IRSTI 20.23.17 DOI: https://doi.org/10.26577/JMMCS.2021.v110.i2.12

A.T.Rakhymova[1*], **M.B.Gabbassov**[2], **K.M.Shapen**[1]
[1]L.N. Gumilyov Eurasian National University, Kazakhstan, Nur-Sultan
[2]System research company «Factor», Kazakhstan, Nur-Sultan
*E-mail: aigerim_rakhimova@mail.ru

FUNCTIONS IN ONE SPACE OF FOUR-DIMENSIONAL NUMBERS

For the first time, the theory of functions of four-dimensional numbers with commutative product was described in works of Abenov M.M., in which the mathematical apparatus was defined, algebraic operations and their properties were determined, functions of four-dimensional numbers, their limits, continuity and differentiability were found. The continuation was the joint work of Abenov M.M. and Gabbasov M.B., where similar anisotropic four-dimensional spaces (with the notation M2-M7) were defined, which are also commutative with zero divisors. This work is devoted to the study of functions of a four-dimensional variable, definitions and analysis of four-dimensional functions, their properties, as well as the regularity of functions. The purpose of this work is to analyze the definition of functions of four-dimensional variables of the space M5, as well as theorems on the continuity and existence of differentiability of functions of four-dimensional variables. This work is descriptive for comparing the spaces of four-dimensional numbers M5 and M3. In the article, theorems on the continuity and differentiability of functions of four-dimensional variables and their properties are proved, and the Cauchy-Riemann conditions are found. The form of trigonometric, exponential, logarithmic, exponential and power functions of four-dimensional variables is determined and the regularity of functions of M5 space is proved.
Key words: four-dimensional function, continuity, differentiability, regular function, Cauchy-Riemann condition.

А.Т.Рахымова[1*], М.Б.Габбасов[2], К.М.Шапен[1]
[1]Л.Н. Гумилев атындағы Еуразия ұлттық университеті, Қазақстан, Нұр-Сұлтан қ.
[2] «Factor» жүйелік зерттеулер компаниясы, Қазақстан, Нұр-Сұлтан қ.
*E-mail: aigerim_rakhimova@mail.ru

ТӨРТ ӨЛШЕМДІ САНДАР КЕҢІСТІГІНДЕГІ ФУНКЦИЯЛАР

Төрт өлшемді сандардың функциялар теориясы алғаш рет М.М.Абеновтың еңбектерінде сипатталған, онда математикалық аппарат анықталған, алгебралық амалдар және олардың ауыстырымдылық көбейтіндісі, қасиеттері анықталған, сонымен қатар төрт өлшемді сандардың функциялары, олардың шектері, үзіліссіздігі және дифференциалдануы зерттелді. Бұл жұмыстың жалғасы М.М. Әбенов пен М.Б. Ғаббасовтың бірлескен зетрреу мақаласы болды, ол жерде ұқсас нөлдік бөлгіштері бар коммутативті болатын анизотропты төртөлшемді кеңістіктер (M2-M7 белгісімен белгіленген) анықталған. Бұл жұмыс төртөлшемді айнымалылы функцияларын, сол функциялардың анықтамалары мен талдауларын, олардың қасиеттерін, сонымен қатар олардың регулярлығын зерттеуге арналған. Бұл жұмыстың мақсаты M5 кеңістігінің төрт өлшемді айнымалыларының функцияларының анықтамасын, сондай-ақ төрт өлшемді айнымалылар функцияларының дифференциалдануы мен үздіксіздігі туралы теоремаларды талдау болып табылады. Бұл жұмыс M5 және M3 төрт өлшемді сандарының кеңістіктерін салыстыру арқылы жүзеге асырылған. Мақалада төрт өлшемді айнымалылар функцияларының үздіксіздігі мен дифференциалдылығы, олардың қасиеттері туралы теоремалар дәлелденген, Коши-Риман шарттары анықталған. Төрт өлшемді айнымалылардың тригонометриялық, экспоненциалдық, логарифмдік, көрсеткіштік және қуаттық функциясының түрлері анықталған, M5 кеңістігінің төртөлшемді айнымалыларының функцияларының заңдылығы дәлелденген.
Түйін сөздер: төртөлшемді функция, үзіліссіздік, дифференциалдылық, регуляр функция, Коши-Риман шарты.

А.Т.Рахымова[1*], М.Б.Габбасов[2], К.М.Шапен[1]
[1]Евразийский национальный университет имени Л.Н. Гумилева, Казахстан, г. Нур-Султан
[2]Компания системных исследований «Factor», Казахстан, г. Нур-Султан
*E-mail: aigerim_rakhimova@mail.ru

ФУНКЦИИ В ОДНОМ ПРОСТРАНСТВЕ ЧЕТЫРЕХМЕРНЫХ ЧИСЕЛ

Впервые теория функций четырехмерных чисел с коммутативным произведением были описаны в работах Абенова М.М., в которых был определен математический аппарат, определены алгебраические операции и их свойства, были найдены функции четырехмерных чисел, их пределы, непрерывность и дифференцируемость. Продолжением была совместная работа Абенова М.М. и Габбасова М.Б., где были определены подобные анизотропные четырехмерные пространства (с обозначениями М2-М7), которые также являются коммутативными с делителями нуля. Данная работа посвящена изучению функций четырехмерного переменного, определений и анализа четырехмерных функций, их свойств, а также регулярности функций. Целью данной работы являются анализ определения функций четырехмерных переменных пространства М5, а также теоремы о непрерывности и существования дифференцируемости функций четырехмерных переменных. Данная работа имеет описательный характер для сравнения пространств четырехмерных чисел М5 и М3. В статье доказаны теоремы о непрерывности и дифференцируемости функций четырехмерных переменных, их свойства, а также найдены условия Коши-Римана. Определен вид тригонометрических, экспоненциальной, логарифмической, показательной и степенной функций четырехмерных переменных и доказана регулярность функций четырехмерных переменных пространства М5.

Ключевые слова: четырехмерная функция, непрерывность, дифференцируемость, регулярная функция, условие Коши-Римана.

1 Introduction

The existence of the theory of functions of four-dimensional numbers originates from the investigations of M.M. Abenov, where four-dimensional numbers, functions of four-dimensional numbers, their limit, continuity and differentiability were found [1]. In work [2], Abenov M.M. and Gabbasov M.B. identified all the existing six (M2, M3, M4, M5, M6, M7) anisotropic four-dimensional spaces, which are also associative and commutative with zero divisors. In paper [3], the space of four-dimensional numbers M5 was investigated, where algebraic operations on four-dimensional numbers were described, the eigenvalues for finding the norm were found, and the metric is defined. In this paper we study the concept of the four-dimensional function in the space M5, their continuity and differentiability, as well as analysis of their properties.

2 Material and methods

It is known from the researches of many authors [4-9] that complex analysis is an extension of real analysis, i.e. all mathematical operations, definitions, functions, their properties, differentiability and continuity are performed by analogy with real analysis. In papers [2-3], complex analysis is generalized by the analysis of functions of four-dimensional variables. Let us define functions, their properties, continuity and differentiability of functions in the four-dimensional space M5.

2.1 Functions given in the space of four-dimensional numbers

Definition 1 *A function of a four-dimensional variable of the space M5 is a mapping F of some four-dimensional number from the set D into a four-dimensional number of the set G.*

If only one value $X \in D$ corresponds to each value of $Y \in G$, then the function is called single-valued, if more than one value of Y corresponds to some X, then the function is called multivalued.

To describe the function, we use the notation $Y = F(X)$ and define functions that map four-dimensional numbers to four-dimensional numbers, that is $F : R^4 \to R^4$.

