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SOLVABILITY OF A NONLINEAR INVERSE PROBLEM FOR A PSEUDOPARABOLIC EQUATION WITH P-LAPLACIAN

Inverse problems of determining the right-hand side of a differential equation arise in the mathematical modeling of many physical phenomena, when an external source or some of its parameters acting to the motion of the process are unknown or unacceptable for measurement, for example, the source is in a high-temperature environment or underground, etc. This paper deals to study the solvability of an inverse problem for a nonlinear pseudoparabolic equation (sometimes they called Sobolev-type equations) with p-Laplacian and damping term with variable exponent. The inverse problem consists of determining a coefficient of the right hand side depending only on time. An additional information for this investigated inverse problem is given as an integral overdetermination condition. Under the suitable conditions on the exponents and on the data the global and local in time existence of a weak solutions to the inverse problem are established. The existence of weak solution proved by Faedo-Galerkin method. The global and local in time a priori estimates for the Galerkin approximations are obtained. On the basis of a priori estimates and by using compactness theorems and the monotonicity method, the convergence of the Galerkin approximations to the solution of the initial inverse problem is proved.
Key words: inverse problem, pseudoparabolic equations, existence, weak solution.

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p-Лапласианды псевдопараболалық теңдеу үшін сызықты емес кері есептің шешімділігі

Дифференциалдық теңдеудің оң жағын анықтау кері есептер көп жағдайда физикалық құбылыстың қозғалысына әсер сыртқы күштер немесе олардың кейбір параметрлері белгісіз әлде өлшеуге қол жетімсіз болатын, мәселен, әсер етуші жылу көзі жоғары температуралы ортада немесе жердің астында болған үрдістерді математикалық тұрғыдан модельдеуде туындайды. Бұл ұсынылған мақалада сызықты емес p-Лапласианды және айнымалы көрсеткішті псевдопараболалық (кейбір жұмыстарда мұндай теңдеулер соболев типті теңдеулер деп аталады) теңдеу үшін кері есептің шешімділігі толыққанды зерттелінеді. Қарастырылатын кері есеп теңдеудің оң жақ бөлігіндегі тек қана уақытқа тәуелді коэффициентті анықтаудан тұрады. Бұл зерттелінетін кері есеп үшін қосымша ақпарат интегралдық қосымша шарт түрінде қойылды. Теңдеудегі көрсеткіштер мен есептің бастапқы берілгендері үшін қолайлы шарттар орындалған кездегі кері есептің әлсіз шешімдердің глобальды және локальды бар болуы көрсетілді. Шешімнің бар болуы Федо-Галеркин әдісі көмегімен дәлелденді. Галеркинге жуық шешімдер үшін уақыт бойынша глобальды және локальды априорлық бағалаулар алынды. Осы алынған априорлық бағалаулар негізінде компактылық теоремалар мен монотондық әдістерін қолдана отырып, галеркиннің жуық шешімдердің бастапқы есептің шешіміне жинақталуы дәлелденді.
Түйін сөздер: Кері есеп, псевдопараболалық теңдеу, шешімнің бар болуы, әлсіз шешім.

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Разрешимость нелинейной обратной задачи для псевдопараболического уравнения с p-Лапласианом

Обратные задачи определения правой части дифференциального уравнения возникают при математическом моделировании многих физических явлений, когда внешний источник или некоторые его параметры, влияющие на движение процесса, неизвестны или неприемлемы для измерения, например, источник находится в высокотемпературной среде или под землей и т. д. В данной работе исследуется разрешимость обратной задачи для одного нелинейного псевдопараболического уравнения (в некоторых работах такие уравнения называются уравнениями типа соболева) с p-лапласианом и демпфирующим членом с переменным показателем степени. Исследуемая обратная задача состоит в определения коэффициента правой части, зависящего только от времени. Дополнительная информация для этой исследуемой обратной задачи задается в виде интегрального условия переопределения. При подходящих условиях на показатели и на данные задачи, установлены глобальное и локальное существование слабых решений. Существование решения доказано с помощью методом Фаэдо-Галеркина. Получены глобальные и локальные по времени априорные оценки для галеркинских приближений. На основе полученных этих априорных оценок и используя теоремы компактности а также метода монотонности, доказаны сходимости галеркинских приближенных решений к решению исходной задачи.
Ключевые слова: Обратная задача, псевдопараболические уравнения, существования решения, слабое решение.

