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CREATION AND EVALUATION OF THE STRUCTURES GRID IN CURVILINEAR AREAS

The article concerns methods of a structural curvilinear grid constructing in areas of geometrically complex shape and its evaluation from the quality point of view. Equidistribution methods based on differential equations were used to construct the grid at the boundary and inside the region. The numerical solution of differential equations was realized by the finite difference method. For the problems of uniform arrangement of grid nodes on the boundary and for the problems of constructing curved grids inside the region, implicit difference schemes were constructed and methods of scalar sweep and alternating directions were used. The results of numerical calculations are obtained and graphs of curved grids are presented for different numbers of grid nodes. The quality of the grid was studied according to four criteria such as orthogonality, elongation, convexity and adaptability, which corresponds to the division of the considered area into equal subdomains, i.e. cells.

Key words: numerical solution, curvilinear area, sweep method, alternating direction method, partial differential equations, curved mesh, difference schemes.

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Қисықсызықты облыстарда құрылымдық тор құру және оны бағалау

Мақалада геометриялық күрделі пішінді облыстарда құрылымды қисықсызықты торды құру әдістері және оны сапа тұрғысынан бағалау қарастырылған. Қисықсызықты облыстың шекарасында және ішінде құрылымды тор құру үшін дифференциалдық теңдеулерге негізделген эквиүлестірім әдістері қолданылды. Дифференциалдық теңдеулерді сандық шешу ақырлы айырмдар әдісімен жүзеге асырылды. Қисықсызықты шекарада тор тораптарын біркелкі орналастыру және облыстың ішінде қисық сызықты тор құру есептері үшін айқын емес айырымдық схемалар құрылып, сколярлық қуалау және айнымалы бағыттар әдістері қолданылды. Сандық есептеулердің нәтижелері алынды және тор тораптарының әртүрлі саны үшін қисықсызықты торлардың графиктері келтірілді. Тордың сапасы ортогоналдылық, созылу, дөңес және қарастырылып отырған облыстың бірдей бөліктерге, яғни ұяшықтарға бөлінуіне жауап беретін бейімделу сияқты төрт критерилер бойынша зерттеулер жүргізілді.

Түйін сөздер: сандық шешім, қисықсызықты облыс, қуалау әдісі, айнымалы бағыттар әдісі, дербес туындылы дифференциалдық теңдеу, қисық сызықты тор, айырымдық схема.

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Построение структурированных сеток в криволинейных областях и ее оценка

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В статье рассмотрены методы построения структурной криволинейной сетки в областях геометрически сложной формы и ее оценка с точки зрения качества. Для построения сетки на границе и внутри области использовались методы эквираспределения, основанные на дифференциальных уравнениях. Численное решение дифференциальных уравнений реализовались методом конечных разностей. Для задач равномерного расположения узлов сетки на границе и для задач построения криволинейных сеток внутри области были построены неявные разностные схемы и использховались методы сколярной прогонки и переменных направлений. Получены результаты численных расчетов и приведены графики криволинейных сеток при различных количествах узлов сетки. Проводились исследование качество сетки по четырем критериям как ортогональность, вытянутость, выпуклость и адаптивность которое отвечает разделения рассматирваемой области на равные подобласти, т.е. ячейки.

Ключевые слова: численное решение, криволинейная область, метод прогонки, метод переменных направлений, уравнения в частных производных, криволинейная сетка, разностные схемы.

1 Introduction

Modern computers for researchers became an effective tool for mathematical modeling of complex problems of science and technology. Therefore, nowadays qualitative research methods are considered of usability in all life spheres, and mathematical modeling is a tool for research.

In recent years, it is often necessary to consider problems in various fields in complex geometric areas. The first thing to do for numerical modeling in complex geometric areas is to sample the physical area, that is to model the physical geometry with the help of a set of cells of difference grids. It is also possible to qualitatively describe the necessary characteristics of the physical process under study, even in a small number of well-defined physical area nodes. It should be noted that the use of uneven grid layouts can lead to the appearance of sources of non-physical mass and momentum in the calculation schemes, as well as the loss of important properties inherent in differential equations. Equations written in curvilinear coordinates have a more complex form than the original equations. Particularly, they contain coefficients of variables, additional components, non-zero right parts, etc. Therefore, the question of approximation of equations in curvilinear grids occurs relevant and requires careful attention. Moreover, the requirements for difference grids lead to a complex mathematical problem of grid construction.