Definition 2 *Let a function $f : C \to C$ has the following property: if $f(x + yi) = c(x, y) + d(x, y)i$, then $f(x - yi) = c(x, y) - d(x, y)i$, that is, it maps complex conjugate numbers to complex conjugate numbers. Let us call such functions self-adjoint functions.*

Theorem 1 *Let the function $f(x + yi) = c(x, y) + d(x, y)i : C \to C$ be differentiable, $c(x, y) = c(x, -y)$ for $\forall (x, y) \in C$ and there is a point $(x_0, 0) \in C$ such that $d(x_0, 0) = 0$. Then it is self-adjoint.*

Proof. Since the function f is differentiable, then it satisfies the Cauchy-Riemann conditions

$$\frac{\partial c(x,y)}{\partial x} = \frac{\partial d(x,y)}{\partial y}$$

$$\frac{\partial c(x,y)}{\partial y} = -\frac{\partial d(x,y)}{\partial x}$$

and the function $f(x - yi) = c(x, -y) + d(x, -y)i$ satisfies the following conditions

$$\frac{\partial c(x,-y)}{\partial x} = -\frac{\partial d(x,-y)}{\partial y}$$

$$\frac{\partial c(x,-y)}{\partial y} = \frac{\partial d(x,-y)}{\partial x}$$

considering that $c(x, y) = c(x, -y)$ and equating the right sides, we get

$$\frac{\partial d(x,y)}{\partial y} = -\frac{\partial d(x,-y)}{\partial y}$$

$$-\frac{\partial d(x,y)}{\partial x} = \frac{\partial d(x,-y)}{\partial x}$$

that is

$$\begin{array}{l} \dfrac{\partial\left(d(x,y) + \partial d(x,-y)\right)}{\partial y} = 0 \\ \dfrac{\partial\left(d(x,y) + d(x,-y)\right)}{\partial x} = 0 \end{array}.$$

Therefore, $d(x, y) = -d(x, -y) + \text{const}$.
Substituting $x = x_0, y = 0$ we get $\text{const} = 0$ which corresponds to the equality

$$f(x, -y) = c(x, y) - d(x, y)i.$$

The theorem is proved.

Remark 1 *There is a differentiable mapping* $f : C \to a + ib = \text{const}$, *which has a generalization to the four-dimensional space M5* $F : R^4 \to (a, b, 0, 0) = \text{const}$.

Consequence 1 *Any complex polynomial with zero intercept is a self-adjoint function.*

Proof. As a point $(x_0, 0) \in C$ it is sufficient to take the point *(0,0)*.

Further, we extend the definition of a self-adjoint function to the four-dimensional space.

Let $S : R^4 \to C^4$ be a bijection assigning to each four-dimensional number (x_1, x_2, x_3, x_4) its spectrum $(\mu_1, \mu_2, \mu_3, \mu_4)$, defined in [4]

$$\mu_1 = x_1 - x_4 + (x_2 + x_3)\, i, \quad \mu_2 = x_1 - x_4 - (x_2 + x_3)\, i,$$

$$\mu_3 = x_1 + x_4 + (x_2 - x_3)\, i, \quad \mu_4 = x_1 + x_4 - (x_2 - x_3)\, i. \tag{1}$$

For each component of the spectrum μ_i, we can apply the function *f(*μ*)* and denote the resulting numbers by $f(\mu_1) = f_1 + f_2 i$, $f(\mu_2) = f_1 - f_2 i$, $f(\mu_3) = f_3 + f_4 i$, $f(\mu_4) = f_3 - f_4 i$. It is easy to understand that this number is the spectrum of some four-dimensional number $Y = (y_1, y_2, y_3, y_4)$, the elements of which are found as follows:

$$y_1 = \frac{f_1 + f_3}{2},\ y_2 = \frac{f_2 + f_4}{2},\ y_3 = \frac{f_2 - f_4}{2},\ y_4 = \frac{-f_1 + f_3}{2}. \tag{2}$$

Thus, we have defined a function $F(X)$ that assigns to each $X = (x_1, x_2, x_3, x_4) \in R^4$ the number $Y = (y_1, y_2, y_3, y_4) \in R^4$. The function defined in this way can be briefly written as

$$F(X) = S^{-1}(f(S(X)), \tag{3}$$

where $f(\mu_1, \mu_2, \mu_3, \mu_4)$ means $(f(\mu_1), f(\mu_2), f(\mu_3), f(\mu_4))$, and S^{-1} is the inverse function to S.

Theorem 2 *The function defined by equality* (3) *in the space M5 is a generalization of the functions of real and complex analysis.*

Proof.

1. Consider the real analysis case. Let $X = (x_1, 0, 0, 0) \in R^1$, then by (1)

$$\mu_1 = \mu_2 = \mu_3 = \mu_4 = x_1.$$

Applying the mapping $f(\mu) : C \to C$ we obtain

If $f(x_1 + 0i) = f_1 + f_2 i$ and $f(x_1 - 0i) = f_1 - f_2 i$, then $f(x_1) = f_1 + f_2 i = f_1 - f_2 i = f_1$ and $f_2 = 0$

$$f(\mu_1) = f(\mu_2) = f(\mu_3) = f(\mu_4) = f(x_1) = f_1.$$

By formula (2), we obtain

$y_1 = f_1,\ y_2 = y_3 = y_4 = 0$ or $Y = (f_1, 0, 0, 0)$.

2. Consider the complex analysis case. Let $X = (x_1, x_2, 0, 0) \in C$, then by (1)

$$\mu_1 = \mu_3 = x_1 + x_2 i, \mu_2 = \mu_4 = x_1 - x_2 i.$$

Applying the mapping $f(\mu) : C \to C$ and, due to the complex conjugacy of the function, we obtain

$$f(\mu_1) = f(\mu_3) = f_1 + f_2 i, f(\mu_2) = f(\mu_4) = f_1 - f_2 i.$$

By formula (3), we obtain
$y_1 = f_1, \quad y_2 = f_2, \quad y_3 = y_4 = 0$ or $Y = (f_1, f_2, 0, 0)$.
The theorem is proved.

This theorem states that if we take $(x_1, 0, 0, 0)$ as the argument of the function f, then their image will be numbers of the form $(f_1, 0, 0, 0)$ and this function coincides with the corresponding one-dimensional function. And if we take $(x_1, x_2, 0, 0)$ as the argument of the function f, then it coincides with the corresponding complex-valued function from which it generated.

Let us define the form of elementary functions in the space M5. Consider a complex exponential function that is a self-adjoint function. Let $X = (x_1, x_2, x_3, x_4)$ be a four-dimensional number. Then the spectrum of this number $S(X) = ((x_1 - x_4) + (x_2 + x_3) i, \ (x_1 - x_4) - (x_2 + x_3) i, \ (x_1 + x_4) + (x_2 - x_3) i, (x_1 + x_4) - - (x_2 - x_3) i)$. Let us apply an exponent to each component.

$$\begin{aligned} exp\left((x_1 - x_4) + (x_2 + x_3) i\right) &= exp\left(x_1 - x_4\right) cos\left(x_2 + x_3\right) + exp\left(x_1 - x_4\right) sin\left(x_2 + x_3\right) i \\ exp\left((x_1 - x_4) - (x_2 + x_3) i\right) &= exp\left(x_1 - x_4\right) cos\left(x_2 + x_3\right) - exp\left(x_1 - x_4\right) sin\left(x_2 + x_3\right) i \\ exp\left((x_1 + x_4) + (x_2 - x_3) i\right) &= exp\left(x_1 + x_4\right) cos\left(x_2 - x_3\right) + exp\left(x_1 + x_4\right) sin\left(x_2 - x_3\right) i \\ exp\left((x_1 + x_4) - (x_2 - x_3) i\right) &= exp\left(x_1 + x_4\right) cos\left(x_2 - x_3\right) - exp\left(x_1 + x_4\right) sin\left(x_2 - x_3\right) i \end{aligned}$$

consequently

$$\left\{ \begin{array}{l} f_1 = exp\left(x_1 - x_4\right) cos\left(x_2 + x_3\right) \\ f_2 = exp\left(x_1 - x_4\right) sin\left(x_2 + x_3\right) \\ f_3 = exp\left(x_1 + x_4\right) cos\left(x_2 - x_3\right) \\ f_4 = exp\left(x_1 + x_4\right) sin\left(x_2 - x_3\right) \end{array} \right. .$$