1 Statement of the problem

In this work, we consider the following nonlinear inverse problem for the pseudoparabolic equation with $p-$Laplacian diffusion and damping term with variable exponents

$$u_t - \Delta u_t - \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right) + |u|^{m(x)-2} u = f(t) \cdot g(x,t), \text{ in } Q_T, \tag{1}$$

$$u(x,0) = u_0(x) \text{ in } \Omega, \tag{2}$$

$$u(x,t) = 0 \text{ on } \Gamma_T, \tag{3}$$

$$\int\limits_{\Omega} (u \cdot \omega + \nabla u \cdot \nabla \omega)\, dx = e(t), \ \ t \geq 0. \tag{4}$$

where Ω is a bounded domain in $\mathbb{R}^d$ with smooth boundary $\partial\Omega$, and $Q_T = \{(x,t) : \ x \in \Omega, \ 0 < t \leq T\}$ is a cylinder with lateral Γ_T. The functions $g(x,t), \ u_0(x), \ \omega(x)$, and $e(t)$ are given. The exponents p is given positive number and m is given function, such that

$$1 < p < \infty, \ \ 1 < m_- \leq m(x) \leq m_+ < \infty, \ \ \forall x \in \Omega; \tag{5}$$

where

$$m_- = \inf_x m(x) \ \text{ and } \ m_+ = \sup_x m(x)$$

The inverse problem (1)-(4) consists of determining the coefficient $f(t)$ of the right hand side and a solution $u(x,t)$ from (1) and additional information (4) which given by integral overdetermination condition, and the initial-boundary conditions (2)-(3).

2 Introduction

Inverse problems of determining the right-hand side of a differential equation arise in the mathematical modeling of many physical phenomena, when an external source or some of its parameters acting to the motion of the process are unknown or unacceptable for measurement, for example, the source is in a high-temperature environment or underground, etc.

Equations like (1) with a one time derivative appearing in the highest order term are called pseudo-parabolic or Sobolev equations, and arise in many areas of mathematics and physics. For instance, they have been used, to model thermodynamics processes [19], filtration in porous media [8], and nonsteady flow of second order fluids [10], the motion of non-Newtonian fluids [3], [21], and many other physical phenomena.

In the case $p = 2$ and $m = 2$, the equation (1) becomes the classical pseudoparabolic equation. To our knowledge, the inverse problems for pseudoparabolic equations have not been studied a lot, see for classical pseudoparabolic equations [1], [7], [9], [14], [11], [16], [17,18], and for pseudoparabolic equations with p-Laplacian and other related equations [4], [2], [12,13], [20], and references therein.

Recently, Antontsev and et. in [4] have been considered the inverse problem (1)-(4) with $m = const$ and with the right-hand side $F(x,t) = f(t)\cdot(\omega(x) - \Delta\omega(x))$, where $\omega(x)$ is the same function appearing also in the overdetermination condition (4). In this paper, we consider the inverse problem (1)-(4) with variable exponent $m = m(x)$ and with the right-hand side $F(x,t) = f(t)g(x,t)$, where $g(x,t)$ is an arbitrary function in $L^\infty(0,T;L^2(\Omega))$. Under suitable assumptions on the exponents and data, we prove the global and local existence theorems as analogical results in [4].

3 Preliminaries

Let $q:\Omega\to[1,\infty]$ be a measurable function. We define the Lebesgue space with variable exponent $q(.)$ by

$$L^{q(.)}(\Omega) := \{u:\Omega\to R \ \ measurable \ \ and \ \ \int_\Omega |\lambda u(x)|^{q(x)}dx < \infty \ \ for\ some \ \ \lambda > 0\}.$$

Equipped with the following Luxembourg-type norm([22]):

$$\|u(x)\|_{q(\cdot)} := inf\{\lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{q(x)} dx \le 1\}$$

is a Banach space.