The work on the creation of structural curvilinear grids in complex geometric areas is considered in the works of many domestic and foreign scientists. The uniform arrangement of grids along the curve is described in detail in [1-3] works. It is widely considered in the work on the construction of the grid by the elliptical method [1, 4-6] in the inner regions. Methods of evaluation the created grids by different criteria are given in [1, 7].

2 Grid in curvilinear areas

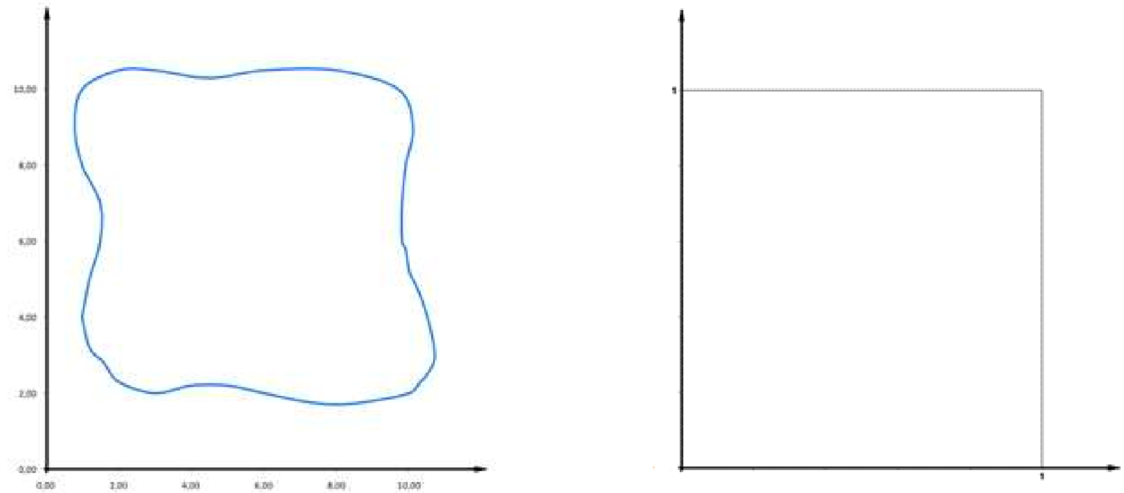
In this paper, the method of creating a curvilinear lattice ∂D in a connected area D with a curvilinear boundary is considered (Fig. 1 (a)).

Border interpolation is carried out to ensure continuity and monotony of boundary points.

In the research, first, the ways of uniform placement of grids within the boundaries of the curvilinear region, secondly, the creation of a grid with a mutually orthogonal structure within the region and the assessment of the created curvilinear grids are considered.

Since the physical region under consideration is complex and has a curvilinear boundary, we use differential methods to create curvilinear grids.

The physical region in the coordinate system (x, y) is carried out by the method of drawing to the computational area in the coordinate system (ξ, η) (Fig. 1).



a) b)

Figure 1: A related domain a) and the computational area b)

Creating the grid in a one-dimensional area starts with creation of the grid within it is boundaries. Since the boundary is not monotonous, the boundary is described by the given parametric form.

$$x = f^1(p), \quad y = f^2(p), \quad 0 \le p \le 1 \tag{1}$$

where l – the length of the border.

To create a grid at the boundary, we use the method of one-dimensional equivalence, it means that the differential equation is given by [1]:

$$\frac{\partial}{\partial \xi}\left(\vartheta(p)\frac{\partial p}{\partial \xi}\right) = 0, \qquad \xi \in (0,1), \qquad p(0) = 0, \qquad p(1) = l \tag{2}$$

where $\vartheta(p) = \sqrt{\left(\frac{\partial f^1(p)}{\partial p}\right)^2 + \left(\frac{\partial f^2(p)}{\partial p}\right)^2} > 0$, $p \in [0, l]$.

To create a grid at a related plane, let us use the following equation of the method of equivalence with the assumption that the search coordinate system is orthogonal [1]:

$$\frac{\partial}{\partial \xi}\left(g_{22}\frac{\partial \overrightarrow{x}}{\partial \xi}\right) + \frac{\partial}{\partial \eta}\left(g_{11}\frac{\partial \overrightarrow{x}}{\partial \eta}\right) = 0 \tag{3}$$

here $\overrightarrow{x} = (x, y)$ – the coordinates of the physical area, $g_{11} = x_\xi^2 + y_\xi^2$, $g_{22} = x_\eta^2 + y_\eta^2$ – the components of the metric tensor.