Then, by formula (1), we obtain

$$exp\left(X\right) = \frac{1}{2} \left(\begin{array}{c} exp\left(x_1 - x_4\right) cos\left(x_2 + x_3\right) + exp\left(x_1 + x_4\right) cos\left(x_2 - x_3\right) \\ exp\left(x_1 - x_4\right) sin\left(x_2 + x_3\right) + exp\left(x_1 + x_4\right) sin\left(x_2 - x_3\right) \\ exp\left(x_1 - x_4\right) sin\left(x_2 + x_3\right) - exp\left(x_1 + x_4\right) sin\left(x_2 - x_3\right) \\ -exp\left(x_1 - x_4\right) cos\left(x_2 + x_3\right) + exp\left(x_1 + x_4\right) cos\left(x_2 - x_3\right) \end{array} \right).$$

Define the logarithmic function $U(X) = Ln(X)$ through the transformation $X = exp(U)$. Let us write this formula componentwise

$$\begin{gathered} x_1 = \tfrac{1}{2}\left(exp\left(u_1 - u_4\right)cos\left(u_2 + u_3\right) + exp\left(u_1 + u_4\right)cos\left(u_2 - u_3\right)\right) \\ x_2 = \tfrac{1}{2}\left(exp\left(u_1 - u_4\right)sin\left(u_2 + u_3\right) + exp\left(u_1 + u_4\right)sinu_2 - u_3\right) \\ x_3 = \tfrac{1}{2}\left(exp\left(u_1 - u_4\right)sin\left(u_2 + u_3\right) - exp\left(u_1 + u_4\right)sin\left(u_2 - u_3\right)\right) \\ x_4 = \tfrac{1}{2}\left(-exp\left(u_1 - u_4\right)cos\left(u_2 + u_3\right) + exp\left(u_1 + u_4\right)cos\left(u_2 - u_3\right)\right) \end{gathered}$$

Hence, by simple calculations, we obtain

$$x_1 + x_4 = \frac{1}{2}\left(exp\left(u_1 - u_4\right)cos\left(u_2 + u_3\right) + exp\left(u_1 + u_4\right)cos\left(u_2 - u_3\right)\right.$$

$$\left.-exp\left(u_1 - u_4\right)cos\left(u_2 + u_3\right) + exp\left(u_1 + u_4\right)cos\left(u_2 - u_3\right)\right)$$

$$x_2 + x_3 = \frac{1}{2}\left(exp\left(u_1 - u_4\right)sin\left(u_2 + u_3\right) + exp\left(u_1 + u_4\right)sin\left(u - u_3\right)\right.$$

$$\left.+exp\left(u_1 - u_4\right)sin\left(u_2 + u_3\right) - exp\left(u_1 + u_4\right)sin\left(u_2 - u_3\right)\right)$$

$$x_1 - x_4 = \frac{1}{2}\left(exp\left(u_1 - u_4\right)cos\left(u_2 + u_3\right) + exp\left(u_1 + u_4\right)cos\left(u_2 - u_3\right)\right.$$

$$\left.+exp\left(u_1 - u_4\right)cos\left(u_2 + u_3\right) - exp\left(u_1 + u_4\right)cos\left(u_2 - u_3\right)\right)$$

$$x_2 - x_3 = \frac{1}{2}\left(exp\left(u_1 - u_4\right)sin\left(u_2 + u_3\right) + exp\left(u_1 + u_4\right)sin\left(u_2 - u_3\right)\right.$$

$$\left.-exp\left(u_1 - u_4\right)sin\left(u_2 + u_3\right) \ + exp\left(u_1 + u_4\right)sin\left(u_2 - u_3\right)\right)$$

$$x_1 + x_4 = exp\left(u_1 + u_4\right)cos\left(u_2 - u_3\right), x_2 + x_3 = exp\left(u_1 - u_4\right)sin\left(u_2 + u_3\right)$$

$$x_1 - x_4 = exp\left(u_1 - u_4\right)cos\left(u_2 + u_3\right), x_2 - x_3 = exp\left(u_1 + u_4\right)sin\left(u_2 - u_3\right)$$

$$\left(x_1 - x_4\right)^2 + \left(x_2 + x_3\right)^2 = exp\left(2(u_1 - u_4)\right)cos^2\left(u_2 + u_3\right) + exp\left(2(u_1 - u_4)\right)sin^2\left(u_2 + u_3\right)$$

$$= exp\left(2\left(u_1 - u_4\right)\right)\left(cos^2\left(u_2 + u_3\right) + sin^2\left(u_2 + u_3\right)\right) = exp\left(2\left(u_1 - u_4\right)\right)$$

$$\left(x_1 + x_4\right)^2 + \left(x_2 - x_3\right)^2 = exp\left(2(u_1 + u_4)\right)cos^2\left(u_2 - u_3\right) + exp\left(2(u_1 + u_4)\right)sin^2\left(u_2 - u_3\right)$$

$$= exp\left(2\left(u_1 + u_4\right)\right)\left(cos^2\left(u_2 - u_3\right) + sin^2\left(u_2 - u_3\right)\right) = exp\left(2\left(u_1 + u_4\right)\right)$$

Further, simplifying the above formulas, we calculate the components of the function $U = (u_1, u_2, u_{3,}u_4)$. From the relations

$$exp\left(u_1 - u_4\right) = \sqrt{\left(x_1 - x_4\right)^2 + \left(x_2 + x_3\right)^2}, exp\left(u_1 + u_4\right) = \sqrt{\left(x_1 + x_4\right)^2 + \left(x_2 - x_3\right)^2}$$

we obtain

$$u_1 = ln\sqrt[4]{\left[\left(x_1 + x_4\right)^2 + \left(x_2 - x_3\right)^2\right]\left[\left(x_1 - x_4\right)^2 + \left(x_2 + x_3\right)^2\right]}$$

$$u_4 = \frac{1}{4}ln\frac{\left(x_1 + x_4\right)^2 + \left(x_2 - x_3\right)^2}{\left(x_1 - x_4\right)^2 + \left(x_2 + x_3\right)^2}$$

From the relations

$$x_2 - x_3 = exp\left(u_1 + u_4\right)sin\left(u_2 - u_3\right)$$

$$x_1 + x_4 = exp\,(u_1 + u_4)\,cos\,(u_2 - u_3)$$

obtain

$$\frac{sin\,(u_2 - u_3)}{cos\,(u_2 - u_3)} = tg\,(u_2 - u_3) = \frac{x_2 - x_3}{x_1 + x_4}.$$

And from

$$x_2 + x_3 = exp\,(u_1 - u_4)\,sin\,(u_2 + u_3)$$

$$x_1 - x_4 = exp\,(u_1 - u_4)\,cos\,(u_2 + u_3)$$

obtain

$$\frac{sin\,(u_2 + u_3)}{cos\,(u_2 + u_3)} = tg\,(u_2 + u_3) = \frac{x_2 + x_3}{x_1 - x_4}.$$

Simplifying the expressions and due to the periodicity of the trigonometric functions, we obtain

$$u_2 = \frac{1}{2}(arctg\frac{x_2 + x_3}{x_1 - x_4} + arctg\frac{x_2 - x_3}{x_1 + x_4}) + 2\pi k, k = 0, \pm 1, \pm 2, \ldots$$

$$u_3 = \frac{1}{2}(arctg\frac{x_2 + x_3}{x_1 - x_4} - arctg\frac{x_2 - x_3}{x_1 + x_4}) + 2\pi k, k = 0, \pm 1, \pm 2, \ldots$$

Thus [12]

$$Ln\,(X) = \begin{pmatrix} ln\sqrt[4]{\left[(x_1 + x_4)^2 + (x_2 - x_3)^2\right]\left[(x_1 - x_4)^2 + (x_2 + x_3)^2\right]} \\ \frac{1}{2}\left(arctg\frac{x_2+x_3}{x_1-x_4} + arctg\frac{x_2-x_3}{x_1+x_4}\right) + 2\pi k \\ \frac{1}{2}\left(arctg\frac{x_2+x_3}{x_1-x_4} - arctg\frac{x_2-x_3}{x_1+x_4}\right) + 2\pi k \\ \frac{1}{4}ln\frac{(x_1+x_4)^2+(x_2-x_3)^2}{(x_1-x_4)^2+(x_2+x_3)^2} \end{pmatrix}$$

where $k = 0, \pm 1, \pm 2, \ldots$ It is a multivalued function, as in the complex analysis.