We use the classical and the following nonlinear Gronwall's inequality ([5]) to establish the first and second local estimates.

Lemma 1 *If $y:\mathbb{R}^+\longrightarrow[0,\infty)$ is a continuous function such that*

$$y(t) \le C_1\int_0^t y^\mu(s)ds + C_2, \quad t\in\mathbb{R}^+, \quad \mu > 1$$

for some positive constants C_1 *and* C_2*, then*

$$y(t) \leq C_2 \left(1 - (\mu - 1)C_1 C_2^{\mu-1} t\right)^{-\frac{1}{\mu-1}} \quad \textit{for} \;\; 0 \leq t < t_{\max} := \frac{1}{(\mu - 1)C_1 C_2^{\mu-1}}.$$

4 Weak formulation

Assume that the data of the problem satisfy the following conditions

$$u_0(x) \in H_0^1(\Omega) \cap W^{1,p}(\Omega) \cap L^{m(x)}(\Omega); \tag{6}$$

$$|g_0(t)| := \left| \int\limits_{\Omega} g(x,t)\omega(x)dx \right| \geq l_0 > 0 \text{ for all } \; t \geq 0; \tag{7}$$

$$g(x,t) \in L^{\infty}(0,T; L^2(\Omega)); \tag{8}$$

$$\omega(x) \in W^{1,p}(\Omega) \cap L^{m(x)}(\Omega) \cap W_0^{1,2}(\Omega); \tag{9}$$

$$e(t) \in W^{1,2}([0,T]), \quad \text{and} \quad \int\limits_{\Omega} u_0 \cdot \omega dx = e(0). \tag{10}$$

Lemma 2 *Under the conditions (5) and (7)-(10), the inverse problem (1)-(4) is equivalent to the following problem for a nonlinear parabolic equation containing the nonlinear nonlocal operator of the function u*

$$u_t - \Delta u_t - \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right) + |u|^{m(x)-2} u = f(t,u)g(x,t), \quad Q_T, \tag{11}$$

$$u(x,0) = u_0(x), \;\; \Omega, \tag{12}$$

$$u(x,t) = 0, \quad \Gamma_T. \tag{13}$$

Here

$$f(t,u) = \frac{1}{g_0(t)} \left(e'(t) + \int\limits_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \omega \; dx + \int\limits_{\Omega} |u|^{m(x)-2} u \cdot \omega dx \right). \tag{14}$$

The prove is analogical as in [4].

Definition 1 *A function $u(x,t)$ is a weak solution to the problem (11)-(14), if:*

1. $u \in L^\infty(0,T; W_0^{1,2} \cap W_p^1 \cap L^{m(x)}) \cap L^p(Q_T) \cap L^{m(x)}(Q_T), \quad u_t \in L^2(0,T; W_0^{1,2}(\Omega))$;
2. $u(0) = u_0$ *a.e. in* Ω;
3. *For every* $\varphi \in W_0^{1,2} \cap W_p^1 \cap L^{m(x)}(\Omega)$ *and for a.a.* $t \in (0,T)$ *holds*

$$\frac{d}{dt}\int_\Omega (u\varphi + \nabla u \cdot \nabla\varphi)\, dx + \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi dx + \int_\Omega |u|^{m(x)-2} u dx = \int_\Omega f(t,u) g\varphi dx. \quad (15)$$

5 Main result

Let be

$$u_0 \in H_0^1(\Omega) \cap W^{1,p}(\Omega) \cap L^{m(x)}(\Omega) \quad (16)$$

The following theorems are valid.

Theorema 1 *Assume that the conditions (7)-(10) and* (16) *be fulfilled and*

$$1 < p \le 2 \quad \text{and} \quad 1 < m_- \le m(x) \le m_+ \le 2, \ \forall x \in \Omega. \quad (17)$$

Then the problem (11)-(14) has at least one weak solution in the sense of definition 1 global in time.