To create the grid on the boundary of the computational area let us solve problem (1) – (3) by the method of the finite-difference schemes. The finite-difference scheme for (2) is written as follows:

$$\frac{1}{h_1}\left(\vartheta_{i+1/2}\frac{p_{i+1}-p_i}{h_1}+\vartheta_{i-1/2}\frac{p_i-p_{i-1}}{h_1}\right)=0, \quad p_1=0, \ \ p_n=l, \ \ i=\overline{2,n_1-1} \tag{4}$$

where

$$\vartheta_{i+1/2}=\sqrt{\left(\frac{f^1(p_{i+1})-f^1(p_i)}{p_{i+1}-p_i}\right)^2+\left(\frac{f^2(p_{i+1})-f^2(p_i)}{p_{i+1}-p_i}\right)^2}$$

If the boundary of the region $A_k(x_k, y_k)$ $(k = 1, \ldots, M)$, $(x_k, y_k) \in \Gamma_l$ $(l = 1, 2)$ is given, then the extension of the point set is determined as follows:

$$l_1=0; \quad l_k=\sum_{i=2}^{k}\sqrt{(x_i-x_{i-1})^2+(y_i-y_{i-1})^2}, \quad k=2,\ldots,M$$

If $p_i \in [l_k, l_{k+1}]$ then the parametric equation for determining the coordinates of the boundary nodes in linear interpolation is as follows

$$\begin{aligned} f^1(p_i) &= x_k+\frac{x_{k+1}-x_k}{l_{k+1}-l_k}(p_i-l_k) \\ f^2(p_i) &= y_k+\frac{y_{k+1}-y_k}{l_{k+1}-l_k}(p_i-l_k) \end{aligned} \tag{5}$$

The resulting finite-difference problem (4) is solved by the following iterative method.As an initial approximation p_i^0, we obtain a uniform grid from the part $[0, l]$. Let us suppose at the n-th iteration a grid p_i^n is constructed. In the grid we obtain

$$\vartheta_{i+1/2}=\sqrt{\left(\frac{f^1(p_{i+1})-f^1(p_i)}{p_{i+1}-p_i}\right)^2+\left(\frac{f^2(p_{i+1})-f^2(p_i)}{p_{i+1}-p_i}\right)^2}$$

and using it we obtain the next approximations. The following linear problem is solved:

$$\begin{gathered} \frac{1}{h_1}\left(\vartheta^n_{i+1/2}\frac{p^{n+1}_{i+1}-p^{n+1}_i}{h_1}+\vartheta^n_{i-1/2}\frac{p^{n+1}_i-p^{n+1}_{i-1}}{h_1}\right)=0, \\ p^{n+1}_1=0, \ \ p^{n+1}_{n_i}=l, \ \ i=2,\ldots,n_1-1. \end{gathered} \tag{6}$$

The iterative process continues to a given accuracy, it means that until the following conditions are met:

$$\max_{1\le i\le n_1}|p_i^{n+1}-p_i^n|\le\varepsilon$$

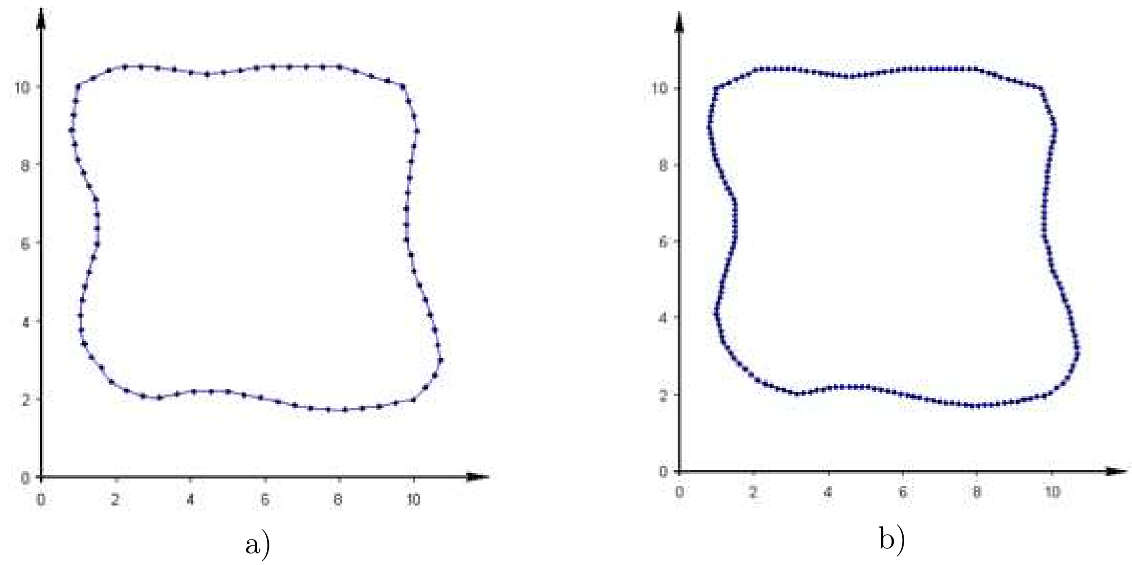


Figure 2: Evenly spaced grid nodes a) (20×20) and b) (50×50)

Based on the results of the last iterative approximation, the coordinates of the nodes at the boundaries of the physical region are calculated using (5).

Fig. 2 shows the results of the calculation of the difference between (6) and (5) evenly spaced at the boundary for a) 20×20 and b) 50×50 grid nodes.

Now we consider the difference problem of equation (3) to find the coordinates of the nodes within the area. The last finite-difference scheme has the following form:

$$\Lambda_{11}\overrightarrow{x}_{i,j} + \Lambda_{22}\overrightarrow{x}_{i,j} = 0 \tag{7}$$

where

$$\Lambda_{11}\overrightarrow{x}_{i,j} = \frac{1}{h_1}\left(g_{22,i+1/2,j}\frac{\overrightarrow{x}_{i+1,j} - \overrightarrow{x}_{i,j}}{h_1} - g_{22,i-1/2,j}\frac{\overrightarrow{x}_{i,j} - \overrightarrow{x}_{i-1,j}}{h_1}\right)$$
$$\Lambda_{22}\overrightarrow{x}_{i,j} = \frac{1}{h_2}\left(g_{11,i,j+1/2}\frac{\overrightarrow{x}_{i,j+1} - \overrightarrow{x}_{i,j}}{h_2} - g_{11,i,j-1/2}\frac{\overrightarrow{x}_{i,j} - \overrightarrow{x}_{i,j-1}}{h_2}\right).$$

Central differences in integer nodes were used to identify metric tensor components.

$$x_{\xi,i,j} = \frac{x_{i+1,j} - x_{i-1,j}}{2h_1}, \quad x_{\eta,i,j} = \frac{x_{i,j+1} - x_{i,j-1}}{2h_2}$$
$$y_{\xi,i,j} = \frac{y_{i+1,j} - y_{i-1,j}}{2h_1}, \quad y_{\eta,i,j} = \frac{y_{i,j+1} - y_{i,j-1}}{2h_2}$$
$$g_{11,i,j} = x^2_{\xi,i,j} + y^2_{\xi,i,j}, \quad g_{22,i,j} = x^2_{\eta,i,j} + y^2_{\eta,i,j}$$

The cells are averaged in the middle of the pages as follows:

$$g_{11,i+1/2,j} = \frac{g_{11,i+1,j} + g_{11,i,jj}}{2}, \quad g_{11,i-1/2,j} = \frac{g_{11,i,j} + g_{11,i-1,j}}{2}$$

The remaining coefficients are determined similarly. To find the numerical solution of equation (7), the method of alternating directions was used, considering the solutions of equation (6) as a boundary condition. Methodological calculations for the construction of curved grids using the method described above are considered for grids of different number of nodes. Fig. 3 and Fig. 4 show the results of the curvilinear grids.