In a similar way, the following elementary functions can be defined

$$sin\,(X) \;= \frac{1}{2}\begin{pmatrix} sin\,(x_1 - x_4)\,ch\,(x_2{+}x_3) + sin\,(x_1 + x_4)\,ch\,(x_2{-}x_3) \\ cos\,(x_1 - x_4)\,sh\,(x_2{+}x_3) + cos\,(x_1 + x_4)\,sh\,(x_2{-}x_3) \\ cos\,(x_1 - x_4)\,sh\,(x_2{+}x_3) - cos\,(x_1 + x_4)\,sh\,(x_2{-}x_3) \\ -sin\,(x_1 - x_4)\,ch\,(x_2{+}x_3) + sin\,(x_1 + x_4)\,ch\,(x_2{-}x_3) \end{pmatrix},$$

$$cos\,(X) \;= \frac{1}{2}\begin{pmatrix} cos\,(x_1 - x_4)\,ch\,(x_2{+}x_3) + cos\,(x_1 + x_4)\,ch\,(x_2{-}x_3) \\ -sin\,(x_1 - x_4)\,sh\,(x_2{+}x_3) - sin\,(x_1 + x_4)\,sh\,(x_2{-}x_3) \\ -sin\,(x_1 - x_4)\,sh\,(x_2{+}x_3) + sin\,(x_1 + x_4)\,sh\,(x_2{-}x_3) \\ -cos\,(x_1 - x_4)\,ch\,(x_2{+}x_3) + cos\,(x_1 + x_4)\,ch\,(x_2{-}x_3) \end{pmatrix},$$

$$sh\,(X)\ =\frac{1}{2}\begin{pmatrix} sh\,(x_1-x_4)\,cos\,(x_2{+}x_3)+sh\,(x_1+x_4)\,cos\,(x_2{-}x_3)\\ ch\,(x_1-x_4)\,sin\,(x_2{+}x_3)+ch\,(x_1+x_4)\,sin\,(x_2{-}x_3)\\ ch\,(x_1-x_4)\,sin\,(x_2{+}x_3)-ch\,(x_1+x_4)\,sin\,(x_2{-}x_3)\\ -sh\,(x_1-x_4)\,cos\,(x_2{+}x_3)+sh\,(x_1+x_4)\,cos\,(x_2{-}x_3) \end{pmatrix},$$

$$ch\,(X)\ =\frac{1}{2}\begin{pmatrix} ch\,(x_1-x_4)\,cos\,(x_2{+}x_3)+ch\,(x_1+x_4)\,cos\,(x_2{-}x_3)\\ sh\,(x_1-x_4)\,sin\,(x_2{+}x_3)+sh\,(x_1+x_4)\,sin\,(x_2{-}x_3)\\ sh\,(x_1-x_4)\,sin\,(x_2{+}x_3)-sh\,(x_1+x_4)\,sin\,(x_2{-}x_3)\\ -ch\,(x_1-x_4)\,cos\,(x_2{+}x_3)+ch\,(x_1+x_4)\,cos\,(x_2{-}x_3) \end{pmatrix},$$

$$a^X=\frac{1}{2}\begin{pmatrix} a^{x_1-x_4}cos\,((x_2+x_3)\,lna)\ +a^{x_1+x_4}cos\,((x_2-x_3)\,lna)\\ a^{x_1-x_4}sin\,((x_2+x_3)\,lna)\ +a^{x_1+x_4}sin\,((x_2-x_3)\,lna)\\ a^{x_1-x_4}sin\,((x_2+x_3)\,lna)\ -a^{x_1+x_4}sin\,((x_2-x_3)\,lna)\\ -a^{x_1-x_4}cos\,((x_2+x_3)\,lna)\ +a^{x_1+x_4}cos\,((x_2-x_3)\,lna) \end{pmatrix}.$$

2.2 Continuity of four-dimensional functions in the space of four-dimensional numbers

Let $F\,(X)=(f_1,f_2,f_3,f_4)$, $G\,(X)=(g_1,g_2,g_3,g_4)$ be four-dimensional functions of the space M5. $X_0=(x_{10},x_{20},x_{30},x_{40})$ - specified four-dimensional point.

Definition 3 *A four-dimensional function $F(X)$ is called continuous at the point X_0 if for any $\varepsilon>0$ there exists a number $\delta>0$ such that, under the condition $0<\|X-X_0\|_C<\delta$, the following inequality holds*

$$\|F\,(X)-F(X_0)\|_C<\varepsilon,$$

where

$$\|X-X_0\|_C=\frac{1}{2}\sqrt{((x_1-x_1{}^0)-(x_4-x_4{}^0))^2+((x_2-x_2{}^0)+(x_3-x_3{}^0))^2}+$$

$$+\frac{1}{2}\sqrt{((x_1-x_1{}^0)+(x_4-x_4{}^0))^2+((x_2-x_2{}^0)-(x_3-x_3{}^0))^2}$$

is a spectral norm of the four-dimensional space M5 [3].

Definition 4 *The value $F(X_0)$ is called the limit of the function $F(X)$ at the point X_0 if, for any sequence of points $\left\{X^{(n)}\right\}\to X_0$, the corresponding sequence of numbers will be $F(X^{(n)})\to F(X_0)$.*

Or we can write it as follows

$$\lim_{X\to X_0}F(X)\ =F(X_0).$$

Theorem 3 *The four-dimensional function* $F(X) = (f_1, f_2, f_3, f_4)$ *is continuous at the point* $X_0 = (x_{10}, x_{20}, x_{30}, x_{40})$ *if and only if each component of the function* $F(X)$ *is continuous at the point* X_0.

Otherwise $lim_{X\to X_0} F(X) = F(X_0) = (f_1^0, f_2^0, f_3^0, f_4^0)$ *if and only if*

$$\lim_{X\to X_0} f_1 = f_1^0, \lim_{X\to X_0} f_2 = f_2^0,$$

$$\lim_{X\to X_0} f_3 = f_3^0, \lim_{X\to X_0} f_4 = f_4^0.$$

Proof There is a limit $lim_{X\to X_0} F(X) = F(X_0)$ if and only if if for any $\varepsilon > 0$ there exists $\delta > 0$, which under the condition $\|X - X_0\|_C < \delta$, the inequality $\|F(X) - F(X_0)\|_C < \varepsilon$ holds. This estimate is reduced to the following form

$$\left\|(f_1, f_2, f_3, f_4) - \left(f_1^0, f_2^0, f_3^0, f_4^0\right)\right\|_C < \varepsilon$$

$$\frac{1}{2}\sqrt{\left(f_1 - f_4 - f_1^0 + f_4^0\right)^2 + \left(f_2 + f_3 - f_2^0 - f_3^0\right)^2} +$$

$$+\frac{1}{2}\sqrt{\left(f_1 + f_4 - f_1^0 - f_4^0\right)^2 + \left(f_2 - f_3 - f_2^0 + f_3^0\right)^2} < \varepsilon.$$

$$\sqrt{\left(f_1 - f_4 - f_1^0 + f_4^0\right)^2 + \left(f_2 + f_3 - f_2^0 - f_3^0\right)^2} < 2\varepsilon$$

$$\sqrt{\left(f_1 + f_4 - f_1^0 - f_4^0\right)^2 + \left(f_2 - f_3 - f_2^0 + f_3^0\right)^2} < 2\varepsilon.$$

$$\left|f_1 - f_1^0 - f_4 + f_4^0\right| < 2\varepsilon, \left|f_2 - f_2^0 + f_3 - f_3^0\right| < 2\varepsilon,$$

$$\left|f_1 - f_1^0 + f_4 - f_4^0\right| < 2\varepsilon, \left|f_2 - f_3 - f_2^0 + f_3^0\right| < 2\varepsilon.$$

After summation and subtraction, we get

$$\left|f_1 - f_1^0\right| < 4\varepsilon, \left|f_2 - f_2^0\right| < 4\varepsilon, \left|f_3 - f_3^0\right| < 4\varepsilon, \left|f_4 - f_4^0\right| < 4\varepsilon \Longleftrightarrow$$

$$\lim_{X\to X_0} f_1 = f_1^0, \lim_{X\to X_0} f_2 = f_2^0,$$

$$\lim_{X\to X_0} f_3 = f_3^0, \lim_{X\to X_0} f_4 = f_4^0.$$

The theorem is proved.

