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EQUIVALENCE OF THE FREDHOLM SOLVABILITY CONDITION FOR THE NEUMANN PROBLEM TO THE COMPLEMENTARITY CONDITION

The methods of complex analysis constitute the classical direction in the study of elliptic equations and mixed-type equations on the plane and fundamental results have now been obtained. In the early 60s of the last century, a new theoretical-functional approach was developed for elliptic equations and systems based on the use of functions analytic by Douglis. In the works of A.P. Soldatov and Yeh, it turned out that in the theory of elliptic equations and systems, Douglis analytic functions play an important role. These functions are solutions of a first-order elliptic system generalizing the classical Cauchy-Riemann system. In this paper, the Fredholm solvability of the generalized Neumann problem for a high-order elliptic equation on a plane is investigated. The equivalence of the solvability condition of the generalized Neumann problem with the complementarity condition (Shapiro-Lopatinsky condition) is proved. The formula for the index of the specified problem in the class of functions under study is calculated.

Key words: higher order elliptic equations, generalized Neumann problem, Fredholm solvability of the problem, normal derivatives on the boundary.

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Нейман есебінің фредгольмді шешімділік шартының толықтыру шартымен пара-парлығы

Кешенді талдау істері жазықтықтағы эллипстік теңдеулер мен аралас типтегі теңдеулерді зерттеуде классикалық бағытты құрайды же қазіргі уақытта іргелі нижелер алынды. өткен ғасырдың 60-жылдарының басында эллипстік теңдеулер мен жүйелер үшін Дуглистің аналитикалық функцияларын қолдануға негізделген жаңа теориялық же функционалды тіл пайда болды. А. П. Солдатовтың же Yeh еңбектерінде эллипстік теңдеулер мен жүйелер теориясында Дуглис аналитикалық функциялары маңызды рөл атқаратыны белгілі болды. Бұл функциялар классикалық Коши-Риман жүйесін жалпылайтын бірінші ретті эллипстік жүйенің шешімдері болып табылады. Бұл мақалада жазықтықтағы жоғары ретті эллипстік теңдеу үшін Нейманның жалпыланған есебінің фредгольмдік шешілуі зерттелген. Толықтыру шартымен (Шапиро-Лопатинский шартымен) Нейманның жалпыланған есебінің шешілу шартының эквиваленттілігі делденді. Зерттелетін функциялар класындағы көрсетілген есеп индексінің формуласы есептеледі.

Түйін сөздер: жоғары ретті эллиптикалық тедеулер, жалпыланған Нейман есебі, есептің фредгольмді шешімділігі, шекарадағы нормал туындылар.

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Эквивалентность условии фредгольмовой разрешимости задачи Неймана с условием дополнительности

Методы комплексного анализа составляют классическое направление в исследовании эллиптических уравнений и уравнений смешанного типа на плоскости и в настоящее время получены фундаментальные результаты. В начале 60-х годов прошлого столетия для эллиптических уравнений и систем был развить новый теоретико-функциональный подход, основанный на использовании функций, аналитических по Дуглису. В работах А.П. Солдатова, и Yeh выяснилось, что в теории эллиптических уравнений и систем важную роль играют функции, аналитические по Дуглису. Эти функции являются решениями эллиптической системы первого порядка, обобщающей классическую систему Коши-Римана. В данной статье исследована фредгольмовая разрешимость обобщенной задачи Неймана для эллиптического уравнения высокого порядка на плоскости. Доказана эквивалентность условии разрешимости обобщенной задачи Неймана с условием дополнительности (условием Шапиро-Лопатинского). Вычислена формула для индекса указанной задачи в исследуемой классе функций.

Ключевые слова: эллиптические уравнения высокого порядка, обобщенная задача Неймана, фредгольмова разрешимость задачи, нормальные производные на границе.

Introduction

Complex analysis methods constitute a classical direction in the study of elliptic equations and equations of mixed type on the plane. At present, active research is being carried out in this direction in many mathematical centers of the world.