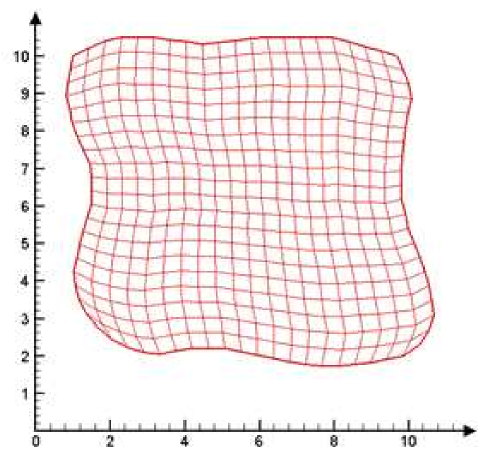


Figure 3: The curvilinear grid 20×20

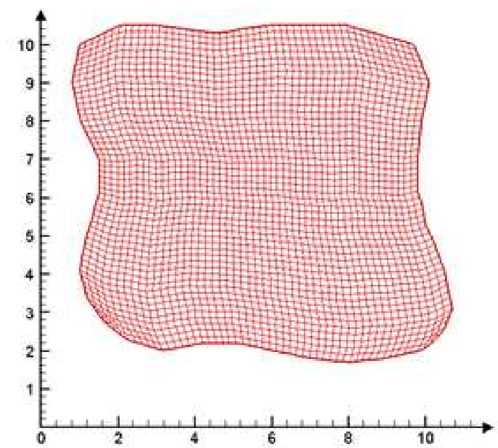
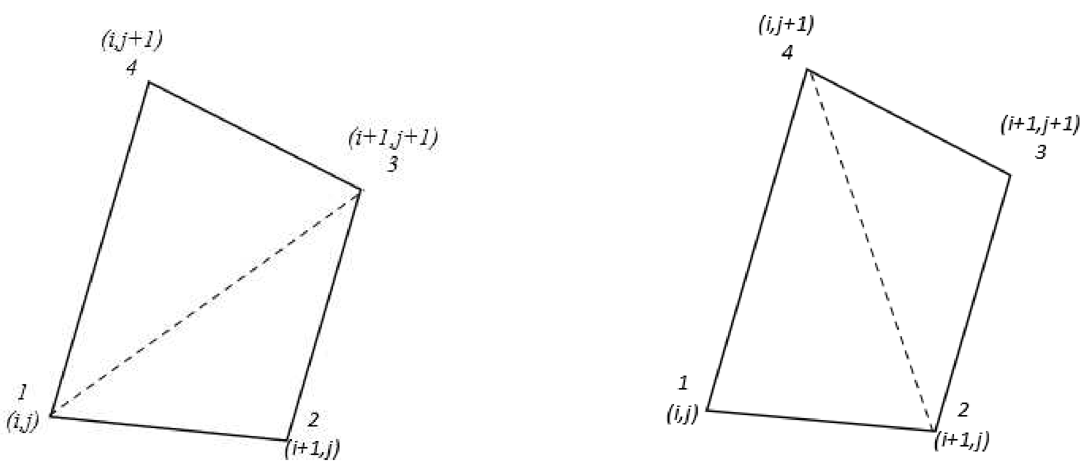


Figure 4: The curvilinear grid 50×50



a) b)

Figure 5: Triangulation of cells

It is not enough to check the quality of the created curved grids only by visual inspection. This is due to unnoticeable non-convex or crossed nodes potential occurrence during mesh nodes multiplication. Therefore, as considered in [1] let us consider four types of criteria for the assessment of grid networks: orthogonal, local uniformity, non-convex and convex of the formed network. Let us give a number to each grid as is shown at Fig. 6.

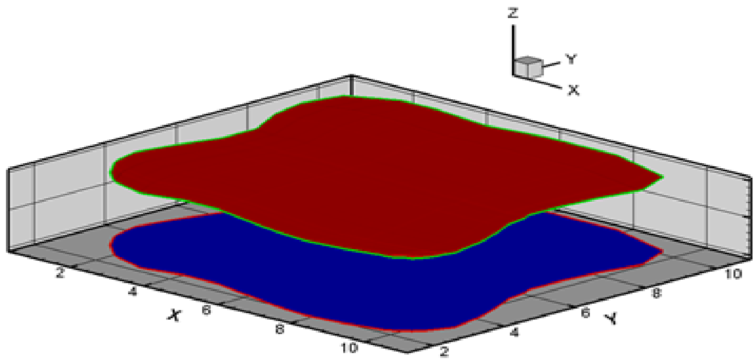


Figure 6: Convexity criterion estimation graph for the created curvilinear grid

Each grid cell is considered and divided into triangles diagonally. The following values are responsible for the convexity criterion estimation:

$$Q^1_{i,j} = \frac{\min\left\{S_{(i,j),(i+1,j),(i+1,j+1)}, S_{(i,j),(i,j+1),(i+1,j+1)}, S_{(i,j),(i+1,j),(i,j+1)}, S_{(i+1,j),(i,j+1),(i+1,j+1)}\right\}}{0.5(S_{(i,j),(i+1,j),(i+1,j+1)} + S_{(i,j),(i,j+1),(i+1,j+1)})} \tag{8}$$

where

$$S_{(i,j),(i+1,j),(i+1,j+1)} = \frac{1}{2}[(x_{i+1,j} - x_{i,j})(y_{i+1,j+1} - y_{i,j}) - (x_{i+1,j+1} - x_{i,j})(y_{i+1,j} - y_{i,j})]$$

$$S_{(i,j),(i,j+1),(i+1,j+1)} = \frac{1}{2}[(x_{i+1,j+1} - x_{i,j})(y_{i,j+1} - y_{i,j}) - (x_{i,j+1} - x_{i,j})(y_{i+1,j+1} - y_{i,j})]$$

$$S_{(i,j),(i+1,j),(i,j+1)} = \frac{1}{2}[(x_{i+1,j} - x_{i,j})(y_{i,j+1} - y_{i,j}) - (x_{i,j+1} - x_{i,j})(y_{i+1,j} - y_{i,j})]$$

$$S_{(i+1,j),(i,j+1),(i+1,j+1)} = \frac{1}{2}[(x_{i+1,j+1} - x_{i+1,j})(y_{i,j+1} - y_{i,j}) - (x_{i,j+1} - x_{i+1,j})(y_{i+1,j+1} - y_{i+1,j})]$$

the area of the corresponding triangles formed by the diagonals. The value of $Q^1_{i,j}$ may lie in $(-\infty, 1]$, for convex cells is $0 < Q^1_{i,j} \leq 1$, for triangular and intersecting cells is $-\infty < Q^1_{i,j} \leq 0$.

The next evaluation criterion is orthogonality. To determine the value of the orthogonality criterion, let apply the sine angle to the minimum value as follows:

$$Q^2_{i,j} = \min_{k=(i,j),(i+1,j),(i,j+1),(i+1,j+1)} \{sin\varphi_k\} \tag{9}$$

where

$$\sin\varphi_{i,j} = \frac{2S_{(i,j),(i+1,j),(i,j+1)}}{l_{(i,j),(i+1,j)} l_{(i,j),(i,j+1)}}$$

$$\sin\varphi_{i+1,j} = \frac{2S_{(i,j),(i+1,j),(i+1,j+1)}}{l_{(i,j),(i+1,j)}l_{(i+1,j),(i+1,j+1)}}$$

$$\sin\varphi_{i,j+1} = \frac{2S_{(i,j),(i,j+1),(i+1,j+1)}}{l_{(i,j),(i,j+1)}l_{(i,j+1),(i+1,j+1)}}$$

$$\sin\varphi_{i+1,j+1} = \frac{2S_{(i+1,j),(i,j+1),(i+1,j+1)}}{l_{(i+1,j),(i+1,j+1)}l_{(i,j+1),(i+1,j+1)}}$$

and the lengths of the nodes sides can be determined by the following equation

$$l_{(i,j),(i+1,j)} = \sqrt{(x_{i+1,j} - x_{i,j})^2 + (y_{i+1,j} - y_{i,j})^2}$$

Function $Q^2_{i,j}$ takes values at $[-1,1]$ section. So for the convex cells it takes positive (right-hand) values, for triangular cells it takes zero values and for non-convex and intersecting cells it takes negative (left-hand) values.

The next criterion for the quality of the grid is the elongation of the cell, the length of which is determined as follows:

$$Q^3_{i,j} = \frac{\min\limits_{k=[(i,j),(i+1,j)],[(i+1,j),(i+1,j+1)],[(i+1,j+1),(i,j+1)],[(i,j+1),(i,j)]}\{l_k\}}{\max\limits_{k=[(i,j),(i+1,j)],[(i+1,j),(i+1,j+1)],[(i+1,j+1),(i,j+1)],[(i,j+1),(i,j)]}\{l_k\}} \tag{10}$$

The value of $Q^3_{i,j}$ changes in the interval $[0,1]$.