Definition 5 A four-dimensional function $F(X)$ is called continuous in some domain $E \subset R^4$ if it is continuous at every point of this domain.

Theorem 4 Let the four-dimensional functions $F(X) = (f_1, f_2, f_3, f_4)$ and $G(X) = (g_1, g_2, g_3, g_4)$ are continuous in the domain $E \subset R^4$. Then the functions

1) $cF(X)$, $c = (c_1, c_2, c_3, c_4)$ – four-dimensional constant,

2)$F(X) \pm G(X)$,

3)$F(X) \cdot G(X)$,

4) $\frac{F(X)}{G(X)}$ for $G(X_0) \neq 0$

are also continuous in the domain $E \subset R^4$ [4, 8, 9].

Proof Let us prove, for example, 3)

According to Definition 3, for each $X_0 \in \Omega \subset R^4$, functions $F(X_0)$ and $G(X_0)$ are continuous at a given point in some domain $E \subset R^4$.

The components of the function $W(X) = F(X) \cdot G(X) = (f_1, f_2, f_3, f_4)(g_1, g_2, g_3, g_4) = (w_1, w_2, w_3, w_4)$ in the M5 space have the form

$$w_1 = f_1 g_1 - f_2 g_2 - f_3 g_3 + f_4 g_4,$$

$$w_2 = f_2 g_1 + f_1 g_2 - f_4 g_3 - f_3 g_4,$$

$$w_3 = f_3 g_1 - f_4 g_2 + f_1 g_3 - f_2 g_4,$$

$$w_4 = f_4 g_1 + f_3 g_2 + f_2 g_3 + f_1 g_4.$$

According to Theorem 2, $\lim_{X \to X_0} F(X) G(X)$ consider limit componentwise

$$\lim_{X\to X_0} w_1 = \lim_{X\to X_0} f_1 \lim_{X\to X_0} g_1 - \lim_{X\to X_0} f_2 \lim_{X\to X_0} g_2 - \lim_{X\to X_0} f_3 \lim_{X\to X_0} g_3 + \lim_{X\to X_0} f_4 \lim_{X\to X_0} g_4 =$$
$$= f_1^0 g_1^0 - f_2^0 g_2^0 - f_3^0 g_3^0 + f_4^0 g_4^0 = w_1^0,$$
$$\lim_{X\to X_0} w_2 = \lim_{X\to X_0} f_2 \lim_{X\to X_0} g_1 + \lim_{X\to X_0} f_1 \lim_{X\to X_0} g_2 - \lim_{X\to X_0} f_4 \lim_{X\to X_0} g_3 - \lim_{X\to X_0} f_3 \lim_{X\to X_0} g_4 =$$
$$= f_2^0 g_1^0 + f_1^0 g_2^0 - f_4^0 g_3^0 - f_3^0 g_4^0 = w_2^0,$$
$$\lim_{X\to X_0} w_3 = \lim_{X\to X_0} f_3 \lim_{X\to X_0} g_1 - \lim_{X\to X_0} f_4 \lim_{X\to X_0} g_2 + \lim_{X\to X_0} f_1 \lim_{X\to X_0} g_3 - \lim_{X\to X_0} f_2 \lim_{X\to X_0} g_4 =$$
$$= f_3^0 g_1^0 - f_4^0 g_2^0 + f_1^0 g_3^0 - f_2^0 g_4^0 = w_3^0,$$
$$\lim_{X\to X_0} w_4 = \lim_{X\to X_0} f_4 \lim_{X\to X_0} g_1 + \lim_{X\to X_0} f_3 \lim_{X\to X_0} g_2 + \lim_{X\to X_0} f_2 \lim_{X\to X_0} g_3 + \lim_{X\to X_0} f_1 \lim_{X\to X_0} g_4 =$$
$$= f_4^0 g_1^0 + f_3^0 g_2^0 + f_2^0 g_3^0 + f_1^0 g_4^0 = w_4^0.$$

Then

$$\lim_{X\to X_0} F(X) \cdot G(X) = \left(f_1^0, f_2^0, f_3^0, f_4^0\right)\left(g_1^0, g_2^0, g_3^0, g_4^0\right) = F(X_0) \cdot G(X_0)$$

The theorem is proved.

2.3 Differentiable functions defined in the space of four-dimensional numbers

After we have determined the continuity of four-dimensional functions in the space of four-dimensional numbers M5, we next study the differentiability.

Definition 5 *The derivative of the function* $F(x_1, x_2, x_3, x_4) = (f_1, f_2, f_3, f_4)$ *at the point* $X = (x_1, x_2, x_3, x_4)$ *is called the limit* $\lim_{\Delta X \to 0} \frac{F(X+\Delta X) - F(X)}{\Delta X}$ *, if it exists as* $\Delta X = (\Delta x_1, \Delta x_2, \Delta x_3, \Delta x_4) \to 0$ *tends to zero along any path consisting of non-degenerate points.*

Theorem 4 *Let all components of the four-dimensional function* $F(x_1, x_2, x_3, x_4) = (f_1, f_2, f_3, f_4) \in R^4$ *have continuous derivatives in a neighborhood of the point* X. *Then the necessary and sufficient conditions for the differentiability of the function* $F(x_1, x_2, x_3, x_4) = (f_1, f_2, f_3, f_4)$ *at point* X *are the following generalized Cauchy-Riemann conditions:*

$$\left\{\begin{array}{l} \frac{\partial f_1}{\partial x_1} = \frac{\partial f_2}{\partial x_2} = \frac{\partial f_3}{\partial x_3} = \frac{\partial f_4}{\partial x_4} \\ \frac{\partial f_2}{\partial x_1} = -\frac{\partial f_1}{\partial x_2} = \frac{\partial f_4}{\partial x_3} = -\frac{\partial f_3}{\partial x_4} \\ \frac{\partial f_3}{\partial x_1} = \frac{\partial f_4}{\partial x_2} = -\frac{\partial f_1}{\partial x_3} = -\frac{\partial f_2}{\partial x_4} \\ \frac{\partial f_4}{\partial x_1} = -\frac{\partial f_3}{\partial x_2} = -\frac{\partial f_2}{\partial x_3} = \frac{\partial f_1}{\partial x_4} \end{array}\right. . \tag{4}$$

Proof. Necessity. Let the derivative exists. Then it does not depend on the way $\triangle X \to 0$ tends to zero, and consider the following ways of tending the increment to zero.