In a simply connected domain D on a plane bounded by a simple smooth contour $\Gamma \in C^{2l,\mu},\ l \geq 2,\ 0 < \mu < 1$, for an elliptic equation of the $2l$-order

$$\sum_{r=0}^{2l} a_r \frac{\partial^{2l} u}{\partial x^{2l-r} \partial y^r} + \sum_{0 \leq r \leq k \leq 2l-1} a_{rk}(x) \frac{\partial^k u}{\partial x^{k-r} \partial y^r} = F \tag{1}$$

with constant highest coefficients $a_r \in \mathbb{R}$ and lower coefficients $a_{rk} \in C^\mu(\overline{D})$, consider the boundary value problem

$$\left. \frac{\partial^{k_j - 1} u}{\partial n^{k_j - 1}} \right|_\Gamma = f_j, \quad j = 1, \ldots, l, \tag{2}$$

where $n = n_1 + i n_2$ means the unit outward normal and natural k_j are subject to the condition $1 \leq k_1 < k_2 < \ldots < k_l \leq 2l$.

Here and below, the normal derivative $(\partial/\partial n)^k$ of order k is understood as the boundary operator

$$\left(n_1 \frac{\partial}{\partial x} + n_2 \frac{\partial}{\partial y} \right)^k = \sum_{r=0}^{k} \binom{k}{r} n_1^r n_2^{k-r} \frac{\partial^k}{\partial x^r \partial y^{k-r}}$$

and a similar meaning has the boundary operator $(\partial/\partial e)^k$ with respect to the unit tangent vector $e = e_1 + i e_2 = i(n_1 + i n_2)$.

This problem turns into the Dirichlet problem if $k_j = j$, and turns into the Neumann problem if $k_j = j + 1$.

Materials and methods. The statement of this problem for $k_{j+1} - k_j \equiv 1$ for a polyharmonic equation originates from [1], where for $k_1 \geq 2$ it is called the generalized Neumann problem. This name is further retained for an arbitrary set of indicators k_j. Another variant of the Neumann problem, based on the variational principle, was proposed in [2]. If lower coefficients and the right-hand side are equel to zero, then problem (1), (2) was studied in [3, 4]. The case where they are different from zero, is studied detail in [5] in the space

$$C_a^{2l-1,\mu}(\overline{D}) \equiv \{u : u \in C^{2l}(D) \cap C^{2l-1,\mu}(\overline{D}), \quad \sum_{r=0}^{2l} a_r \frac{\partial^{2l} u}{\partial x^{2l-r} \partial y^r} \in C^{\mu}(\overline{D})\},$$

in particular, a necessary and sufficient condition for its Fredholm property is found.

Results and discussion. The study of equation (1) in the multidimensional case is of great scientific interest. In the model case, equation (1) is called the polyharmonic equation $\Delta^l u(x) = F(x)$, $x = (x_1, ..., x_n) \in D \subseteq \mathbb{R}^n$. It is known that for this polyharmonic equation, the Dirichlet problem is uniquely solvable for any right-hand side of the equation. In [6, 7], a new representation of the Green function of the Dirichlet problem for a polyharmonic equation in a multidimensional sphere is constructed explicitly. In [8, 9], a representation of the Green function of the Neumann problem for the Poisson equation in a multidimensional unit ball is obtained. In [10–12], Green functions of Dirichlet, Neumann, and Robin problems for biharmonic and polyharmonic equations in a circle, semicircle, semi-ring, triangle, and other standard plane domains are constructed. The results of these works are based on the classical theory of integral representations for analytic, harmonic and polyharmonic functions on the plane.

The paper [13] describes well-posed boundary value problems for a polyharmonic operator.

In this article, for a higher order elliptic equation, it is proposed to develop a new functional-theoretical approach based on the use of functions that are analytic according to Douglis [14–16].

In the early 60s it became clear [17,18], that in the theory of elliptic equations and systems an important role is played by functions analytic in the sense of Douglis. These functions are solutions of a first-order elliptic system generalizing the classical Cauchy-Riemann system. In [19,20], this approach has already been successfully applied to problems of the plane theory of elasticity (including the general anisotropic case). However, for domains with piecewise smooth boundaries and equations with continuous coefficients and, especially, for problems with nonlocal boundary conditions, this approach requires its further development.

In this paper, under the assumption $\Gamma \in C^{2l,\mu}$ that the results obtained in [5] are extended to the more standard class $C^{2l,\mu}(\overline{D})$.