Theorema 2 *If instead of* (17) *holds the condition*

$$1 < p < \infty \quad \text{and} \quad 1 < m_- \le m(x) \le m_+ \le 2^*, \ x \in \Omega \quad (18)$$

then the problem (11)-(14) has at least one weak solution in the sense of definition 1 local in time.

In order to prove these theorems it is enough to establish the first and second a priori estimates. After then by using these a priori estimates and the monotonicity method [6], we establish the passage to limit for Galerkin's approximation, and as a result, we get that the limit function is a weak solution to the our investigated problem.

5.1 Galerkin's approximations

Let $\{\psi_k\}_{k\in N}$ be an orthonormal family in $L^2(\Omega)$ and a linear combinations are dense in $V := W_0^{1,2} \cap W^{1,p} \cap L^{m(x)}(\Omega)$ [15]. Given $n \in N$, let us consider the n-dimensional space V^n spanned by $\psi_1, \ldots, \psi_n$. For each $n \in N$, we search for approximate solutions

$$u^n(x,t) = \sum_{j=1}^n c_j^n(t)\psi_j(x), \quad \psi_j \in V^n, \quad (19)$$

where the coefficients $c_1^n(t), \dots, c_n^n(t)$ are defined as the solutions of the following n ordinary differential equations derived from

$$\int_\Omega (u_t^n \psi_k + \nabla u_t^n \nabla \psi_k)\, dx + \int_\Omega |\nabla u^n|^{p-2} \nabla u^n \cdot \nabla \psi_k \, dx + \int_\Omega |u^n|^{m(x)-2} u^n \cdot \psi_k \, dx = f(t, u^n) \int_\Omega g \psi_k dx, \tag{20}$$

for $k = 1,\ 2,\ \dots,\ n$.

The system (20) of ODEs is supplemented with the following Cauchy data

$$u^n(0) = u_0^n \quad \text{in } \Omega. \tag{21}$$

and assume that

$$u_0^n \to u_0(x) \text{ as } n \to \infty \ \text{ in } \ W_0^{1,2} \cap W^{1,p} \cap L^{m(x)}(\Omega). \tag{22}$$

According to the general theory of nonlinear ODE, the problem (20)-(21) has a solution $c_j^n(t)$ in $[0, t_0]$, where $t_0 \in (0, T]$. The solution can be extended to $[0, T]$ by a priory estimate which we shall obtain below.

5.2 First and second a priory estimates

First priory estimate. Let be now $1 < p, \quad 2 \geq m_+ \geq m(x) \geq m_- > 1$. Multiplying (20) by $c_j^n(t)$ and summing with respect to j, from 1 to n, we have

$$\frac{1}{2} \frac{d}{dt} \left(\|u^n\|_{2,\Omega}^2 + \|\nabla u^n\|_{2,\Omega}^2 \right) + \|\nabla u^n\|_{L^p(\Omega)}^p + \int_\Omega |u^n|^{m(x)} dx = R, \tag{23}$$

where

$$R = \frac{1}{g_0(t)} \left(e'(t) + \int_\Omega |\nabla u^n|^{p-2} \nabla u^n \cdot \nabla \omega \ dx + \int_\Omega |u^n|^{m(x)-2} u^n \cdot \omega dx \right) \int_\Omega g(x,t) u^n(x) dx = \sum_{i=1}^{3} R_{1i}, \tag{24}$$