Let $\triangle X = (\triangle x_1, 0, 0, 0) \to 0 = (0, 0, 0, 0)$. Then

$$\frac{dF}{dX} = \lim_{(\triangle x_1,0,0,0)\to(0,0,0,0)} \frac{(\triangle f_1(x_1,x_2,x_3,x_4), \triangle f_2(x_1,x_2,x_3,x_4), \triangle f_3(x_1,x_2,x_3,x_4), \triangle f_4(x_1,x_2,x_3,x_4))}{(\triangle x_1, 0, 0, 0)},$$

where $\triangle f_i(x_1,x_2,x_3,x_4) = f_i(x_1 + \triangle x_1, x_2, x_3, x_4) - f_i(x_1,x_2,x_3,x_4), \ i = 1,2,3,4.$

By virtue of Theorem 2 from [3] $\frac{1}{(\triangle x_1,0,0,0)} = \left(\frac{1}{\triangle x_1}, 0, 0, 0\right).$

Consequently $\frac{dF}{dX} = \lim_{\triangle x_1\to 0}\left(\frac{\triangle f_1}{\triangle x_1}, \frac{\triangle f_2}{\triangle x_1}, \frac{\triangle f_3}{\triangle x_1}, \frac{\triangle f_4}{\triangle x_1}\right) = \left(\frac{\partial f_1}{\partial x_1}, \frac{\partial f_2}{\partial x_1}, \frac{\partial f_3}{\partial x_1}, \frac{\partial f_4}{\partial x_1}\right).$

Now let us choose another way of $\triangle X$ tending to zero, namely, $\triangle X = (0, \triangle x_2, 0, 0) \to 0 = (0, 0, 0, 0)$. Then

$$\frac{dF}{dX} = \mathop{lim}_{(0,\triangle x_2,0,0)\to(0,0,0,0)} \frac{(\triangle f_1(x_1,x_2,x_3,x_4), \triangle f_2(x_1,x_2,x_3,x_4), \triangle f_3(x_1,x_2,x_3,x_4), \triangle f_4(x_1,x_2,x_3,x_4))}{(0, \triangle x_2, 0, 0)},$$

where $\triangle f_i(x_1,x_2,x_3,x_4) = f_i(x_1, x_2 + \triangle x_2, x_3, x_4) - f_i(x_1,x_2,x_3,x_4), \ i = 1,2,3,4.$

By virtue of Theorem 2 from [3] $\frac{1}{(0,\triangle x_2,0,0)} = \left(0, -\frac{1}{\triangle x_2}, 0, 0\right).$

Therefore, by the rule of multiplication of four-dimensional numbers

$$\frac{dF}{dX} = \mathop{lim}_{\triangle x_2\to 0}\left(\frac{\triangle f_2}{\triangle x_2}, -\frac{\triangle f_1}{\triangle x_2}, \frac{\triangle f_4}{\triangle x_2}, -\frac{\triangle f_3}{\triangle x_2}\right) = \left(\frac{\partial f_2}{\partial x_2}, -\frac{\partial f_1}{\partial x_2}, \frac{\partial f_4}{\partial x_2}, -\frac{\partial f_3}{\partial x_2}\right).$$

Choose the third way of $\triangle X$ tending to zero, $\triangle X = (0, 0, \triangle x_3, 0) \to 0 = (0, 0, 0, 0)$. Then

$$\frac{dF}{dX} = \mathop{lim}_{(0,0,\triangle x_3,0)\to(0,0,0,0)} \frac{(\triangle f_1(x_1,x_2,x_3,x_4), \triangle f_2(x_1,x_2,x_3,x_4), \triangle f_3(x_1,x_2,x_3,x_4), \triangle f_4(x_1,x_2,x_3,x_4))}{(0, 0, \triangle x_3, 0)},$$

where $\triangle f_i(x_1,x_2,x_3,x_4) = f_i(x_1, x_2, x_3 + \triangle x_3, x_4) - f_i(x_1,x_2,x_3,x_4), \ i = 1,2,3,4.$

By virtue of Theorem 2 from [3] $\frac{1}{(0,0,\triangle x_3,0)} = \left(0, 0, \frac{1}{\triangle x_3}, 0\right).$

Therefore, by the rule of multiplication of four-dimensional numbers

$$\frac{dF}{dX} = \mathop{lim}_{\triangle x_3\to 0}\left(\frac{\triangle f_3}{\triangle x_3}, \frac{\triangle f_4}{\triangle x_3}, -\frac{\triangle f_1}{\triangle x_3}, -\frac{\triangle f_2}{\triangle x_3}\right) = \left(\frac{\partial f_3}{\partial x_3}, \frac{\partial f_4}{\partial x_3}, -\frac{\partial f_1}{\partial x_3}, -\frac{\partial f_2}{\partial x_3}\right).$$

Finally, choose the fourth way of $\triangle X$ tending to zero, $\triangle X = (0,0,0,\triangle x_4) \to 0 = (0,0,0,0)$. Then

$$\frac{dF}{dX} = \lim_{(0,0,0,\triangle x_4)\to(0,0,0,0)} \frac{(\triangle f_1(x_1,x_2,x_3,x_4), \triangle f_2(x_1,x_2,x_3,x_4), \triangle f_3(x_1,x_2,x_3,x_4), \triangle f_4(x_1,x_2,x_3,x_4))}{(0,0,0,\triangle x_4)},$$

where $\triangle f_i(x_1,x_2,x_3,x_4) = f_i(x_1,x_2,x_3,x_4+\triangle x_4) - f_i(x_1,x_2,x_3,x_4)$, $i=1,2,3,4$.

By virtue of Theorem 2 from [3] $\frac{1}{(0,0,0,\triangle x_4)} = \left(0,0,0,-\frac{1}{\triangle x_4}\right)$.

Therefore, by the rule of multiplication of four-dimensional numbers

$$\frac{dF}{dX} = \lim_{\triangle x_4\to 0}\left(\frac{\triangle f_4}{\triangle x_4}, -\frac{\triangle f_3}{\triangle x_4}, -\frac{\triangle f_2}{\triangle x_4}, \frac{\triangle f_1}{\triangle x_4}\right) = \left(\frac{\partial f_4}{\partial x_4}, -\frac{\partial f_3}{\partial x_4}, -\frac{\partial f_2}{\partial x_4}, \frac{\partial f_1}{\partial x_4}\right).$$

Thus, we got that

$$\frac{dF}{dX} = \left(\frac{\partial f_1}{\partial x_1}, \frac{\partial f_2}{\partial x_1}, \frac{\partial f_3}{\partial x_1}, \frac{\partial f_4}{\partial x_1}\right) = \left(\frac{\partial f_2}{\partial x_2}, -\frac{\partial f_1}{\partial x_2}, \frac{\partial f_4}{\partial x_2}, -\frac{\partial f_3}{\partial x_2}\right) =$$

$$= \left(\frac{\partial f_3}{\partial x_3}, \frac{\partial f_4}{\partial x_3}, -\frac{\partial f_1}{\partial x_3}, -\frac{\partial f_2}{\partial x_3}\right) = \left(\frac{\partial f_4}{\partial x_4}, -\frac{\partial f_3}{\partial x_4}, -\frac{\partial f_2}{\partial x_4}, \frac{\partial f_1}{\partial x_4}\right).$$

Equating the components according to the rule of equality of four-dimensional numbers, we obtain (4).

Sufficiency. Let conditions (4) hold. By definition $F(X+\triangle X) - F(X) = (f_1(x_1+\triangle x_1, x_2+\triangle x_2, x_3+\triangle x_3, x_4+\triangle x_4) - f_1(x_1,x_2,x_3,x_4), \dots)$. Since the components f_i are continuously differentiable functions of four variables, for each of them we can write $f_i(x_1+\triangle x_1, x_2+\triangle x_2, x_3+\triangle x_3, x_4+\triangle x_4) - f_i(x_1,x_2,x_3,x_4) = \frac{\partial f_i}{\partial x_1}\triangle x_1 + \frac{\partial f_i}{\partial x_2}\triangle x_2 + \frac{\partial f_i}{\partial x_3}\triangle x_3 + \frac{\partial f_i}{\partial x_4}\triangle x_4 + o(\sqrt{\triangle x_1^2 + \triangle x_2^2 + \triangle x_3^2 + \triangle x_4^2})$, i=1,2,3,4. This easily implies the existence of a limit from the definition of the derivative.

The theorem is proved.