1 Description of the Fredholm property of problem (1),(2)

An operator $A \in \mathfrak{L}(X, Y)$ is called *Fredholm* if its kernel $KerA$ and cokernel are finite-dimensional, and

$$\dim ImA = \dim (cokerA)^{\perp}.$$

The Fredholm property and the index of a problem are understood in relation to its bounded operator, in our case

$$X \equiv C^{2l,\mu}(\overline{D}) \to Y \equiv C^{\mu}(\overline{D}) \times \prod_{j=1}^{l} C^{2l-k_j+1,\mu}(\Gamma).$$

Let $\nu_k,\ 1 \le k \le m$ be all different roots of the characteristic equation

$$\chi(z) = a_{2l} \prod\nolimits_{k=1}^{m} (z-\nu_k)^{l_k} \prod\nolimits_{k=1}^{m} (z-\overline{\nu_k})^{l_k} = 0,$$

in the upper half-plane and the l_k- multiplicity of the kth root, so that their total multiplicity $l_1 + \ldots + l_m$ is equal to l. The ellipticity condition is that $a_{2l} \neq 0$ and the roots of the characteristic polynomial $\chi(z) = a_0 + a_1 z + \ldots + a_{2l} z^{2l}$ do not lie on the real axis.

Let us introduce functions that are fractionally linear in z

$$\omega(e,z) = \frac{e_2 - e_1 z}{e_1 + e_2 z} = \frac{n_1 + n_2 z}{e_1 + e_2 z}, \tag{3}$$

where the dependence on the unit tangent vector $e = e_1 + ie_2$ to the contour Γ is indicated explicitly. For definiteness, the vector e is oriented positively with respect to the domain D, i.e. D lies to the left of this vector.

For an analytic l-vector-function $g(z) = (g_1(z), \ldots, g_n(z))$, in a neighborhood of points $z_1, \ldots, z_m$, we introduce the block $l \times l$-matrix

$$W_g(z_1, \ldots, z_m) = (W_g(z_1), \ldots, W_g(z_m)), \tag{4}$$

where matrix $W_g(z_k) \in \mathbb{C}^{l \times l_k}$ is composed of column vectors

$$g(z_k), g'(z_k), \ldots, \frac{1}{(l_k-1)!} g^{(l_k-1)}(z_k).$$

As g below we use a vector with components

$$g_j(z) = z^{k_j - 1},\ 1 \le j \le l. \tag{5}$$

In this notation, the following theorem was proved in [5].

Theorem 1 *a) Problem (1), (2) is Fredholm if and only if*

$$\det W_g[\omega(e,\nu_1), \ldots, \omega(e,\nu_m)] \neq 0, \quad e \in \mathbb{T}, \tag{6}$$

where $\mathbb{T}$ stands for the unit circle. Accordingly, the index of this problem is given by the formula

$$æ = -2 \left[\frac{1}{2\pi} \arg \det W_g[\omega(e,\nu_1), \ldots \omega(e,\nu_m)] \Big|_{\mathbb{T}} + l^2 - \sum\nolimits_{j=1}^{m} l_j^2 \right], \tag{7}$$

where the increment is taken along the counterclockwise unit circle.

b) In each of the following two cases

$$k_j = k_1 + j - 1,\ 1 \le j \le l; \tag{8i}$$

$$m = 1,\ \nu_1 = \nu. \tag{8ii}$$

problem (1), (2) is Fredholm and its index is zero.

Obviously, condition (6) depends only on the set of numbers $k_1, k_2, \ldots, k_l$, so that for fixed k_j, when it is satisfied, problem (1), (2) is Fredholm in any domain.

2 Equivalence of the Fredholm property condition for problem (1), (2) to the complementarity condition

From the point of view of the general elliptic theory [21] problem (1), (2) is Fredholm in the space $C^{2l,\mu}(\overline{D})$ if and only if its boundary conditions satisfy as follows called the complementarity condition (or the Shapiro-Lopatinski condition [22]). In this case, [23] also says that the boundary conditions (2) cover the differential operator

$$L = \sum_{r=0}^{2l} a_r \frac{\partial^{2l}}{\partial x^{2l-r} \partial y^r},$$

corresponding to the main part (1). The indicated condition is as follows: starting from the fixed point $t \in \Gamma$ differentiation with respect to x and y in the expressions of the operators L and B_j, we replace, respectively, $e_1(t) + zn_1(t)$ and $e_2(t) + zn_2(t)$. As a result, we get polynomials

$$L(n, z) = \sum_{r=0}^{2l} a_r (e_1 + zn_1)^{2l-r} (e_2 + zn_2)^r$$

and

$$B_j(z) = [n_1(e_1 + zn_1) + n_2(e_2 + zn_2)]^{k_j - 1} = z^{k_j - 1}, \quad 1 \le j \le l.$$

Since $n_1 = e_2,\ n_2 = -e_1$, in the notation (3) the polynomial $L(n, z)$ can be written in the form

$$L(n, z) = (e_1 + zn_1)^{2l} \sum_{r=0}^{2l} a_r [-\omega(z)]^r,$$

so $L(z) = 0$ is equivalent to

$$-\omega(z) = \nu, \tag{9}$$

where ν is an arbitrary root of the characteristic equation $\chi(z)$. Moreover, their corresponding multiplicities coincide.