One of the main requirements for curvilinear grids is local smoothness, that is, the areas of all cells in the domain must be equal to each other. The criterion of local smoothness is determined as follows:

$$Q^4_{i,j} = \min\left\{\frac{S_{i+1/2,j+1/2}}{\widetilde{S}}, \frac{\widetilde{S}}{S_{i+1/2,j+1/2}}\right\} \tag{11}$$

here $S_{i+1/2,j+1/2}$ – the area of the cell surrounded by the nodes $\{(i,j),(i+1,j),(i+1,j+1),(i,j+1)\}$ and $\widetilde{S} = \dfrac{\sum\limits_{i=1}^{n_1-1}\sum\limits_{j=1}^{n_2-1} S_{i+1/2,j+1/2}}{(n_1-1)(n_2-1)}$ – the average area of one cell. Here the value of $Q^4_{i,j}$ changes at the interval $[0,1]$. It can be seen from the graph that all the curvilinear grids are sufficiently convex.

From the criteria of orthogonality one can see the areas tapered fitted at grid nodes.

The low estimation values at the elongated areas of the grid nodes can be seen from the Figure as well.

Since the inclination curve divides the area into mutually equal areas, then it can be seen that the value of the corresponding criterion is low in the areas where the grid nodes are compressed.

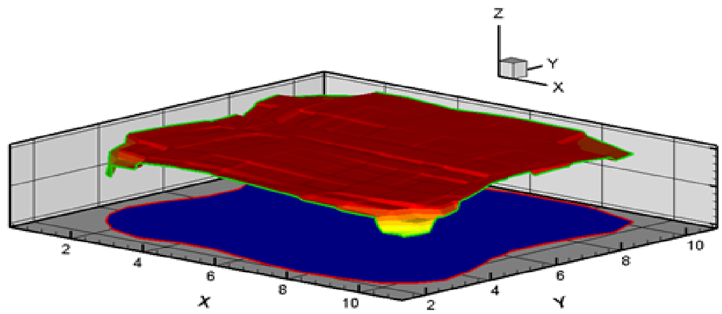


Figure 7: Mutual orthogonality criterion estimation graph for the created curvilinear grid

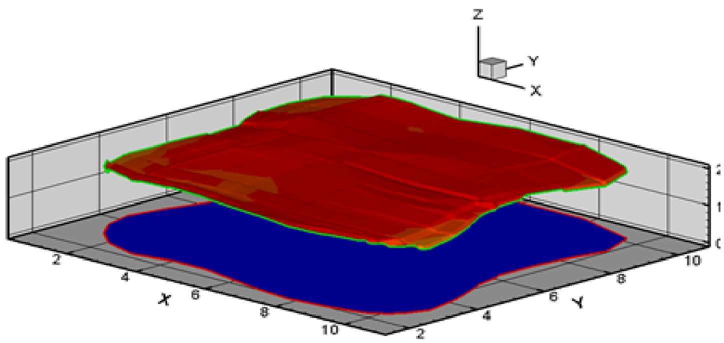


Figure 8: Durability criterion estimation graph for the created curvilinear grid

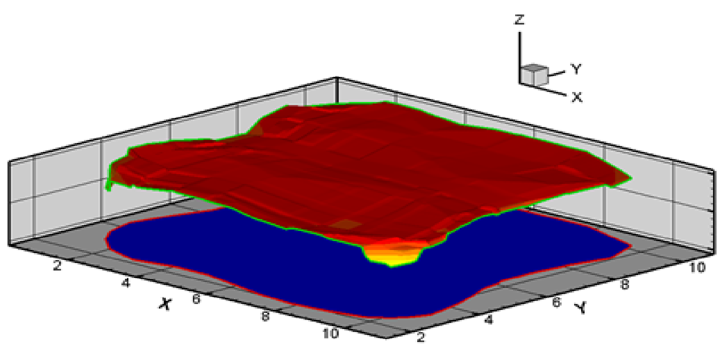


Figure 9: Inclination criterion estimation graph for the created curvilinear grid

3 Conclusion

In order to determine the best grid model, the grid quality criteria were determined at each iteration by the methods described above. At each iteration, the worst (lowest estimation value) and the best of the worst were selected for a certain grid quality criterion. Thus, the most optimal lattice was determined by the convexity, due to the fact that, the convexity and orthogonality are similar criteria.

The methods of creating a curvilinear grid and determining its quality considered in this paper, allowing us to smooth out and evenly distribute the difference grid nodes in a complex geometric area, as well as automatically create a new grid in case of changes in the number of nodes.

In addition, a qualitative description of the necessary characteristics of the physical process, which is studied in a small number of well-defined physical area nodes, is possible to be made.

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