Consequence 2 *To calculate the derivative of a four-dimensional function can be used one of the following formulas:*

$$\frac{dF}{dX} = \left(\frac{\partial f_1}{\partial x_1}, \frac{\partial f_2}{\partial x_1}, \frac{\partial f_3}{\partial x_1}, \frac{\partial f_4}{\partial x_1}\right) = \left(\frac{\partial f_2}{\partial x_2}, -\frac{\partial f_1}{\partial x_2}, \frac{\partial f_4}{\partial x_2}, -\frac{\partial f_3}{\partial x_2}\right) =$$

$$= \left(\frac{\partial f_3}{\partial x_3}, \frac{\partial f_4}{\partial x_3}, -\frac{\partial f_1}{\partial x_3}, -\frac{\partial f_2}{\partial x_3}\right) = \left(\frac{\partial f_4}{\partial x_4}, -\frac{\partial f_3}{\partial x_4}, -\frac{\partial f_2}{\partial x_4}, \frac{\partial f_1}{\partial x_4}\right). \quad (5)$$

Example 1 *Find the derivative of the function X^3. By definition*

$$X^3 = \left(x_1\left(x_1^2 - 3x_2^2 - 3x_3^2 + 3x_4^2\right) + 6x_2x_3x_4, x_2\left(x_1^2 - x_2^2 - 3x_3^2 + 3x_4^2\right)\right.$$

$$\left.-6x_1x_3x_4, x_3\left(3x_1^2 - 3x_2^2 - x_3^2 + 3x_4^2\right) - 6x_1x_2x_4, x_4\left(3x_1^2 - 3x_2^2 - 3x_3^2 + x_4^2\right) + 6x_1x_2x_3\right).$$

Applying formula (5), *we obtain* $\frac{dX^3}{dX} = \left(3\left(x_1^2 - x_2^2 - x_3^2 + x_4^2\right), 3\left(x_1x_2 - x_3x_4\right),\right.$
$\left. 3\left(x_1x_3 - x_2x_4\right), 3(x_1x_4 + x_2x_3)\right) = 3 \cdot X^2.$
Find the derivative of the function $sin(X)$. *By definition*
$sin(X) = \left(\frac{1}{2}\left(sin\left(x_1 - x_4\right) ch\left(x_2{+}x_3\right) + sin\left(x_1 + x_4\right) ch\left(x_2{-}x_3\right)\right), \frac{1}{2}\left(cos\left(x_1 - x_4\right)\right.\right.$
$sh\left(x_2{+}x_3\right) + cos\left(x_1 + x_4\right) sh\left(x_2{-}x_3\right)\Big), \frac{1}{2}\Big(cos\left(x_1 - x_4\right) sh\left(x_2{+}x_3\right) - cos\left(x_1 + x_4\right)$
$\left.sh\left(x_2{-}x_3\right)\Big), \frac{1}{2}\left(-sin\left(x_1 - x_4\right) ch\left(x_2{+}x_3\right) + sin\left(x_1 + x_4\right) ch\left(x_2{-}x_3\right)\right)\right).$

Applying any one of (5) we obtain the formula
$\frac{dsin(X)}{dX} = \left(\frac{1}{2}\left(cos\left(x_1 - x_4\right) ch\left(x_2{+}x_3\right) + cos\left(x_1 + x_4\right) ch\left(x_2{-}x_3\right)\right), \frac{1}{2}\left(-sin\left(x_1 - x_4\right)\right.\right.$
$sh\left(x_2{+}x_3\right) - sin\left(x_1 + x_4\right) sh\left(x_2{-}x_3\right)\Big), \frac{1}{2}\Big(-sin\left(x_1 - x_4\right) sh\left(x_2{+}x_3\right) + sin\left(x_1 + x_4\right)$
$\left.sh\left(x_2{-}x_3\right)\Big), \frac{1}{2}\left(-cos\left(x_1 - x_4\right) ch\left(x_2{+}x_3\right) + cos\left(x_1 + x_4\right) ch\left(x_2{-}x_3\right)\right)\right) = cos\left(X\right).$

Above, we have defined an explicit formula for the four-dimensional exponent. Applying the obtained formulas for determining the derivative, it is easy to make sure that $\frac{dexp(X)}{dX} = exp(X)$.

Definition 6 *7 A four-dimensional function that has a derivative at all points of a certain domain is called regular in this domain.*

Let us now investigate the differential properties of regular functions.

Theorem 5 *Let* $F(X) = (f_1, f_2, f_3, f_4)$, $G(X) = (g_1,g_2, g_3, g_4)$ *be regular functions from the space M5,* $A = (a_1, a_2, a_3, a_4)$ *and* $B = (b_1, b_2, b_3, b_4)$ *are four-dimensional constants. Then the following equalities hold:*

$$1. \frac{d(AF(X) + BG(X))}{dX} = A\frac{dF(X)}{dX} + B\frac{dG(X)}{dX},$$

$$2. \frac{d\left(F(X) \cdot G(X)\right)}{dX} = \frac{dF(X)}{dX} \cdot G\left(X\right) + F\left(X\right) \cdot \frac{dG(X)}{dX}.$$

3. $\frac{d\left(\frac{F(X)}{G(X)}\right)}{dX} = \frac{\frac{dF(X)}{dX} \cdot G(X) - F(X) \cdot \frac{dG(X)}{dX}}{G^2(X)}$, *where* $G(X)$ *non-degenerate function.*

Proof.

1. From the definition of the derivative obviously follows, that $\frac{d(F(X)+G(X))}{dX} = \frac{dF(X)}{dX} + \frac{dG(X)}{dX}$. Applying the first equality from (4) from the definition of multiplication in the space M5, we can write
$\frac{d(AF(X))}{dX} = \Big(a_1\frac{\partial f_1}{\partial x_1} - a_2\frac{\partial f_2}{\partial x_1} - a_3\frac{\partial f_3}{\partial x_1} + a_4\frac{\partial f_4}{\partial x_1},\quad a_2\frac{\partial f_1}{\partial x_1} + a_1\frac{\partial f_2}{\partial x_1} - a_4\frac{\partial f_3}{\partial x_1} - a_3\frac{\partial f_4}{\partial x_1},$
$a_3\frac{\partial f_1}{\partial x_1} - a_4\frac{\partial f_2}{\partial x_1} + a_1\frac{\partial f_3}{\partial x_1} - a_2\frac{\partial f_4}{\partial x_1}, a_4\frac{\partial f_1}{\partial x_1} + a_3\frac{\partial f_2}{\partial x_1} + a_2\frac{\partial f_3}{\partial x_1} + a_1\frac{\partial f_4}{\partial x_1}\Big) = A\frac{dF(X)}{dX}.$
These two equalities imply the validation of the theorem.

2. The components of the function $W\left(X\right) = F(X) \cdot G(X)$ in the space M5 have the following form

$$w_1 = f_1g_1 - f_2g_2 - f_3g_3 + f_4g_4,$$

$$w_2 = f_2g_1 + f_1g_2 - f_4g_3 - f_3g_4,$$

$$w_3 = f_3g_1 - f_4g_2 + f_1g_3 - f_2g_4,$$

$$w_4 = f_4g_1 + f_3g_2 + f_2g_3 + f_1g_4.$$

Then $\frac{\partial w_1}{\partial x_1} = \frac{\partial f_1}{\partial x_1}g_1 + f_1\frac{\partial g_1}{\partial x_1} - \frac{\partial f_2}{\partial x_1}g_2 - f_2\frac{\partial g_2}{\partial x_1} - \frac{\partial f_3}{\partial x_1}g_3 - f_3\frac{\partial g_3}{\partial x_1} + \frac{\partial f_4}{\partial x_1}g_4 + f_4\frac{\partial g_4}{\partial x_1}$, and $\frac{\partial w_2}{\partial x_2} = \frac{\partial f_2}{\partial x_2}g_1 + f_2\frac{\partial g_1}{\partial x_2} + \frac{\partial f_1}{\partial x_2}g_2 + f_1\frac{\partial g_2}{\partial x_2} - \frac{\partial f_4}{\partial x_2}g_3 - f_4\frac{\partial g_3}{\partial x_2} - \frac{\partial f_3}{\partial x_2}g_4 - f_3\frac{\partial g_4}{\partial x_2}$.

Taking into account that the functions $F(X)$ and $G(X)$ satisfy the Cauchy-Riemann conditions (4), we make sure that $\frac{\partial w_1}{\partial x_1} = \frac{\partial w_2}{\partial x_2}$. In a similar way it is proved that the function $W(X)$satisfies all other equalities of (4), that is, it is a regular function. Proofs of 2 and 3 are carried out by direct verification.