Obviously, transformation (3) takes the upper half-plane onto itself, so the transformation $z \to -\omega(\bar{z})$ also has a similar property. In particular, the polynomial l-degree

$$L^+(z) = (z - z_1)^{l_1} \dots (z - z_m)^{l_m}, \quad -\omega(z_j) = \bar{\nu}_j, \tag{10}$$

is formed by the roots of the equation $L(n, z) = 0$ lying in the upper half-plane.

In the adopted notation, the **complementarity condition** consists in the linear independence of the polynomials $B_j(z),\ 1 \le j \le l$, modulo the polynomial $L^+(z)$. Thus, this condition should be equivalent to condition (6) obtained in another way. This fact can be established directly.

Lemma 1 *Condition (6) is satisfied if and only if the polynomials $B_j(z) = z^{k_j - 1},\ 1 \le j \le l$, are linearly independent in to the modulus of the polynomial $L^+(z)$.*

Proof 1 *Suppose that these polynomials are linearly dependent modulo $L^+(z)$, that is, there is a nontrivial linear combination of them $B = \alpha_1 B_1 + \ldots + \alpha_l B_l$, a multiple of L^+. In the notation (5), the polynomial $B_j = g_j$, so this fact can be written in the form*

$$B(z) = \sum_{j=1}^{l} \alpha_j z^{k_j - 1} = Q(z) L^+(z)$$

with some polynomial Q. In accordance with (10), this relation means that the polynomial B at the points z_k has a zero of order l_k or, which is equivalent,

$$\sum_{j=1}^{l} \alpha_j g_j^{(s)}(z_k) = 0, \quad 0 \le s \le l_k - 1, \ 1 \le k \le m. \tag{11}$$

These equalities represent a homogeneous system of l equations for $\alpha_1, \ldots, \alpha_l$. It can be seen from definition (4) that the matrix of this system coincides with the matrix transposed to $W_g(z_1, \ldots, z_m)$. Therefore, a nonzero solution to system (11) is possible if and only if

$$\det W_g(z_1, \ldots, z_m) = 0. \tag{12}$$

According to definition (3), equality (9) is equivalent to $\overline{z}_j = \omega(\nu_j)$; therefore, equality (12) can be expressed in the form of vanishing of the determinant on the left-hand side of (6). Thus, violation of the complementarity condition is equivalent to violation of condition (6), which completes the proof of the lemma.

3 Continuation of the description of the Fredholm property condition for problem (1), (2)

Note that formulas (6) and (7) will not change if we go from the vector g to the vector q defined by the relation

$$g(z) = z^{k_1 - 1} q(z), \quad q(z) = (1, z^{s_1}, \ldots, z^{s_{l-1}}), \quad s_j = k_{j+1} - k_1. \tag{13}$$

Moreover, as noted in [3,5], the determinants of the matrices W with these vectors are related by the relation

$$\det W_g(z_1, \ldots, z_m) = \prod_j z_j^{l_j(l_1 - 1)} W_q(z_1, \ldots, z_m).$$

In these designations, condition (6) and the index formula (7) can be given a different form, more convenient for use. Let us introduce the fractional-linear functions

$$\gamma_k(z) = \frac{\nu_k - z}{1 + \nu_k z}, \quad 1 \le k \le m, \tag{14}$$

and the function

$$R(z) = \det W_g(\gamma_1(z), \ldots, \gamma_m(z)). \tag{15}$$

This transformation $\gamma_k(z)$ swaps the points $\pm i$ and is involutive:

$$\gamma(\pm i) = \mp i, \quad \gamma[\gamma(z)] \equiv z. \tag{16}$$

Moreover, for $\nu_k = i$, the identity $\gamma_k(z) \equiv i$ holds.

