Since the set $C_0^{(2)}(\mathbb{R}^+)$ is dense in $H_p(r, \mathbb{R}^+)$, then without loss of generality we can assume that $u_n(x) = 0$ for $x \in (0, a)$ for some $a > 0$. Let $v_n(x) = \begin{cases} 0, & x \in (-\infty, 0], \\ u_n(x), & x \in \mathbb{R}^+. \end{cases}$

The sequence $\{v_n\}_{n=1}^{+\infty}$ is a Cauchy sequence in the $H_p(r, \mathbb{R})$ therefore it converges in the $L_p(\mathbb{R})$ since the operator E is compact. Then by construction, the sequence $\{u_n\}_{n=1}^{+\infty}$ converges in the $L_p(\mathbb{R}^+)$, hence E_+ is also compact operator.

The compactness of the operator E_- is proved similarly. The lemma is proved.

Theorem 2 *Let the conditions of Lemma 2 hold. Then the resolvent l^{-1} is compact in $L_p(\mathbb{R})$ if and only if*

$$\lim_{t\to+\infty}\alpha_{1,r}(t)=0,\qquad \lim_{\tau\to-\infty}\beta_{1,r}(\tau)=0. \tag{7}$$

Proof. According to the Lemma 2, the resolvent l^{-1} is bounded from the space $L_p(\mathbb{R})$ into the space $H_p(r,\mathbb{R})$. Due to the Lemma 9 condition (7) and compactness of the embedding operator $E: H_p(r,\mathbb{R}) \to L_p(\mathbb{R})$ are equivalent. The theorem is proved.

5 Conclusion

In the current paper we have investigated an operator l corresponding to a second-order differential equation (1) with unbounded coefficients, with an intermediate term that does not subordinate to the sum of the extreme terms. The main results of the paper are Theorem 1 on the estimation of the Fredholm radius of the resolvent l^{-1}, as well as Theorem 2, which gives the compactness conditions of the resolvent l^{-1} in L_p.

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