The theorem is proved.

Theorem 6 *Let $F(X) = F(x_1, x_2, x_3, x_4) = (f_1, f_2, f_3, f_4)$ be a regular function in certain domain. Then, at all points of this domain, the following equalities hold:*

$$\frac{\partial^2 f_i}{\partial x_1^2} + \frac{\partial^2 f_i}{\partial x_2^2} = 0,\ \frac{\partial^2 f_i}{\partial x_1^2} + \frac{\partial^2 f_i}{\partial x_3^2} = 0,\ \frac{\partial^2 f_i}{\partial x_1^2} - \frac{\partial^2 f_i}{\partial x_4^2} = 0,$$
$$\frac{\partial^2 f_i}{\partial x_2^2} - \frac{\partial^2 f_i}{\partial x_3^2} = 0,\ \frac{\partial^2 f_i}{\partial x_2^2} + \frac{\partial^2 f_i}{\partial x_4^2} = 0,\ \frac{\partial^2 f_i}{\partial x_3^2} + \frac{\partial^2 f_i}{\partial x_4^2} = 0,$$
$$\frac{\partial^2 f_i}{\partial x_1^2} + \frac{\partial^2 f_i}{\partial x_2^2} + \frac{\partial^2 f_i}{\partial x_3^2} + \frac{\partial^2 f_i}{\partial x_4^2} = 0,\ \frac{\partial^2 f_i}{\partial x_1^2} + \frac{\partial^2 f_i}{\partial x_2^2} + \frac{\partial^2 f_i}{\partial x_3^2} + \frac{\partial^2 f_i}{\partial x_4^2} = 0,$$
$$\frac{\partial^2 f_i}{\partial x_1^2} + \frac{\partial^2 f_i}{\partial x_2^2} - \frac{\partial^2 f_i}{\partial x_3^2} - \frac{\partial^2 f_i}{\partial x_4^2} = 0.$$

where $i = 1, 2, 3, 4$.

The proof follows directly from the Cauchy-Riemann conditions.

Definition 7 *The integral of a four-dimensional regular function is the antiderivative of this function of the following form*

$$\int F(X)dX = W(X) + C$$

where $W(X)$ is any of the antiderivatives, $C = (C_1, C_2, C_3, C_4)$ is a four-dimensional arbitrary constant.

Define the basic properties of the integral [1,2]:

1. $\int (AF(X) \pm BG(X))dX = A\int F(X)dX \pm B\int G(X)dX + C,$

2. $\frac{d\left(\int F(X)dX\right)}{dX} = \int F(X)dX,$

3. $\int \frac{F(X)}{dX}dX = F(X) + C.$

Based on the definitions and properties of the derivative and antiderivative, below is presented the table of four-dimensional functions

1. Derivative

$$\frac{dC}{dX} = \theta; \frac{X^n}{dX} = \frac{X^{n-1}}{n-1}; \frac{LnX}{dX} = \frac{J_1}{X}; \frac{a^X}{dX} = a^X \ lna;$$

$$\frac{sinX}{dX} = cosX \ ; \frac{cosX}{dX} = -sinX \ ; \frac{e^X}{dX} = e^X;$$

$$\frac{tanX}{dX} = \frac{J_1}{cos^2 X}; \frac{arcsinX}{dX} = \frac{J_1}{\sqrt{J_1 - X^2}}; \frac{arctanX}{dX} = \frac{J_1}{J_1 + X^2}.$$

2. Antiderivative

$$\int \theta dX = C; \int X^n dX = \frac{X^{n+1}}{n+1} + C, \ n \neq -1; \int \frac{dX}{X} = LnX + C; \int a^X dX = \frac{a^X}{lna} + C;$$

$$\int \sin X \ dX = -\cos X \ + C; \int \cos X \ dX = \sin X \ + C; \int e^X dX = e^X + C;$$

$$\int \frac{dX}{cos^2 X} = \tan X \ + C; \int \frac{dX}{\sqrt{J_1 - X^2}} = \arcsin X \ + C; \int \frac{dX}{J_1 + X^2} = \arctan X \ + C.$$

where $\theta = (0,0,0,0)$ is four-dimensional zero, $J_1 = (1,0,0,0)$ is four-dimensional unit.

3 Conclusion

In this article, functions of a four-dimensional variable in the space M5 and their properties, as well as continuity and differentiability have been investigated. The types of elementary functions, such as sine, cosine, hyperbolic sine and cosine, exponential, logarithmic, exponential and power functions are defined using spectral values. Theorems on the continuity and differentiability of functions of a four-dimensional variable in the space M5 are proved. The regularity of functions of four-dimensional variables is proved, and the Cauchy-Riemann conditions for the differentiability are defined. This work is of an overview type and is a continuation of research paper [3]. The results of the study show that the analysis of functions of a four-dimensional variable, their properties, continuity and differentiability of functions have an analogy with the studies in work [1]. Furthermore, the obtained results show that the theory of functions of a four-dimensional variable of space M5 is a generalization of the theories of real and complex analyzes.

References

[1] Abenov M.M. Chetirehmernaya matematika. Metody i prilozheniya. Nauchnaya monographia [Four-dimensional mathematics: Methods and applications. Scientific monograph]. Almaty.: Publishing House Kazakh University, 2019. - 176.

[2] Abenov M.M., Gabbassov M.B. Anyzotropnie chetirehmernie prostranstva ili novie kvaternioni [Anisotropic four-dimensional spaces or new quaternions]. Preprint, Nur-Sultan. 2020.

[3] Rakhymova A.T., Gabbassov M.B., Shapen K.M., "On one space of four-dimensional numbers," *Journal of Mathematics, Mechanics and Computer Science (Vol 4)* (2020): 199-225.

[4] Kudryavsev L.D. Kurs matematicheskogo analyza [Mathematical Analysis Course]. – M.: Vishaya shkola, 1981. -687.

[5] Lavrent'ev M.A., Shabat B.V. Metody teorii funksii kompleksnogo peremennogo [Methods of the theory of functions of a complex variable]. M.: Nauka, 1965. -716.

[6] Bitsadze A.V. Osnovi teorii analyticheskih funksii kompleksnogo peremennogo [Fundamentals of the theory of analytic functions of a complex variable]. M.: Nauka, 1984. -280.

[7] Kolmogorov A.N., Fomin S.V. Elemenri teorii funksii i funksional'nogo analyza [Elements of function theory and functional analysis]. M.: Nauka, 1989. -624.

[8] Fikhtengol'ts G.M. Osnovi matematicheskogo analyza [The Fundamentals of Mathematical Analysis]. M.: Nauka, 1968. -441.

[9] Il'in V.A., Sadovnichii V.A., Sendov B.Kh. Matematicheskii analyz. Nachal'nii kurs [Mathematical analysis. Initial course]. M.: Izdatel'stvo MGU, 1985. -660.

[10] Polovinin E.S. Teoriya funksii kompleksnogo preremennogo: uchebnik [Theory of functions of complex variables: book]. M.: MFTI, 2014. -253.

[11] Sidorov V.Yu., Fedoryuk M.I., Shabunin M. Leksii po teorii funksii kompleksnogo peremennogo [Theory of functions of complex variables]. M.: Nauka,1982. -488.

[12] Markushevich A.I. Teotiya analyticheskih funksii [Theory of analytical functions]. M.: Nauka, 1967. -491.

[13] C. Caratheodory, *The Theory of Functions of a Complex Variable*. Vol. 1 (Chelsea publishing company, 1954), -319.

[14] M Giaquinta and G Modica, *Mathematical analysis. An introduction to functions of several variable* (Birkhauser. Boston, 2009), -348.

[15] S.L. Green, *The theory and use of the complex variable. An introduction* (New York: SIR ISAAC Pitman & Sons, 1939), -136.

[16] R. Remmert, *Theory of complex functions* (New York: Springer, 1989), - 480.