Thus, the rational function $R(z)$ admits poles only at the points $\nu_k' = -1/i \neq i$ in the upper half-plane and, in particular, is analytic in the lower half-plane. In addition, for $m \geq 2$ ($m \geq 3$) at the points $\zeta = -i$ ($\zeta = \pm i$) vanishes. This follows from (34) and the fact that for $z_1 = ... = z_m = -i$ ($z_1 = ... = z_m = \pm i$) matrix (4) has the same columns, so its determinant is zero. In particular, the function $R(z)$ is completely divisible by $z^2 + 1$.

Theorem 2 *Problem (1), (2) is Fredholm if and only if the rational function $R(\zeta)$ has no real roots on the extended real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, and under this condition, its index is given by the formula*

$$æ = 4(n - \sum\nolimits_{i<j} l_i l_j), \tag{17}$$

where n is the number of zeros of this function in the lower half-plane of the function, taking into account their multiplicity.

Proof 2 *Comparison of definitions (4) and (14) implies that*

$$\det W_g[\omega(e, \nu_1), \ldots, \omega(e, \nu_m)] = R(e_2/e_1).$$

The function $\omega(e, \nu)$ in (3) is even in the variable $e \in \mathbb{T}$ and therefore the quantity

$$\arg\, \det W_g[\omega(e, \nu_1), \ldots \omega(e, \nu_m)]\Big|_{\mathbb{T}} = 2\arg\, \det W_g[\omega(e, \nu_1), \ldots \omega(e, \nu_m)]\Big|_{\mathbb{T}^+},$$

where $\mathbb{T}^+$ is a semicircle in the right half-plane. The mapping $e = e_1 + ie_2 \to t = e_2/e_1$ realizes a homeomorphism of this semicircle onto the extended real line $\overline{\mathbb{R}}$, and bypassing it from the point $e = -i$ to $e = i$ corresponds to movement on a straight line in the positive direction. Therefore, condition (6) is equivalent to the fact that the function R has no real roots on the extended real line, and the equality

$$\arg\, \det W_g[\omega(e, \nu_1), \ldots \omega(e, \nu_m)]\Big|_{\mathbb{T}} = 2\arg\, \det W_g[\gamma_1(t), \ldots \gamma_m(t)]\Big|_{-\infty}^{+\infty}.$$

As a result, formula (7) becomes

$$æ = -\frac{2}{\pi}\arg R(t)\Big|_{-\infty}^{+\infty} - 2(l^2 - \sum\nolimits_{j=1}^{m} l_j^2).$$

The rational function R has no poles in the lower half-plane, so, taking into account Rouche's theorem, the previous equality coincides with (36). It is only necessary to take into account that the lower half-plane remains on the left when traversing the straight line in the negative direction and that

$$l^2 - \sum\nolimits_{j=1}^{m} l_j^2 = 2\sum_{i<j} l_i l_j.$$

The theorem is proved.

Let us consider in more detail the function $\gamma(z)$, defined by (14) with $\nu = \nu_k$. For $\nu = i$ this function is identically equal to i, so we can assume $\nu \neq i$.

Lemma 2 *Let $\nu \neq i$. The transformation $z \to \gamma(z)$ takes the lower half-plane to the circle*

$$B = \{z : |z|^2 + 1 - 2\rho Imz < 0\}, \ \rho = \frac{|\nu|^2 + 1}{2Im\nu}. \tag{18}$$

This circle has the center point $i\rho$ radius $r = \sqrt{\rho^2 - 1}$, lies entirely in the upper half-plane, contains the point $z = i$ and is invariant under the involution $z \mapsto z' = -1/z$.

In addition, the points ν u $\nu' = -1/\nu$ lie on its boundary circle $L = \partial B$.

Proof 3 *By (14), we have*

$$Im[\gamma(z)] = \frac{(1 + |z|^2)Im\nu - (1 + |\nu|^2)Imz}{|1 + \nu z|^2}.$$

Hence the image of the lower half-plane is the disc B, which lies entirely in the upper half-plane and contains the point $z = i$. By the symmetry principle, the points pmi are symmetric as relative to the straight line $\mathbb{R}$, and to the circle $L = \partial B$. In particular, the center of this circle must lie on the imaginary axis. Denoting the center and radius of this circle, respectively, $i\rho$ and r, we come to the relation $|i - i\rho||i + i\rho| = r^2$, whence $r^2 = \rho^2 - 1$. The equation $|z - i\rho|^2 = r^2$ of the circle L can be written in the form $|z|^2 + 1 - 2\rho Imz = 0$, which proves the description (18) of the circle B.
Obviously, the points $\gamma(0) = \nu$ and $\gamma(\infty) = -1/\nu$ lie on L. In particular, substituting $z = \nu$, into this equation, we arrive at the expression for ρ in (18). The fact that the circle L is invariant under the transformation $z \mapsto z' = -1/z$ follows directly from its equation. The lemma is proved.