Using the Holder's and Youn's inequalities and the assumptions (17), we estimate each term on the right hand side of (23)

$$|R_{11}| = \left| \frac{1}{g_0(t)} \int_\Omega g u^n e'(t) dx \right| \leq \frac{1}{8} \|u^n\|_{2,\Omega}^2 + \frac{1}{2 l_0^2} \|g(t)\|_{2,\Omega}^2 |e'(t)|^2. \tag{25}$$

$$
\begin{aligned}
&|R_{12}| \leq \frac{1}{l_0} \|g(t)\|_{2,\Omega} \|u^n\|_{2,\Omega} \|\nabla \omega\|_{p,\Omega} \|\nabla u^n\|_{p,\Omega}^{p-1} \leq \\
&\frac{1}{2} \|\nabla u^n\|_{p,\Omega}^p + M(p) \left(\frac{1}{l_0} \|g(t)\|_{2,\Omega} \|\nabla \omega\|_{p,\Omega} \|u^n\|_{2,\Omega} \right)^p \leq \\
&\frac{1}{2} \|\nabla u^n\|_{p,\Omega}^p + \frac{1}{8} \|u^n\|_{2,\Omega}^2 + M'(t)
\end{aligned} \tag{26}
$$

where $M'(t) = \left(M(p) \frac{1}{l_0} \|g(t)\|_{2,\Omega} \|\nabla \omega\|_{p,\Omega} \right)^{\frac{2}{2-p}}$

Estimate the term R_{13} by using Holder, Young inequalities and the Poincare's and the Sobelev inequality

$$
\|u\|_{m_+,\Omega} \leq M(\Omega) \|\nabla u\|_{2,\Omega}, \text{which holds for } m_+ \leq 2^*,
$$

where $2^* = \frac{2d}{d-2}$ if $d > 2$ and $2^* \in (1, \infty)$ if $d = 2$. Using algebraic inequality $(a_1 + ... + a_k^s) \leq L(a_1^s + ... + a_k^s)$, where L=const depending only k and s.

$$
\begin{aligned}
&|R_{13}| \leq \frac{1}{l_0} \|u^n\|_{2,\Omega} \|g\|_{2,\Omega} \|\omega\|_{m_+,\Omega} \left(\int_\Omega \left(|u^n|^{m(x)-1} \right)^{\frac{m_+}{m_+-1}} dx \right)^{\frac{m_+-1}{m_+}} \leq \\
&\frac{1}{l_0} \|u^n\|_{2,\Omega} \|g\|_{2,\Omega} \|\omega\|_{m_+,\Omega} \left[\int_{\Omega_-} |u^n|^{m_- - 1 \cdot \frac{m_+}{m_+-1}} dx + \|u^n\|_{m_+,\Omega}^{m_+} \right]^{\frac{m_+-1}{m_+}} \leq \\
&\frac{1}{l_0} \|u^n\|_{2,\Omega} \|g\|_{2,\Omega} \|\omega\|_{m_+,\Omega} \left[\|u^n\|_{m_+,\Omega}^{m_+} + M(\Omega, m_-, m_+) \|u^n\|_{m_+,\Omega}^{m_+ \cdot \frac{m_- - 1}{m_+-1}} \right]^{\frac{m_+-1}{m_+}} \leq \\
&M'' \|\nabla u^n\|_{2,\Omega} \|g\|_{2,\Omega} \|\omega\|_{m_+,\Omega} \|\nabla u^n\|_{2,\Omega}^{m_+-1} \leq M''' \|\nabla u^n\|_{2,\Omega}^{m_+}
\end{aligned} \tag{27}
$$

where

$$
\Omega_+ := \left\{ x \in \Omega : \quad |u|^{m(x)} > 1 \right\}, \quad \Omega_- := \left\{ x \in \Omega : \quad |u|^{m(x)} \leq 1 \right\},
$$

$$
M'' := \frac{1}{l_0} L \left[1 + M'(\Omega, m_-, m_+) \right], \quad M''' := M'' \sup_{t \in [0,T]} \|g\|_{2,\Omega} \|\omega\|_{m_+,\Omega}.
$$