Lemma 2 is used for the case $m = 2$ of two points ν_1, ν_2 which, according to Theorem 2, can be considered different without loss of generality. Let their numbering be such that $\nu_1 \neq i$. Then, by virtue of (16), the transformation γ_1 takes the disc B to the lower half-plane, and we can introduce the function

$$S(z) = R[\gamma_1(z)] = (\det W_g)[z, \delta(z)], \ \delta(z) = \gamma_2[\gamma_1(z)], \tag{19}$$

analytic in the disc B. In explicit form,

$$\delta(z) = \frac{1 + \tau z}{\tau - z}, \ \tau = \frac{1 + \nu_1\nu_2}{\nu_2 - \nu_1} \in B. \tag{20}$$

The fact that the point τ does not belong to the closed circle $\overline{B}$ is a consequence of Lemma 2. Indeed, $\tau = -1/[\gamma_1(\nu_2)]$, and by Lemma 2 the point $z = \gamma_1(\nu_2)$ lies outside $\overline{B}$, so this is true and for $\tau = z' = 1/z$. With respect to the function S Theorem 3 takes the following form.

Theorem 3 *Let $m = 2$ with $\nu_2 \neq \nu_1 \neq i$ and the notation of Lemma 2 is adopted. Then the Fredholm property of problem (1), (2) is equivalent to the fact that the function $S(z)$ has no zeros on the circle $L = \partial B$. When this condition is satisfied, its index is given by formula (17), in which n is the number of zeros of the function S in the circle B, taken with account of their multiplicity.*

Note that, like R, the function S vanishes at the points $\pm i$. This function is especially simplified if $1 + \nu_1\nu_2 = 0$, then the transformation δ in (20) is an involution $z \mapsto z' = -1/z$. In this case, Theorem 1 turns into Theorem 3 from the work [5].

4 Application of the results to the general equation of the fourth and sixth orders

Let us illustrate the application of Theorem 3 by the example of fourth-order equation (1). Since the lower-order terms do not affect the Fredholm property and the index of the problem, we can restrict ourselves to the main part of the equation with $\nu_2 \neq \nu_1 \neq i$. This equation can be written in the form

$$L_1 L_2 u = 0 \tag{21}$$

with second-order operators

$$L_k = \frac{\partial^2}{\partial y^2} - 2(\operatorname{Re}\nu_k)\frac{\partial^2}{\partial y \partial x} + |\nu_k|^2 \frac{\partial^2}{\partial x^2}, \quad k = 1, 2.$$

With respect to the difference $s = k_2 - k_1$, which in the considered case takes three values $s = 1, 2, 3$, problem (2) is written in the form

$$\left.\frac{\partial^i u}{\partial n^i}\right|_\Gamma = f_1, \quad \left.\frac{\partial^{i+s} u}{\partial n^{i+s}}\right|_\Gamma = f_2, \quad 0 \leq i \leq 3 - s. \tag{22_s}$$

According to (4), (11), in the case under consideration, the matrix W_q takes the form

$$W_q(z_1, z_2) = \begin{pmatrix} 1 & 1 \\ z_1^s & z_2^s \end{pmatrix}, \quad \det W_q(z_1, z_2) = z_2^s - z_1^s,$$

so that $S(z) = [\delta(z)]^s - z^s$. Explicitly form

$$S(z) = \frac{(1 + z^2) P_s(z)}{(\tau - z)^s},$$

where $P_1(z) = 1,\ P_2(z) = -z^2 + 2\tau z + 1$ and

$$P_3(z) = [qz^2 + (1 - q)\tau z + 1][q^2 z^2 + (1 - q^2)\tau z + 1], \ q = e^{2\pi i/3}. \tag{23}$$

Note that the polynomial P_2 is nonzero in $\overline{B}$. Indeed, let $z^2 - 2\tau z - 1 = 0$ for some $z \in \overline{B}$. So the point $z' = -1/z$ also belongs to $\overline{B}$, then the point $\tau = (z + z')/2 \in \overline{B}$, which contradicts (20).

Since in the considered case $\sum_{i>j} l_i l_j = 1$, then, based on Theorem 1, we obtain the following conclusion.

Remark 1 *For $s \leq 2$ problem (21), (22_s) is Fredholm and its index is zero, and for $s = 3$ it is Fredholm if and only if the zeros of P_3 polynomial do not lie on the boundary circle L of the disc B, defined by Lemma 2 by $\nu = \nu_1$. Under this condition, its index is $æ = 4k$, where k is the number of these zeros in the disc B, taken with multiplicity.*

As the following lemma shows, with a suitable choice of ν_1 and ν_2 , it is always possible to achieve that one of the zeros of the polynomial P_3 lies on the circle L.

Lemma 3 *Let the point $\nu = \nu_1$ lie in the upper half-plane and in the notation of Lemma 2*

$$\tau = -i\rho - \sqrt{(\rho^2 - 1)/3}. \tag{24}$$

Then the point

$$\nu_2 = \frac{1 + \tau\nu_1}{\tau - \nu_1},$$

also lies in the upper half-plane, and for these points the Fredholm property of problem (21), (22_s) is violated.

Proof 4 *First of all, check that the point ν_2 lies in the upper half-plane. Indeed, it is clear from the definition of ν_2 that $\tau = -1/\gamma_1(\nu_2)$. Therefore, if $Im\nu_2 \leq 0$, then, by Lemma 2, the point τ must belong to $\overline{B}$, which is impossible.*

Let it_1 and it_2, $t_2 > t_1$, be the intersection points of the circle L with the imaginary axis. Then, according to (18), the equalities $t_k^2 + 1 - 2\rho t_k = 0, k = 1, 2,$ we have

$$t_1 + t_2 = 2\rho,\ t_1 t_2 = 1,\ t_2 - t_1 = 2\sqrt{\rho^2 - 1}. \tag{25}$$

It is asserted that the point $z = it_2$ is the root of the first factor in (23) and, therefore, problem (21), (22_s) is not Fredholm.

Indeed, since $1/z = -it_1$, the equation $e^{2\pi i/3}z^2 - \tau(1 - e^{2\pi i/3})z + 1 = 0$ can be rewritten in the form

$$e^{\pi i/3} i t_2 - e^{-\pi i/3} i t_1 = -i\tau\sqrt{3},$$

which, taking into account relations (25), is equivalent to equality (24).

For elliptic equations of orders higher than the fourth, it is already difficult to describe explicitly the roots of the corresponding polynomials. As example, let us consider a sixth-order equation, i.e. $l = 3$. In accordance with Theorem 2, it suffices to restrict ourselves to considering two cases: (i) all roots are pairwise distinct, i.e. $l_1 = l_2 = l_3 = 1$ and (ii) one of these roots is multiple, for example, $l_1 = 1,\ l_2 = 2$. Accordingly to these cases, similarly to (21), we have the equations

$$L_1 L_2 L_3 u = f, \tag{26i}$$

$$L_1 L_2^2 u = f, \tag{26ii}$$

the corresponding operators of the second order. With respect to the positive differences $r = k_2 - k_1$ и $s = k_3 - k_2$, for which $r + s \leq 5$, problem (2) is written in the form

$$\left.\frac{\partial^i u}{\partial n^i}\right|_\Gamma = f_1,\quad \left.\frac{\partial^{i+s} u}{\partial n^{i+s}}\right|_\Gamma = f_2,\quad \left.\frac{\partial^{i+s+1} u}{\partial n^{i+s+1}}\right|_\Gamma = f_3,\quad 0 \leq i \leq 5 - r - s. \tag{$27_{r,s}$}$$

In accordance with this, vector (13) should be taken in the form $q = (1, z^r, z^{r+s})$, so that for the matrix W_q in definition (4), we have the expressions

$$(i) W_q = \begin{pmatrix} 1 & 1 & 1 \\ z_1^r & z_2^r & z_3^r \\ z_1^{r+s} & z_2^{r+s} & z_3^{r+s} \end{pmatrix},\ (ii) W_q = \begin{pmatrix} 1 & 1 & 0 \\ z_1^r & z_2^r & r z_2^{r-1} \\ z_1^{r+s} & z_2^{r+s} & (r+s) z_2^{r+s-1} \end{pmatrix}.$$

In case (i) the determinant of the matrix $W_q(z_1, z_2, z_3)$ can be represented in the form

$$-detW_q = (z_1^r - z_2^r)z_3^{r+s} + (z_1^r - z_3^r)z_2^{r+s} + (z_3^r - z_2^r)z_1^{r+s}.$$

Therefore, for function (14) we have the equality

$$-R(z) = \frac{(1+z^2)P(z)}{[(1+\nu_1 z)(1+\nu_2 z)(1+\nu_3 z)]^{r+s}}$$

with some polynomial $P(z)$. Here it is taken into account that for $m \geq 3$ the function $R(z)$ vanishes at the points $z = \pm i$.