Plugging (25), (26), (27) into (23) and using the assumption $m_+ \leq 2^*$, we obtain

$$
\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \left(1 + \|u^n\|_{2,\Omega}^2 + \|\nabla u^n\|_{2,\Omega}^2 \right) + \|\nabla u^n\|_{L^p(\Omega)}^p + \int_\Omega |u^n|^{m(x)} dx \leq \\
&M_1 \left[1 + \|u^n\|_{2,\Omega}^2 + \|\nabla u^n\|_{2,\Omega}^2 \right]^\theta + M_0,
\end{aligned} \tag{28}
$$

where

$$
\theta = \max\{1, \frac{p}{2}, \frac{m_+}{2}\}, \;\; M_0 = \frac{1}{2l_0^2} \sup_{t \in [0,T]} \left(\|g(t)\|_{2,\Omega}^2 |e'(t)|^2 + M'(t) \right), \quad M_1 = \max\{M''', \frac{1}{4}\}.
$$

Omitting second and third terms on left hand side, integrating by $\tau \in (0,t)$, we get and in case $p \leq 2,\ m_+ \leq 2$ applying Gronwall's lemma for $Y(t) := 1 + \|u^n\|^2_{2,\Omega} + \|\nabla u^n\|^2_{2,\Omega}$, we get

$$Y(t) \leq M_0 e^{M_1 T} \tag{29}$$

In case $2 < m_+$ or $2 < p$ applying generalized Gronwall's lemma for $Y(t)$, we have

$$Y(t) \leq M_0 \left(1 - (\theta-1) M_1 M_0^{\theta-1} T\right)^{-\frac{1}{\theta-1}} \quad \text{for} \ \ 0 \leq t \leq T_0 := \frac{1}{(\theta-1)M_1 M_0^{\theta-1}}. \tag{30}$$

Substituting (29) and (30) into (28), and taking supremum by t, we obtain the following first energy estimate

$$\sup_{t\in(0,T_{max}]} \left(\|u^n\|^2_{2,\Omega} + \|\nabla u^n\|^2_{2,\Omega}\right) + \|\nabla u^n\|^p_{L^p(Q_{T_{max}})} + \int\limits_\Omega |u^n|^{m(x)}\, dx \leq K_0, \tag{31}$$

where $T_{max} = T$ if (17) holds, i.e. global in time, and $T_{max} = T_0$ (18) holds, i.e. local in time.

Second priory estimate. Let us now multiply (20) by $\frac{dc_j^n}{dt}$ and sum up the result from $j = 1$ to $j = n$, and integrate by τ from 0 to $t \in [0,T])$.

Then we have

$$\begin{gathered}\int\limits_0^t \left(\|u^n_t(\tau)\|^2_{2,\Omega} + \|\nabla u^n_t(\tau)\|^2_{2,\Omega}\right) d\tau + \frac{1}{p}\|\nabla u^n\|^p_{L^p(\Omega)} + \int\limits_\Omega \frac{1}{m(x)} |u^n|^{m(x)} dx \leq \\ \frac{1}{p}\|\nabla u^n(0)\|^p_{L^p(\Omega)} + G,\end{gathered} \tag{32}$$

$$\begin{gathered}\text{where} \ \ G = \int\limits_0^t \frac{1}{g_0(\tau)} \left(e'(\tau) + \int\limits_\Omega |\nabla u^n|^{p-2}\, \nabla u^n \cdot \nabla\omega \ dx \right. \\ \left. + \int\limits_\Omega |u^n|^{m(x)-2} u^n \cdot \omega dx\right) \int\limits_\Omega g(x,\tau) u^n_t(x,\tau) dx d\tau.\end{gathered} \tag{33}$$

Let we estimate G on the right side

$$|G| \leq \frac{1}{2}\int\limits_0^t \|u^n_t\|^2_{2,\Omega}\, d\tau + \frac{1}{2l_0^2}\int\limits_0^t \|g(t)\|^2_{2,\Omega}\, (|e'(t)| +$$

$$\|\nabla\omega\|_{p,\Omega} \cdot \|\nabla u^n\|^{p-1}_{p,\Omega} + \|\omega\|_{m_+,\Omega} \left(\int\limits_\Omega |u^n|^{m(x)-1\cdot\frac{m_+}{m_+-1}}\, dx\right)^{\frac{m_+-1}{m_+}} \Bigg)^2 d\tau \leq$$

$$\frac{1}{2}\int\limits_0^t \|u^n_t\|^2_{2,\Omega}\, d\tau + \overline{M}\int\limits_0^t \left(|e'(t)|^2 + \|\nabla\omega\|^2_{p,\Omega} \cdot \|\nabla u^n\|^{2(p-1)}_{p,\Omega} + \right.$$

$$\overline{M}(m_+, m_-, \Omega)\|\omega\|^2_{m_+,\Omega}\|\nabla u^n\|^{2(m_+-1)}_{2,\Omega}\Big) \, d\tau \le$$
$$\frac{1}{2}\int_0^t \|u^n_t\|^2_{2,\Omega}\, d\tau + \overline{M}\, \|\nabla\omega\|^2_{p,\Omega} \int_0^t \|\nabla u^n\|^{2(p-1)}_{p,\Omega}\, d\tau + \overline{M}' \tag{34}$$

where $\overline{M}' := \overline{M}\int_0^t \left(|e'(t)|^2 + M(m_+, m_-, \Omega)\|\omega\|^2_{m_+,\Omega}K_0^{m_+-1}\right) d\tau$.

Plugging (34) into (32) we obtain

$$\int_0^t \left(\|u^n_t(\tau)\|^2_{2,\Omega} + \|\nabla u^n_t(\tau)\|^2_{2,\Omega}\right) d\tau + \frac{1}{p}\, \|\nabla u^n\|^p_{p,\Omega} + \int_\Omega \frac{1}{m(x)}|u^n|^{m(x)}dx \le$$
$$\overline{M}_0 \int_0^t \left(\frac{1}{p}\, \|\nabla u^n\|^p_{p,\Omega}\right)^{\frac{2(p-1)}{p}} d\tau + \overline{M}_1 \tag{35}$$

where $\overline{M}_1 := \overline{M}' + \frac{1}{p}\, \|\nabla u_0\|^p_{p,\omega}$. Omitting first and third terms on left hand side and in case $p \le 2$ applying classical Gronwall's lemma for $Z(t) := \frac{1}{p}\, \|\nabla u^n\|^p_{p,\Omega}$ we get

$$Z(t) \le \overline{M}_0 e^{\overline{M}_1 T}. \tag{36}$$

In the case $2 < p$, applying the generalized Gronwall's lemma for $Z(t) := \frac{1}{p}\, \|\nabla u^n\|^p_{p,\Omega}$ we get

$$Z(t) \le \overline{M}_0 \left(1 - (\frac{2(p-1)}{p} - 1)\overline{M}_1 \overline{M}_0^{\frac{2(p-1)}{p}-1} t\right)^{-\frac{1}{\frac{2(p-1)}{p}-1}} \tag{37}$$

for

$$0 \le t < T_1 := \frac{1}{(\frac{2(p-1)}{p} - 1)\overline{M}_1 \overline{M}_0^{\frac{2(p-1)}{p}-1}} < T_0. \tag{38}$$

Substituting (36) and (37) into (35), and taking supremum by t, we obtain the following first energy estimate

$$\int_0^t \left(\|u^n_t(\tau)\|^2_{2,\Omega} + \|\nabla u^n_t(\tau)\|^2_{2,\Omega}\right) d\tau + \frac{1}{p}\, \|\nabla u^n\|^p_{p,\Omega} + \gamma \int_\Omega \frac{1}{m(x)}|u^n|^{m(x)}dx \le K_1, \tag{39}$$

where $T_{max} = T$ if (17) holds, i.e. global in time, and $T_{max} = T_1$ (18) holds, i.e. local in time.

6 Conclusion

In conclusion, we have established the unique solvability of the inverse problem pseudo-parabolic equation with p-Laplacian and damping term with variable exponents. Using the Galerkin method the approximate solutions are constructed. The global and local a priori estimates are obtained for approximate solutions. Using these a priori estimates and the compactness theorems and the monotonicity method the convergence of approximate solutions to the solution of the initial problem is proved. The uniqueness of the posed inverse problem is also obtained. The results of this work can be applied to solve various inverse problems for linear and nonlinear equations of mathematical physics.

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