Since $\sum_{i>j} l_i l_j = 3$, then, based on Theorem 3, we obtain the following conclusion.

Remark 2 *The Fredholm property of the problem (26i), (27) is equivalent to the absence of real zeros of the polynomial $P(\zeta)$ on the circle L and its index is $æ = 4(n-1)$, where n is the number of these zeros in the lower half-plane.*

The polynomial P for $r = 1$ according to

$$\gamma_i(z) - \gamma_j(z) = \frac{(\nu_i - \nu_j)(1+z^2)}{(1+\nu_i z)(1+\nu_j z)}$$

we have

$$P(z) = {\sum}'(\nu_i - \nu_j)[(1+\nu_i z)(1+\nu_j z)]^s(\nu_k - z)^{s+1},$$

where the prime at the sign of the sum means that the summation is performed over cyclic triples

$$(i,j,k) = (1,2,3); \quad (2,3,1); \quad (3,1,2).$$

If in addition $s = 1$, then, as direct verification shows, $P(z) = c(1+z^2)^2$ with the factor

$$c = {\sum}'(\nu_i - \nu_j)\nu_k^2 = (\nu_1 - \nu_2)\nu_3^2 + (\nu_2 - \nu_3)\nu_1^2 + (\nu_3 - \nu_1)\nu_2^2.$$

In this case, the index of the problem is zero, which is consistent with Theorem 2.

Let us turn to case (ii), where we can assume $\nu_2 \neq \nu_1 \neq i$. In this case

$$detW_q = z_2^{r-1}[sz_2^s(z_2^r - z_1^r) - rz_1^r(z_2^s - z_1^s)] = z_1^{r+s-1}z_2^{r-1}(z_2 - z_1)\chi(z_2/z_1),$$

where $(q-1)\chi_{r,s}(q) = sq^s(q^r - 1) - r(q^s - 1)$ with the polynomial

$$\chi_{r,s}(q) = \sum_{j=0}^{r+s-1} \alpha_j q^j, \quad \alpha_j = \begin{cases} -r, & 0 \leq j \leq s-1, \\ s, & s \leq j \leq r+s-1, \end{cases}$$

of degree $r + s - 1 \leq 4$. Explicitly

$$\chi_{1,2}(q) = -1 + 2q + 2q^2, \ \chi_{2,1}(q) = -2 - 2q + q^2,$$
$$\chi_{1,3}(q) = -1 + 3q + 3q^2 + 3q^3, \ \chi_{3,1}(q) = -3 - 3q - 3q^2 + q^3,$$
$$\chi_{2,3}(q) = -2 - 2q + 3q^2 + 3q^3 + 3q^4, \ \chi_{3,2}(q) = -3 - 3q - 3q^2 + 2q^3 + 2q^4,$$
$$\chi_{2,2}(q) = -2 - 2q + 2q^2 + 2q^3 = 2(q+1)^2(q+1).$$

As in the case $l = 2$, from this we arrive at the following expression for the function $S(z)$ of Theorem 3:

$$S(z) = z^{r+s}(1+z^2)\frac{(1+az)^{r-1}}{(a-z)^r}P_{r,s}(z),\ P_{r,s}(z) = [q_j z^2 + (1-q_j)az + 1],$$

where q_j are the roots of the polynomial $\chi_{r,s}(q)$, taken taking into account the multiplicity.

Since $\sum_{i>j} l_i l_j = 2$, then, based on Theorem 3, we obtain the following conclusion.

Remark 3 *The Fredholm property of the problem (26ii), (27) is equivalent to the absence of zeros of the polynomial $P_{r,s}$ on the circle L and, accordingly, its index* $æ = 4(n-1)$, *where n is the number of these zeros in the lower half-plane.*

The final answer can be given only in the case $r = s = 2$. For it

$$P_{2,2}(z) = (z^2 - 2\tau z - 1)^2(z^2+1)$$

and, as shown in the case $l = 2$ of a fourth-order equation, the first factor here has no zeros in the closed circle $\overline{B}$. Therefore, the problem $(26ii)$, $(27_{2,2})$ is Fredholm and its index is zero.

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